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Quasi-optimal energy-efficient leader election algorithms in radio networks

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Abstract

Radio networks (RN) are distributed systems (*ad hoc networks*) consisting in $n \geqslant 2$ radio stations. Assuming the number n unknown, two distinct models of RN without collision detection (*no-CD*) are addressed: the model with *weak no-CD* RN and the one with *strong no-CD* RN. We design and analyze two distributed leader election protocols, each one running in each of the above two (no-CD RN) models, respectively. Both randomized protocols are shown to elect a leader within $\mathcal{O}(\log(n))$ expected time, with no station being awake for more than $\mathcal{O}(\log\log(n))$ time slots (such algorithms are said to be *energy-efficient*). Therefore, a new class of efficient algorithms is set up that match the $\Omega(\log(n))$ time lower-bound established by Kushilevitz and Mansour [E. Kushilevitz, Y. Mansour, An $\Omega(D\log(N/D))$ lower-bound for broadcast in radio networks, SIAM J. Comp. 27 (1998) 702–712).].

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1. Introduction

Electing a leader is a fundamental problem in distributed systems and it is studied in a variety of contexts including radio networks [3]. A radio network (RN, for short) can be viewed as a distributed system of n radio stations with no central controller. The stations are bulk-produced, hand-held devices and are also assumed to be indistinguishable: *no* identification numbers (or IDs) are available (*anonymous* network).

A large body of research has already focused on finding efficient solutions to elect one station among an n-stations RN under various assumptions (see e.g. [3,11,16]). It is also assumed that the stations run on batteries. Therefore, saving battery power is important, since recharging batteries may not be possible in standard working conditions. We are interested in designing *power-saving* protocols (also called *energy-efficient* protocols). The present work is motivated by various applications in emerging technologies: from wireless communications, cellular telephony, cellular data, etc., to simple sensors, or hand-held multimedia services.

1.1. The models

As customary, the time is assumed to be slotted, the stations work synchronously and they have no IDs available. No *a priori* knowledge is assumed on the number $n \geqslant 2$ of stations involved in the RN, neither a (non-trivial) lower-bound nor an upper-bound on n. During a time slot, a station may be awake or sleeping. When sleeping, a station is unable to listen or to send a message. Awake stations communicate globally, i.e., the underlying graph is a clique, by using a unique radio frequency channel with *no collision detection* (*no-CD* for short) mechanism. In each time slot, the status of the unique channel can be in one of the following two states:

- either SINGLE: there is exactly one transmitting station,
- or NULL: there is either no station or more than two ($\geqslant 2$) broadcasting stations.

When the status is NULL, each listening station hears some noise and can not decide whether 0 or more than 2 station are broadcasting. When the status is SINGLE, each listening station hears clearly the message sent by the unique broadcasting station.

In the *weak* no-CD model of RN, during a time slot each awake station may *either* send (broadcast) a message *or* listen to the channel, *exclusively*. By contrast, in the *strong* no-CD model of RN, both operations can be performed *simultaneously* by each awake station during a time slot. Hence, in the strong no-CD model, when exactly one station sends at time slot t, then all the stations that listen at time t, transmitter included, eventually receive the message. In the literature, the no-CD RN usually means the strong model of RN, see e.g. [11,14]. In the weak no-CD case, such a transmitting station is not aware of the channel status.

Such models feature concrete situations; in particular, the lack of feedback mechanism occurs in real-life applications (see e.g. [12]). Usually, the natural noise existing within radio channels makes it impossible to carry out a message collision detection.

1.2. Related works

The model of RN considered is the broadcast network model (see e.g. [3]). In this setting, the results of Willard [16], Greenberg et al. [7] (with collision detection) or Kushilevitz and Mansour [11] (no-CD) for example, are among the most popular leader election protocols.

In such a model, Massey and Mathys [12] serves as a global reference for basic conflict-resolution based protocols. Previous researches on RN with multiple-access channel mainly concern stations that are kept awake during all the running time of a protocol even when such stations are the "very first losers" of a coin flipping game algorithm [15]. In [9], the authors design an energy-efficient protocol (with $o(\log\log(n))$ energy cost) that approximates n up to a constant factor. However, the running time achieved is $\mathcal{O}\left(\log^{2+\epsilon}(n)\right)$ in strong no-CD RN. Also, an important issue in RN is to perform distribution analysis of various randomized election protocols as derived in [4,8] for example.

1.3. Our results

Two leader election protocols are provided in the present paper. The first one (Algorithm 1) is designed for the *strong* no-CD model of RN, while the second one (Algorithm 2) is designed for the *weak* no-CD model of RN. Both are double-loop randomized algorithms, which use a simple coin-tossing procedure (*rejection algorithms*). Our leader election protocols achieve a (quasi) optimal $\mathcal{O}(\log n)$ average running time complexity, with no station in the RN being awake for more than $\mathcal{O}(\log\log(n))$ time slots. Indeed, both algorithms match the $\Omega(\log n)$ time lower-bound established in [11] and also allow the stations to keep sleeping most of the time. In other words, each algorithm greatly reduces the total awake time slots of the n stations: shrinking from the usual $\mathcal{O}(n\log n)$ down to $\mathcal{O}(n\log\log n)$, while their expected time complexity still remains $\mathcal{O}(\log n)$. These protocols are thus "energy-efficient" and suitable for hand-held devices working with batteries.

