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Note

Characterizations of Clifford semigroup digraphs

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Abstract

This paper characterizes directed graphs which are Cayley graphs of strong semilattices of groups and, in particular, strong chains of groups, i.e. of completely regular semigroups which are also called Clifford semigroups.

Keywords: Cayley graph; Digraph; Clifford semigroup; Strong semilattice of groups; Completely regular

1. Introduction

One of the first investigations on Cayley graphs of algebraic structures can be found in Maschke's Theorem from 1896 about groups of genus zero, that is, groups G which possess a generating system A such that the Cayley graph $\mathrm{Cay}(G, A)$ is planar, see for example [15]. In [10] Cayley graphs which represent groupoids, quasigroups, loops or groups are described. The result for groups originates from [14] and is meanwhile folklore, see for example [2]. After this it is natural to investigate Cayley graphs for semigroups which are unions of groups, so-called completely regular semigroups, see for example [13]. In [1] Cayley graphs which represent completely regular semigroups which are right (left) groups are characterized. We now characterize Cayley graphs which represent Clifford semigroups. Recent studies in a different direction investigates transitivity of Cayley graphs of groups and semigroups [7], of right and left groups and of Clifford semigroups [12]. Other related results can be found, for example, in [5,7,9]. Relations to network theory together with many theoretical results are presented in [4]. The concept of the Cayley graph of a groupoid has also been considered in relation to automata theory in a recent book by Kelarev [6].

2. Basic definitions and results

All sets in this paper are assumed to be finite. A *groupoid* is a non-empty set G together with a binary operation on G. A *semigroup* is a groupoid G which is associative. A *monoid* is a semigroup G which contains a (two-sided) identity

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element $e_G \in G$. A *group* is a monoid G such that for every $a \in G$ there exists a group inverse $a^{-1} \in G$ such that $a^{-1}a = aa^{-1} = e_G$.

Let S be a semigroup. The set $C(S) = \{c \in S | cs = sc \text{ for all } s \in S\}$ is called the *center* of S. The set of all idempotents of S is denoted by $E(S)$. An element $s \in S$ is called a *regular element*, if $sxs = s$ for some $x \in S$. One calls S a *regular semigroup* if all of its elements are regular. A regular semigroup S is called a *Clifford semigroup* if $E(S) \subseteq C(S)$, i.e. idempotents of S commute with all elements of S.

If $(Y, \leqslant)$ is a non-empty partially ordered set such that the meet $a \wedge b$ of a and b exists for every a, b in Y we say that $(Y, \leqslant)$ is a (lower) *semilattice*. A semigroup S is said to be a *semilattice of groups* $(G_\alpha, \circ_\alpha)$, $\alpha \in Y$, if Y is a semilattice, $S = \bigcup_{\alpha \in Y} G_\alpha$, and $G_\alpha G_\beta \subseteq G_{\alpha\wedge\beta}$, and a *strong semilattice of groups* (or a *strong chain of groups* if Y is a chain) if, in addition, for all $\beta \geqslant \alpha$ in Y there exists a group homomorphism $f_{\beta,\alpha} : G_\beta \to G_\alpha$ such that $f_{\alpha,\alpha} = \mathrm{id}_{G_\alpha}$ is the identity mapping and for all $\alpha, \beta, \gamma \in Y$ with $\alpha \leqslant \beta \leqslant \gamma$, we have $f_{\beta,\alpha} \circ f_{\gamma,\beta} = f_{\gamma,\alpha}$, where the multiplication $*$ on $S = \bigcup_{\alpha \in Y} G_\alpha$ is defined for $x \in G_\alpha$ and $y \in G_\beta$ by

$$x * y = f_{\alpha,\alpha\wedge\beta}(x) \circ_{\alpha\wedge\beta} f_{\beta,\alpha\wedge\beta}(y).$$

Theorem 2.1 (*Kilp et al. [10], Petrich and Reilly [13]*). *For a semigroup S the following are equivalent*:

(i) *S is a Clifford semigroup*,
(ii) *for every $s \in S$ there exists $x \in S$ such that $sxs = s$ and $sx = xs$*,
(iii) *S is a semilattice of groups*,
(iv) *S is a strong semilattice of groups*.