Besides the algorithms use a parameter α which works as a flexible regulator. By tuning the value of α the running time ratio of each protocol to its energy consumption may be adjusted : the running time and the awake duration are functions of α. It is also worth mentioning that the algorithms include explicit termination detection.

Furthermore, the design of Algorithms 1 and 2 suggests that within both the weak and the strong no-CD RN, the average time complexity of each algorithm only differs by a slight constant factor. Also, our results improve on [13].

1.4. Outline of the paper

In Section 2, we present Algorithm 1 and Algorithm 2, and the main complexity results are given: Theorem 1 and Theorem 2, corresponding to Algorithm 1 and Algorithm 2. Sections 3 and 4 are devoted to the analysis of both algorithms, by means of tight asymptotic techniques.

2. Algorithms and results

Both algorithms rely on the obvious fact that all stations must be awake together within a sequence of predetermined time slots in order to be informed that the election succeeds of fails.

To complete the election a first naive idea is to have stations using probabilities 1/2, 1/4, ... to wake up and broadcast. This solution is not correct since with probability > 0 no station ever broadcasts alone. In order to correct the failure we will plan an unbounded sequence of rounds with predetermined length. Awake time slots are programmed at the end of each round in order to allow all stations to detect the possible termination of the session.

In the following, we let α denote a real (tuning) parameter value, which is required to be >1 (this is explained in Remarks 4 and 6).

2.1. Algorithm 1

In Algorithm 1 the stations work independently. Given a round j in the outer loop (repeat-until loop), during the execution time of the inner loop each station randomly chooses to sleep or to be awake: in this last case it listens and broadcasts, simultaneously. If a unique station is broadcasting, this station knows the status of the radio channel; if the status is SINGLE, it becomes a *candidate*. At the end of round j, every station wakes up and listens to the channel and the candidates broadcast. If there is a single candidate, the status is SINGLE again, and this candidate is elected. Every listening station knows the channel status, and is informed that the election is done. Otherwise, the next round begins.

```
round ← 1;
Repeat
For k from 1 to ⌈α^round⌉ do                                  /* probabilistic phase */
  [ Each station wakes up independently with probability 1/2^k.
  [ An awake station listens and broadcasts
  [ If a unique station broadcasts Then it becomes a candidate station EndIf
EndFor                                                         /* deterministic phase */
  [ At the end of each round, all stations wake up, listen,
  [ and in addition all candidate stations broadcast;
  [ If there is a unique candidate then it is elected EndIf;
round ← round + 1;
until a station is elected
```

Algorithm 1. Leader election protocol for strong no-CD RN

The brackets in both algorithms represent the actions that take place in one time slot. Notice that the content of the broadcasting message is not specified, since it has no importance. The status “candidate” is valid for one round duration only.

Definition and Notation 1. For the sake of simplicity, the following notations are used throughout the paper. We let

$$j^\star \equiv j^\star(n) \stackrel{\text{def}}{=} \lceil \log_\alpha \log_2(n) \rceil. \tag{1}$$

Let also $\mathcal{C} : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\mathcal{C}(x, y) \stackrel{\text{def}}{=} \frac{xy^3}{(y-1)\Big(1 - y(1-x)\Big)}. \tag{2}$$

We have

Theorem 1. *Let* $\alpha \in (1, 1.1743\ldots)$ *and* $p_1^\star = .14846\ldots$ *On the average, Algorithm* 1 *elects a leader in at most* $\mathcal{C}(p_1^\star, \alpha)\log_2(n) + \mathcal{O}(\log\log n)$ *time slots, with no station being awake for more than* $2\log_\alpha \log_2(n)\Big(1 + o(1)\Big)$ *mean time slots.*

2.2. Algorithm 2

In the case of the weak no-CD model of RN a potential candidate alone cannot be aware of its status since it cannot broadcast and listen *simultaneously*. The awake stations choose to broadcast or (exclusively) to listen. In the inner loop of Algorithm 2 any exclusively broadcasting station is called an *initiator*; when there is a unique initiator, a station that is listening to the channel, hears clearly the message. Such a station is said to be a *witness*. Witnesses are intended to acknowledge an initiator in the case when there is exactly one initiator.

---

round ← 1;
Repeat

For k from 1 to $\lceil \alpha^{round} \rceil$ **do** /* probabilistic phase */

- Each station wakes up independently with probability $1/2^k$;
- With probability 1/2, each awake station *either* broadcasts the message$\langle k \rangle$ *or* listens, exclusively;
- **If** there is a unique initiator, the status is SINGLE;
- The witness(es) record(s) the value $\langle k \rangle$ **EndIf**

EndFor /* deterministic phase */

- At the end of each round, all stations wake up;
- All witnesses forward the recorded message;
- **If** the status is SINGLE
- there is a unique witness, the other stations receive the message $\langle k \rangle$
- **Then** the unique initiator of the message $\langle k \rangle$ is *elected*

- and replies to all the stations to advise of its status **EndIf**

round ← *round* + 1;

until a station is elected.