In the sequel, we will use the term strong semilattice of groups instead of Clifford semigroup, since we will use the respective properties and formulate results also for the more special case of strong chains of semigroups. We need the following standard definitions.

Definition 2.2. Let (V_1, E_1) and (V_2, E_2) be digraphs. A mapping $\varphi : V_1 \to V_2$ is called a (*digraph*) *homomorphism* if $(u, v) \in E_1$ implies $(\varphi(u), \varphi(v)) \in E_2$, i.e. φ preserves arcs. We write $\varphi : (V_1, E_1) \to (V_2, E_2)$. A digraph homomorphism $\varphi : (V, E) \to (V, E)$ is called a (*digraph*) *endomorphism*. If $\varphi : (V_1, E_1) \to (V_2, E_2)$ is a bijective digraph homomorphism and φ^{-1} is also a digraph homomorphism, then φ is called a (*digraph*) *isomorphism*. A digraph isomorphism $\varphi : (V, E) \to (V, E)$ is called a (*digraph*) *automorphism*.

Definition 2.3. Let G be a groupoid (semigroup, group, etc.) and $A \subseteq G$. We define the *Cayley graph* $\mathrm{Cay}(G, A)$ as follows: G is the vertex set and (u, v), $u, v \in G$, is an arc in $\mathrm{Cay}(G, A)$ if there exists an element $a \in A$ such that $v = ua$. The set A is called the *connection set* of $\mathrm{Cay}(G, A)$.

Note that this way we do not get multiple edges in Cayley graphs. For groupoids this may in some cases imply a loss of information. Note, moreover, that there exist several slightly different definitions of Cayley graphs, in particular the definition by left action of the elements of A. Using right action we can write mappings (color endomorphisms) on the left to get a biact, see [9].

Definition 2.4. A digraph (V, E) is called a *groupoid* (*semigroup*, *group*, *etc.*) *digraph* or *digraph of a groupoid*, (*semigroup*, *group*, *etc.*) if there exists a groupoid (*semigroup*, *group*, *etc.*) G and a connection set $A \subseteq G$ such that (V, E) is isomorphic to the Cayley graph $\mathrm{Cay}(G, A)$. We speak about G-groupoid digraphs, if we want to consider various subsets $A \subseteq G$ and the respective Cayley graphs $\mathrm{Cay}(G, A)$.

For terms in graph theory not defined here, see for example [3].

Theorem 2.5 (*Na Chiangmai [11], Sabidussi [14]*). *A digraph (V, E) with n vertices is a group digraph if and only if its automorphism group* $\mathrm{Aut}(V, E)$ *contains a subgroup $\triangle$ of order n such that for any two vertices $u, v \in V$ there exists $\sigma \in \triangle$ such that $\sigma(u) = v$.*

3. General lemmas

Lemma 3.1. *Let Y be a finite semilattice, $\left(\bigcup_{\alpha\in Y} G_\alpha; *\right)$ a strong semilattice of groups $(G_\alpha; \circ_\alpha)$, $\alpha \in Y$, and let A be a subset of $\bigcup_{\alpha\in Y} G_\alpha$. If $x'_\beta \in G_\beta$, $y'_\alpha \in G_\alpha$, and if (x'_β, y'_α) is an arc in* Cay $\left(\bigcup_{\alpha\in Y} G_\alpha, A\right)$, *then $\beta \geqslant \alpha$ and for each $x_\beta \in G_\beta$, there exists $y_\alpha \in G_\alpha$ such that (x_β, y_α) is an arc in* Cay $\left(\bigcup_{\alpha\in Y} G_\alpha, A\right)$.