Algorithm 2. Leader election protocol for weak no-CD RN

During the last deterministic phase all stations wake up. An election takes place in a round if this round is the first when there is a unique time slot in which there remains exactly a unique initiator and a unique witness. When a station is a witness, it continues to behave as if it were not. Hence, a witness can be a witness twice or more, or even be an initiator. This is obviously not optimal but simpler for the analysis.

We have

Theorem 2. *Let* $\alpha \in (1, 1.0861\ldots)$ *and* $p_2^\star = .07929\ldots$ *On the average, Algorithm* 2 *elects a leader in at most* $\mathcal{C}(p_2^\star, \alpha)\log_2(n) + \mathcal{O}(\log\log n)$ *time slots, with no station being awake for more than* $5/2\,\log_\alpha \log_2(n)\Big(1 + \mathrm{o}(1)\Big)$ *mean time slots.*

3. Analysis of Algorithms 1 and 2

To prove Theorems 1 and 2, we give lower bounds on the probability of success in each round, by using the following Lemma 1 to control the mean of the quantities of interests.

Lemma 1. *Let* $(X_i)_{i\geqslant 1}$ *and* $(Y_i)_{i\geqslant 1}$ *be two sequences of independent Bernoulli random variables, denoted by* $B(P_i)$ *and* $B(Q_i)$, *respectively, and such that* $P_i \geqslant Q_i$ *for any* i. *By definition,*

$$\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 0) = P_i \quad \text{and} \quad \mathbb{P}(Y_i = 1) = 1 - \mathbb{P}(Y_i = 0) = Q_i.$$

Let $H = \inf\{j, X_j = 1\}$ *and* $K = \inf\{j, Y_j = 1\}$, *which may be regarded as a first success in each sequence* (X_i) *and* (Y_i) *(resp.). Then, the stochastic dominance* $H \leq_S K$ *holds: for any non-negative integer* k, $\mathbb{P}(H \leqslant k) \geqslant \mathbb{P}(K \leqslant k)$.
As a consequence for any non-decreasing function f,

$$\mathbb{E}(f(H)) \leqslant \mathbb{E}(f(K)). \tag{3}$$

Proof. The first part of the Lemma can be proved by constructing a probability space Ω in which the sequences of r.v. (X_i) and (Y_i) "live" and in which, for each $\omega \in \Omega$, $X_i(\omega) \geqslant Y_i(\omega)$. Since for any ω, $K(\omega) \geqslant H(\omega)$, the stochastic order is a simple consequence of this almost sure order on Ω. Next, for any nondecreasing function f, $f(K(\omega)) \geqslant f(H(\omega))$ also holds almost surely, whence Eq. (3). □

3.1. Proof of Theorem 1

Assume that Algorithm 1 begins with its variable *round* $= j$, and let p_j be the probability that one station is *elected* at the end of this round j. One may also view p_j as the conditional probability that an election occurs at round j knowing that it did not occur before. Within that round, that is for k ranging from 1 to $\lceil\alpha^j\rceil$, every station decides to broadcast with the sequence of probabilities $(1/2^k)_{1\leqslant k\leqslant\lceil\alpha^j\rceil}$.

The probability $\rho_{(i,n)}$ that there is exactly one candidate for $k = i$ in a round is $\rho_{(i,n)} = \dfrac{n}{2^i}\left(1 - \dfrac{1}{2^i}\right)^{n-1}$. We then have

$$p_j = \sum_{k=1}^{\lceil\alpha^j\rceil} \rho_{(k,n)} \prod_{\substack{i=1\\ i\neq k}}^{\lceil\alpha^j\rceil} \left(1 - \rho_{(i,n)}\right) = \sum_{k=1}^{\lceil\alpha^j\rceil} \rho_{(k,n)} \left(1 - \rho_{(k,n)}\right)^{-1} \prod_{i=1}^{\lceil\alpha^j\rceil} \left(1 - \rho_{(i,n)}\right) \tag{4}$$

which rewrites $p_j = t_j s_j$ with

$$t_j \equiv \sum_{m=0}^{\infty} \sum_{k=1}^{\lceil \alpha^j \rceil} \left(\rho_{(k,n)}\right)^{(m+1)} \quad \text{and} \quad s_j \equiv \prod_{i=1}^{\lceil \alpha^j \rceil} (1 - \rho_{(i,n)}). \tag{5}$$

The following Lemma 2 provides a lower bound on $\liminf_n p_j$.

Lemma 2. *Let $j \equiv j(n)$. Assuming that $n/2^{\alpha^j} \to 0$ (so that $j(n) \to +\infty$),*

$$\liminf_n p_j > p_1^\star = .14846\ldots$$

We postpone the proof of this Lemma until Appendix A.

Remark 3. Simple considerations show that when $2^{\alpha^j} \ll n$, the probability s_j of having no candidate in the jth round is close to 1 (and it is “far from” 1 when $2^{\alpha^j} \gg n$). This remark explains the occurrences of the crucial values $n/2^{\alpha^j}$ and the definition of $j^\star$ in Eq. (1).

We return to the proof of Theorem 1. It is straightforward to check that, when $n \to \infty, n/2^{\alpha^{(j^\star+1)}} \to 0$. According to Lemma 2, if n is large enough

$$p_j \geqslant p_1^\star \, \mathbb{I}_{j \geqslant j^\star+1}. \tag{6}$$

As a consequence, let N_1 denote the number of rounds in Algorithm 1 and $N_1' = j^\star + G$ where G is a geometric random variable with parameter $p_1^\star$, then

$$N_1 \leq_S N_1'. \tag{7}$$

We recall that the geometric distribution with parameter p is given by $p(1-p)^{(k-1)}$ for any $k \geqslant 1$; this is the law of the first success in a sequence of independent Bernoulli random variables with parameter p.

To prove Eq. (7), notice first that N_1 has the same distribution as $\inf\{i, X_i = 1\}$, where the (X_i) are independent and X_i is $B(p_i)$-distributed. Next, $N_1' = \inf\{i, Y_i = 1\}$, where Y_i is $B(q_i)$-distributed for $q_i = p_1^\star \, \mathbb{I}_{j \geqslant j^\star+1}$. Indeed, the first $j^\star$ trials fail and afterwards, each trial results in a success with probability $p_1^\star$.
Finally, Eq. (6) and Lemma 1 allow to conclude,

$$\mathbb{E}(N_1) \leqslant \mathbb{E}(N_1') = j^\star + 1/p_1^\star = \log_\alpha \log_2(n) + \mathcal{O}(1).$$

Let $T_1 \equiv T_1(n)$ be the time needed to elect a leader in Algorithm 1. Since $N_1 \leq_S N_1'$ and since $r \mapsto \sum_{i=1}^{r} \lceil \alpha^i \rceil$ is non-decreasing, by Lemma 1,

$$\mathbb{E}(T_1) = \mathbb{E}\left(\sum_{j=1}^{N_1} \lceil \alpha^j \rceil\right) \le \mathbb{E}\left(\sum_{j=1}^{N_1'} \lceil \alpha^j \rceil\right) \le \sum_{k=1}^{+\infty} \sum_{j=1}^{j^\star + k} (1+\alpha^j) p_1^\star (1-p_1^\star)^{k-1}$$
$$\leqslant \mathcal{C}(p_1^\star, \alpha) \log_2(n) + \mathcal{O}(\log\log n). \tag{8}$$

During a round, the mean number of awake time slots for a given station is at most 2. (One at the end and one from the contribution of $\sum_k 1/2^k$, $1/2^k$ being the probability to be awake at time slot k.)

Since the number of rounds is smaller than, say $\log_\alpha \log_2(n) + \log\log\log(n)$, with probability going to 1, we get the announced result.

Remark 4. It is easily seen that the algorithm and the convergence of the double sum in Eq. (8) (resp.) require the conditions $\alpha > 1$ and $\alpha(1 - p_1^\star) < 1$, with $p_1^\star = .14846$ (resp.). The value of α may thus be chosen in the range $(1, 1.743\ldots)$, so as to achieve a tradeoff between the average execution time of the algorithm and the global awake time. Thus, the minimum value of the constant $\mathcal{C}(p_1^\star, \alpha)$ is $\mathcal{C}(p_1^\star, \widetilde{\alpha}) \simeq 29.058\ldots$, with $\widetilde{\alpha} = 1.0767\ldots$

3.2. Proof of Theorem 2

Two awake stations are needed in Algorithm 2: the one is only sending (the initiator) and the other is listening. The probability p_j' that one station is elected in round j expresses along the same lines as in Eq. (5) with the corresponding probability. The probability of having exactly one initiator and one witness for $k = j$ in a round is

$$q_j^n \equiv \frac{1}{2} \frac{\binom{n}{2}}{4^j} \left(1 - \frac{1}{2^j}\right)^{n-2}.$$

Hence $p_j' = t_j' s_j'$, where

$$t_j' \equiv \sum_{k=1}^{\lceil \alpha^j \rceil} q_{(k,n)} \left(1 - q_{(k,n)}\right)^{-1} \quad \text{and} \quad s_j' \equiv \prod_{i=1}^{\lceil \alpha^j \rceil} (1 - q_{(i,n)}). \tag{9}$$

The proof of Theorem 2 is the same as the proof of Theorem 1, except that the lower bound on $\liminf_n p_j'$ given by the following Lemma 5, replaces the one on $\liminf_n p_j$ (in Lemma 2).

Lemma 5. *Let $j \equiv j(n)$. Assuming that $n/2^{\alpha^j} \to 0$ (so that $j(n) \to +\infty$),*

$$\liminf_n p_j' > p_2^\star := .07929\ldots$$

The proof of this Lemma is postponed until the Appendix.

Remark 6. The lower bound $p_2^\star$ in Algorithm 2 is already defined in (2). Now, since $p_2^\star = .07929$, α meets the new condition if it belongs to $(1, 1.086\ldots)$, and the minimum value of the constant $\mathcal{C}(p_2^\star, \alpha)$ is $\mathcal{C}(p_2^\star, \widetilde{\alpha}) \simeq 52.516$, with $\widetilde{\alpha} = 1.0404\ldots$

Remark 7. Note also that Algorithms 1 and 2 can be improved by starting from $k = k_0$, $k_0 > 1$ (instead of 1) in the third line of the algorithms. Though the running time of each algorithm remains asymptotically the same, starting from $k = k_0 > 1$ reduces the awake time to $(1 + \epsilon) \log_\alpha \log_2(n)$ time slots in Algorithm 1, and to $(1.5 + \epsilon) \log_\alpha \log_2(n)$ time slots in Algorithm 2 (with $\epsilon = 1/2^{k_0 - 1}$). Yet, this also makes the running time longer for small values of n; whence the (obvious) fact that the knowledge of any lower bound on n greatly helps.