Proof. For $x'_\beta \in G_\beta$, $y'_\alpha \in G_\alpha$ let (x'_β, y'_α) be an arc in the Cayley graph Cay $\left(\bigcup_{\alpha\in Y} G_\alpha, A\right)$. Then there exists $a \in A$ such that $y'_\alpha = x'_\beta * a$ with $a \in G_\gamma$ for some $\gamma \in Y$. Hence, $y'_\alpha = x'_\beta * a = f_{\beta,\beta\wedge\gamma}(x'_\beta)\circ_{\beta\wedge\gamma} f_{\gamma,\beta\wedge\gamma}(a)$ and thus $\beta\wedge\gamma = \alpha$ and $\beta \geqslant \alpha$. Moreover, for any $x_\beta \in G_\beta$ we get

$$x_\beta * a = f_{\beta,\beta\wedge\gamma}(x_\beta)\circ_{\beta\wedge\gamma} f_{\gamma,\beta\wedge\gamma}(a) = f_{\beta,\alpha}(x_\beta)\circ_\alpha f_{\gamma,\alpha}(a).$$

Hence, we have $y_\alpha = x_\beta * a \in G_\alpha$ and (x_β, y_α) is an arc in Cay $\left(\bigcup_{\alpha\in Y} G_\alpha, A\right)$. □

In this lemma we use the concept of group digraph from Definition 2.3 and the concept strong group subdigraph, meaning that the injection is a strong graph homomorphism, or in other words, we mean the induced subdigraph, i.e. the subdigraph having all edges between its vertices which exist in the big digraph between the respective vertices.

In this lemma, to be used in Theorem 4.1 and Corollary 4.2, we restrict our attention to one-element connection sets.

Lemma 3.2. *Let Y be a finite semilattice, $\left(\bigcup_{\alpha\in Y} G_\alpha; *\right)$ a strong semilattice of groups $(G_\alpha; \circ_\alpha)$, $\alpha \in Y$. Take $a \in \bigcup_{\alpha\in Y} G_\alpha$. Then the Cayley graph* Cay $\left(\bigcup_{\alpha\in Y} G_\alpha, \{a\}\right)$ *contains $|Y|$ disjoint strong group subdigraphs (G_α, E_α), $\alpha \in Y$, where $(G_\alpha, E_\alpha) \cong \mathrm{Cay}(G_\alpha, A_\alpha)$ with $A_\alpha := \{f_{\beta,\alpha}(a)\}$ for $a \in G_\beta$, $\beta \geqslant \alpha$, and $A_\alpha := \emptyset$ otherwise.*

Proof. Take $\alpha \in Y$ and $a \in \bigcup_{\alpha\in Y} G_\alpha$. Consider the strong group subdigraph (G_α, E_α) of Cay $\left(\bigcup_{\alpha\in Y} G_\alpha, \{a\}\right)$. We show that (G_α, E_α) is isomorphic to $\mathrm{Cay}(G_\alpha, A_\alpha)$ where $A_\alpha := \{f_{\beta,\alpha}(a)\}$ for $a \in G_\beta$ and $\beta \geqslant \alpha$. Let $\varphi : (G_\alpha, E_\alpha) \to \mathrm{Cay}(G_\alpha, A_\alpha)$ be the identity mapping on the group G_α. We will show that φ and φ^{-1} are digraph homomorphisms.

For $x_\alpha, y_\alpha \in G_\alpha$, take $(x_\alpha, y_\alpha) \in E_\alpha$. We prove that (x_α, y_α) is an arc in $\mathrm{Cay}(G_\alpha, A_\alpha)$. Since (G_α, E_α) is the strong group subdigraph of the Cayley graph Cay $\left(\bigcup_{\alpha\in Y} G_\alpha, \{a\}\right)$, (x_α, y_α) is an arc in Cay $\left(\bigcup_{\alpha\in Y} G_\alpha, \{a\}\right)$. Therefore, we have $y_\alpha = x_\alpha * a$. For the given $a \in G_\beta$, say, for $\beta \in Y$. Hence,

$$y_\alpha = x_\alpha * a = f_{\alpha,\alpha\wedge\beta}(x_\alpha)\circ_{\alpha\wedge\beta} f_{\beta,\alpha\wedge\beta}(a),$$

and thus $\alpha\wedge\beta = \alpha$. Therefore, $\beta \geqslant \alpha$, and thus $y_\alpha = x_\alpha\circ_\alpha f_{\beta,\alpha}(a)$. Hence (x_α, y_α) is an arc in $\mathrm{Cay}(G_\alpha, A_\alpha)$.