A. Appendix

A.1. Two technical results

The following two technical Lemmas are at the basis of the asymptotic complexity analysis of Algorithms 1 and 2, in the proof of Theorems 1 and 2, respectively. They both use Mellin transform techniques [5,6,10]. Note that real asymptotic approximations, such as e.g. Euler–Maclaurin summation, only provide a $\mathcal{O}(1)$ error term when n gets large. In the present case, such an error term is far too imprecise to stay under control when summing on m.

Lemma 8. *Let $r \equiv r(n)$ such that, when $n \to +\infty$, we have $n/2^r \to 0$ (and consequently $r \to \infty$). Then, for all positive integer m,*

$$\sum_{k=1}^{r} \left(\frac{n}{2^k}\right)^m \exp\left(-\frac{nm}{2^k}\right) = \frac{m!}{m^{m+1} \ln 2} + \frac{1}{m\, 2^m} U_m\Big(\log_2(n)\Big) + \mathcal{O}\left(\frac{2^m}{n^m}\right) + \mathcal{O}\left(\frac{n^m}{2^{rm}}\right). \tag{A.1}$$

Denote $\chi_\ell \equiv 2\mathrm{i}\ell\pi/\ln 2$. For any positive integer m, $U_m\Big(\log_2(n)\Big)$ is defined as

$$U_m(z) = \frac{-2^m}{m^{m-1} \ln 2} \sum_{\ell \in \mathbb{Z}\setminus\{0\}} \Gamma(m + \chi_\ell) \exp\Big(-\chi_\ell \ln(m)\Big) \exp(-2\mathrm{i}\ell\pi z). \tag{A.2}$$

The Fourier series $U_m(z)$ has mean value 0 *and the amplitude of its coefficients does not exceed* .024234.

Proof. The asymptotic approximation of the finite sum in Eq. (A.1) is obtained by direct use of the properties of the Mellin transform [5,6].

The sum

$$f(x) = \sum_{k=1}^{+\infty} \left(\frac{x}{2^k}\right)^m \exp\left(-\frac{mx}{2^k}\right)$$

is to be analyzed as $x \to \infty$. Its Mellin transform, with fundamental strip $\langle -1, 0\rangle$, is

$$f^*(s) = \frac{\Gamma(s+m)2^s}{m^{s+m}(1-2^s)} .$$

There is a simple pole at $s = 0$, but also simple imaginary poles at $s = \chi_\ell \equiv 2\mathrm{i}\ell\pi/\ln 2$ (for all non zero integers ℓ), which are expected to introduce periodic fluctuations. The singular expansion of $f^*(s)$ in $\langle -1/2, 2\rangle$ is

$$f^*(s) \asymp \left[\frac{\Gamma(m)}{m^m \ln 2}\,\frac{1}{s}\right] - \frac{1}{m^m \ln 2} \sum_{\ell\in\mathbb{Z}\setminus\{0\}} \frac{\Gamma(m+\chi_\ell)\,\exp\Big(-\chi_\ell\,\ln(m)\Big)}{s-\chi_\ell} .$$

Accordingly, by putting $\sum_{k=1}^{r}\left(\frac{x}{2^k}\right)^m \exp\left(-\frac{mx}{2^k}\right)$ back into $f(x)$, one finds the results stated in Eqs. (A.1) and (A.2). Note that the sign minus appears before $U_m(z)$ since $n \to +\infty$, and so does x in the sum $f(x)$ (see the proof e.g. in [6]).

In Eq. (A.1), the periodic fluctuations are occurring under the form of the Fourier series $U_m\Big(\log_2(n)\Big)$ with mean value 0. Besides, for any positive integer m, the maximum of the amplitude of the Fourier series is taken at $m = 11$ and it is rather small

$$\forall m > 0 \quad |U_m(z)| \leqslant \sum_{\ell\in\mathbb{Z}\setminus\{0\}} \frac{2^m\,|\Gamma(m+\chi_\ell)|}{m^{m-1}\ln 2} < .024234 \tag{A.3}$$

The Fourier coefficients also decrease very fast (see e.g., [6] or [10, p. 131]). The error terms $\mathcal{O}(2^m/n^m)$ and $\mathcal{O}(n^m/2^{rm})$ in Eq. (A.1) result from the "truncated" summation: $1 \leqslant k \leqslant r$. (The Mellin transform itself results in a $\mathcal{O}\left(n^{-1}\right)$ error term.) □

Lemma 9. *Again, let $r \equiv r(n)$ such that, when $n \to +\infty$, we have $n/2^r \to 0$ (and consequently $r \to \infty$). Then, for all positive integer m,*

$$\sum_{k=1}^{r} \frac{1}{4^m}\left(\frac{n^2}{4^k}\right)^m \exp\left(-\frac{nm}{2^k}\right) = \frac{(2m-1)!}{4^m m^{2m+1}\ln 2} + \frac{1}{m^2} V_m\Big(\log_2(n)\Big) + \mathcal{O}\left(\frac{2^m}{n^m}\right) + \mathcal{O}\left(\frac{n^m}{2^{rm}}\right). \tag{A.4}$$

Again denote $\chi_\ell \equiv 2\mathrm{i}\ell\pi/\ln 2$. For any positive integer m, the above Fourier series

$$V_m(z) = -\frac{m}{4^m m^{2m}\ln 2} \sum_{\ell\in\mathbb{Z}\setminus\{0\}} \Gamma(2m+\chi_\ell)\,\exp\Big(-2\mathrm{i}\ell\pi\log_2(m)\Big)\,\exp(-2\mathrm{i}\ell\pi z)$$

has mean value 0 and the amplitude of the coefficients of $V_m(z)$ cannot exceed $9.0054\,10^{-5}$.

Proof. An asymptotic approximation of the finite sum in Eq. (A.4) is computed along the same lines as in Lemma 8. Now,

$$g(x) = \sum_{k=1}^{+\infty} \frac{1}{4^m} \left(\frac{x}{2^k}\right)^{2m} \exp\left(-\frac{mx}{2^k}\right)$$

is to be analyzed when $x \to \infty$. Similarly, its Mellin transform is $g^*(s) = \dfrac{\Gamma(s+2m)2^s}{m^{s+2m}(1-2^s)}$, with fundamental strip $\langle -1, 0\rangle$. Again there is a simple pole at $s = 0$ and simple imaginary poles at $s = \chi_\ell \equiv 2i\ell\pi/\ln 2$ (for all non zero integers ℓ), which are also expected to introduce periodic fluctuations. Also note that the sign minus before $V_m(z)$ appears again for the same reason as in Lemma 8.

As in Lemma 8 the singular expansion of $g^*(s)$ provides an asymptotic approximation of the finite sum in Eq. (A.4). The periodic fluctuations occur under the form of the Fourier series $V_m\Big(\log_2(n)\Big)$. As $U_m(z)$, $V_m(z)$ has mean value 0 and its coefficients have a very tiny amplitude. Their maximum is taken at $m = 2$, and

$$\forall m > 0 \quad |V_m(z)| \leqslant \sum_{\ell \in \mathbb{Z}\setminus\{0\}} \frac{|\Gamma(2m+\chi_\ell)|}{4^m\, m^{2m-1} \ln 2} < 9.0054\, 10^{-5}.$$

The Fourier coefficients also decrease very fast (see e.g., [6] or [10, p. 131]). Last, the "truncated" summation ($1 \leqslant k \leqslant r$) results again in error terms $\mathcal{O}(2^m/n^m)$ and $\mathcal{O}(n^m/2^{rm})$ in Eq. (A.4). □

A.2. Proofs of the lower Bound on $\liminf p_j$ *and* $\liminf p'_j$

Lemma 10 provides a lower bound on the finite product denoted by s_j in Eq. (5).

Lemma 10. *Let $j \equiv j(n)$. Assuming that $n/2^{\alpha^j} \to 0$ (which implies that $j(n) \to +\infty$),*

$$\liminf_n s_j \geqslant .1809\ldots.$$

Proof. We rewrite the product s_j in (5) as follows: for any $r \in \{2, \ldots, \lceil \alpha^j \rceil\}$,

$$s_j = \prod_{i=1}^{r-1} (1 - \rho_{(i,n)}) \times \prod_{i=r}^{\lceil \alpha^j \rceil} (1 - \rho_{(i,n)}), \tag{A.5}$$

where $\rho_{(i,n)} = \dfrac{n}{2^i}\left(1 - \dfrac{1}{2^i}\right)^{n-1}$ (as defined in Subsection 3.1).

Since for any $a \in [0,1]$ and $n \geqslant 1$, $(1-a)^n \leqslant e^{-an}$, we have for any $i \geqslant 1$,

$$1 - \rho_{(i,n)} \geqslant 1 - \frac{n}{2^i} \exp\left(-\frac{n-1}{2^i}\right) \geqslant 1 - \frac{n}{2^i} \exp\left(-\frac{n}{2^i}\right) e^{1/2}; \tag{A.6}$$

and more precisely, for $i \in \{r, \ldots, \lceil \alpha^j \rceil\}$, with r large enough,

$$1-\rho_{(i,n)} \;\geqslant\; 1-\frac{n}{2^i}\exp\Big(-\frac{n}{2^i}\Big)\exp\left(\frac{1}{2^r}\right) \;\geqslant\; 1-\frac{n}{2^i}\exp\Big(-\frac{n}{2^i}\Big)\left(1+\frac{2}{2^r}\right). \tag{A.7}$$

Now we let $r \equiv r(n) = \lfloor 1/2 \log_2(n) \rfloor$, ensuring that $2^r/\sqrt{n}$ is bounded. A simple bounding argument in Eq. (A.6) shows that $\prod_{i=1}^{r-1}(1-\rho_{(i,n)}) \to 1$ when $n \to \infty$.

Thus, using also inequality (A.7), Eq. (A.5) yields

$$\begin{aligned}\liminf_n s_j &\geqslant \exp\left(\sum_{i=r}^{\lceil\alpha^j\rceil}\ln\left(1-\left(1+\frac{2}{2^r}\right)\frac{n}{2^i}\exp\Big(-\frac{n}{2^i}\Big)\right)\right)\\ &\geqslant \exp\left(-\sum_{m\geqslant 1}\frac{\left(1+\frac{2}{2^r}\right)^m}{m}\sum_{i=1}^{\lfloor\alpha^j\rfloor}\frac{n^m}{2^{im}}\exp\Big(-\frac{nm}{2^i}\Big)\right).\end{aligned} \tag{A.8}$$

Denote by A_n the right hand side of the inequality. Computing the second sum (inside the exponential in A_n) is completed through asymptotic approximations using the Mellin transform in Lemma 8 (given in Appendix A.1). This is possible since, by assumption, $n/2^{\alpha^j} \to 0$. And by Lemma 8,

$$A_n \geqslant \exp\left(\mathrm{o}(1) - \sum_{m\geqslant 1}\frac{\left(1+\frac{2}{2^r}\right)^m m!}{m^{m+2}\ln 2} - \frac{\left(1+\frac{2}{2^r}\right)^m}{m^2\,2^m}\sup_\nu |U_\nu(\log_2 n)|\right), \tag{A.9}$$

where $\sup_\nu |U_\nu(z)| < .024234$ for all z, as stated in Lemma 8.
Therefore, the third term in Eq. (A.9) is bounded from below by

$$\exp\Big(-.024234\sum_{m\geqslant 1}\frac{1}{m^2}\Big) = .96092\ldots$$

Next, the second term within the exponential in Eq. (A.9) evaluates to

$$\sum_{m=1}^{\infty}\frac{\left(1+\frac{2}{2^r}\right)^m m!}{(\ln 2)\,m^{m+2}} \leqslant \sum_{m=1}^{n^{1/6}}\frac{\left(1+\frac{2}{2^r}\right)^m m!}{(\ln 2)\,m^{m+2}} + \sum_{m=n^{1/6}+1}^{\infty}\frac{2^m m!}{(\ln 2)\,m^{m+2}}. \tag{A.10}$$

As a consequence of Lebesgue's dominated convergence theorem, the first sum on the right hand side of Eq. (A.10) converges to

$$\sum_{m\geqslant 1}\frac{m!}{(\ln 2)\,m^{m+2}} = 1.6702\ldots,$$

(when n and so r tends to $+\infty$), the numerical results being given by Maple. The second term of the right hand side of Eq. (A.10) goes to 0, as can be derived from Stirling formula.
Finally, combining all these facts we obtain the announced lower bound on s_j. □

Proof of Lemma 2. Since $(1-x) \geqslant \exp(-2x)$ for $x \in [0, 1/2]$, we can derive the following,

$$\begin{aligned}
\sum_{k=1}^{\lceil\alpha^j\rceil} (\rho_{(k,n)})^{(m+1)} &\geqslant \sum_{k=1}^{\lceil\alpha^j\rceil} \left(\frac{n}{2^k}\exp\left(-\frac{n-1}{2^{k-1}}\right)\right)^{(m+1)} \geqslant \sum_{k=1}^{\lceil\alpha^j\rceil} \frac{1}{2^{m+1}}\left(\frac{n}{2^{k-1}}\right)^{m+1} \exp\left(-\frac{n(m+1)}{2^{k-1}}\right) \\
&= \sum_{k=1}^{\lceil\alpha^j\rceil} \frac{1}{2^{m+1}}\left(\frac{n}{2^k}\right)^{m+1} \exp\left(-\frac{n(m+1)}{2^k}\right) \\
&+ \frac{1}{2^{m+1}}\left(n^{m+1}\exp\big(-n(m+1)\big) - \left(\frac{n}{2^{\lceil\alpha^j\rceil}}\right)^{m+1} \exp\left(-\frac{n(m+1)}{2^{\lceil\alpha^j\rceil}}\right)\right) \\
&= \frac{1}{2^{m+1}} \frac{(m+1)!}{(m+1)^{m+2}\ln 2} + \frac{U_{m+1}\big(\log_2(n)\big)}{(m+1)^2\, 4^{m+1}} + \mathcal{O}\left(\frac{n}{2^{\alpha^j+m+1}} + \frac{1}{n\, 2^{m+1}}\right).
\end{aligned}$$

By assumption, $n/2^{\alpha^j} \to 0$. So, the latter expression comes from Lemma 8, where the term $U_m(z)$ is defined in Eq. (A.2). Now, summing on m in Eq. (5) we derive

$$t_j \geqslant \mathcal{O}\left(\frac{n}{2^{\alpha^j}} + \frac{1}{n}\right) + \sum_{m=1}^{\infty} \frac{m!}{2^m\, m^{m+1}\ln 2} - \sum_{m=1}^{\infty} \frac{\sup_\nu |U_\nu(\log_2 n)|}{m^2\, 4^m}.$$

By Eq. (A.3), numerical evaluations of the sums (with Maple) give $\liminf_n t_j \geqslant .82092\ldots$
Finally, by using both lower bounds on s_j and t_j (in Lemma 10 and in the above proof, resp.) we obtain the desired lower bound on $\liminf_n p_j > p_1^\star = .14846\ldots$ □

Proof of Lemma 5. First,

$$\begin{aligned}
s'_j &\geqslant \prod_{i=1}^{\lceil\alpha^j\rceil} \left(1 - \frac{1}{2}\frac{\binom{n}{2}}{4^i}\exp\left(-\frac{n-2}{2^i}\right)\right), \quad \text{since } \forall x \in [0,1]\ \ 1-x \leqslant e^{-x}, \\
&\geqslant \prod_{i=1}^{\lceil\alpha^j\rceil} \left(1 - \frac{n^2}{4^{i+1}}\exp\left(-\frac{n-2}{2^i}\right)\right), \quad \text{since } \binom{n}{2} \leqslant \frac{n^2}{2}, \\
&\geqslant \prod_{i=1}^{\lceil\alpha^j\rceil} \left(1 - \frac{n^2}{4^i}\exp\left(-\frac{n}{2^i}\right)\frac{e}{4}\right) \geqslant \prod_{i=1}^{\lceil\alpha^j\rceil} \left(1 - \frac{n^2}{4^i}\exp\left(-\frac{n}{2^i}\right)\right).
\end{aligned}$$

Using this latter lower bound in Lemma 9 yields

$$\begin{aligned}
s'_j &\geqslant \exp\left(-\sum_{m=1}^{\infty}\sum_{i=1}^{\lceil\alpha^j\rceil} \frac{1}{m}\frac{n^{2m}}{4^{im}}\exp\left(-\frac{nm}{2^i}\right)\right) \\
&\geqslant \exp\left(o(1) - \sum_{m=1}^{\infty} \frac{1}{m}\frac{(2m-1)!}{m^{2m+1}\ln 2} - \sum_{m=1}^{\infty}\left(\frac{1}{m^3}\sup_\nu |V_\nu(\log_2 n)|\right)\right),
\end{aligned}$$

with the following value of the lower bound on s'_j (computed with Maple):

$$\liminf_n s'_j \geqslant .19895\ldots$$

Next, $1 - x^2 \geqslant e^{-x}$ when x is close to 0 and so $\binom{n}{2} \geqslant \frac{n^2}{2} \exp\left(-\frac{1}{\sqrt{n}}\right)$. Whence t'_j also is bounded from below as follows:

$$t'_j \geqslant \sum_{k=1}^{\lceil \alpha^j \rceil} \sum_{m=1}^{\infty} \left(\frac{1}{2} \frac{\binom{n}{2}}{4^k} \left(1 - \frac{1}{2^k}\right)^n \right)^m \geqslant \sum_{m=1}^{\infty} \sum_{k=1}^{\lceil \alpha^j \rceil} \left(\frac{1}{2} \frac{n^2}{2} \frac{\exp\left(-\frac{1}{\sqrt{n}}\right)}{4^k} \left(1 - \frac{1}{2^k}\right)^n \right)^m .$$

Now, $1 - x \geqslant \exp(-x - x^2)$ when $x \in [0, 1/2]$, and

$$\begin{aligned} t'_j &\geqslant \sum_{m=1}^{\infty} e^{-\frac{m}{\sqrt{n}}} \sum_{k=1}^{\lceil \alpha^j \rceil} \frac{1}{4^m} \left(\frac{n^2}{4^k} \exp\left(-\frac{n}{2^k} - \frac{n}{4^k}\right) \right)^m \\ &\geqslant \sum_{m=1}^{\infty} e^{-\frac{m}{\sqrt{n}}} \sum_{k=r}^{\lceil \alpha^j \rceil} \frac{1}{4^m} \left(\frac{n^2}{4^k} \exp\left(-\frac{n}{2^k} - \frac{n}{4^r}\right) \right)^m , \end{aligned} \tag{A.11}$$

where the last summation is starting from $r \equiv r(n) = 3/2 \log_2(n)$. For such a choice of r, we have $2^r \ll n \ll 4^r$ (which is used in the following).

Therefore, we can now use Lemma 9 to deal with the inner sum in Eq. (A.11)

$$\begin{aligned} t'_j &\geqslant \sum_{m=1}^{\infty} \exp\left(-\frac{m}{\sqrt{n}} - m\frac{n}{4^r}\right) \sum_{k=r}^{\lceil \alpha^j \rceil} \frac{1}{4^m} \left(\frac{n^2}{4^k} \exp\left(-\frac{n}{2^k}\right) \right)^m \\ &\geqslant \sum_{m=1}^{n^{1/4}} e^{-\frac{m}{\sqrt{n}} - \frac{mn}{4^r}} \left(\frac{(2m-1)!}{4^m\, m^{2m+1} \ln 2} - \frac{1}{m^2} \sup_\nu |V_\nu(\log_2 n)| \right) + o(1). \end{aligned}$$

By Lebesgue's monotone convergence theorem, this sum converges to

$$\sum_{m=1}^{+\infty} \left(\frac{(2m-1)!}{4^m\, m^{2m+1} \ln 2} - \frac{1}{m^2} \sup_\nu |V_\nu(\log_2 n)| \right),$$

and the numerical value obtained is $\liminf_n t'_j \geqslant .39856\ldots$ Finally, from the lower bounds values on s'_j and t'_j, we find

$$\liminf_n p'_j \geqslant .079294\ldots \qquad \square$$

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