For $x_\alpha, y_\alpha \in G_\alpha$, let now (x_α, y_α) be an arc in $\mathrm{Cay}(G_\alpha, A_\alpha)$. Therefore, there exists $\beta \in Y$ such that $\beta \geqslant \alpha$ and $y_\alpha = x_\alpha\circ_\alpha f_{\beta,\alpha}(a) = x_\alpha * a$. Therefore (x_α, y_α) is an arc in (G_α, E_α). □

4. Clifford semigroup digraphs

Now we proceed to the main results and finally give an example of Cayley graphs of a Clifford semigroup with different connection sets. In the first theorem we describe the structure of Cayley graphs of a given strong semilattice of groups with a given connection set. In view of Lemma 8.3 in [8] we know that $\mathrm{Cay}(G, A) = \bigcup_{a\in A} \mathrm{Cay}(G, \{a\})$. Corollary 4.2 specializes the result of Theorem 4.1 for strong chains of groups.

Theorem 4.1. *Let Y be a finite semilattice, $\left(\bigcup_{\alpha\in Y} G_\alpha; *\right)$ a strong semilattice of groups $(G_\alpha; \circ_\alpha)$, $\alpha \in Y$. Take $a \in G_\gamma \subseteq \bigcup_{\alpha\in Y} G_\alpha$. Then*

(1) *the Cayley graph* Cay $\left(\bigcup_{\alpha\in Y} G_\alpha, \{a\}\right)$ *contains $|Y|$ disjoint strong group subdigraphs (G_α, E_α), $\alpha \in Y$, where $(G_\alpha, E_\alpha) := \mathrm{Cay}(G_\alpha, A_\alpha)$ with $A_\alpha := \{f_{\gamma,\alpha}(a)\}$ if $\gamma \geqslant \alpha$, $A_\alpha := \emptyset$ otherwise, in particular, $A_\gamma = \{a\}$, and*
(2) *if $\alpha \neq \beta$, $x_\beta \in G_\beta$, $y_\alpha \in G_\alpha$, then (x_β, y_α) is an arc in* Cay $\left(\bigcup_{\alpha\in Y} G_\alpha, \{a\}\right)$ *if and only if $\beta > \alpha$, $\beta\wedge\gamma = \alpha$, and $(f_{\beta,\alpha}(x_\beta), y_\alpha)$ is an arc in $\mathrm{Cay}(G_\alpha, A_\alpha)$ for the given $a \in G_\gamma$.*

Note that for $\beta = \gamma$ the condition in (2) is not fulfilled since $\beta\wedge\gamma \neq \alpha$. If $\alpha = \gamma$ we have that $(f_{\beta,\alpha}(x_\beta), y_\alpha)$ is an arc in $\mathrm{Cay}(G_\alpha, a)$.

Proof. By Lemma 3.2, we get (1).

For (2) take $\alpha \neq \beta$ in Y, $y_\alpha \in G_\alpha$, $x_\beta \in G_\beta$.

($\Rightarrow$) Suppose (x_β, y_α) is an arc in $\mathrm{Cay}\left(\bigcup_{\alpha\in Y} G_\alpha, \{a\}\right)$, i.e. $y_\alpha = x_\beta * a$, $a \in G_\gamma$, $\gamma \in Y$. Therefore, $y_\alpha = f_{\beta,\beta\wedge\gamma}(x_\beta)\circ_{\beta\wedge\gamma} f_{\gamma,\beta\wedge\gamma}(a)$ and $\beta\wedge\gamma=\alpha$, i.e. $y_\alpha = f_{\beta,\alpha}(x_\beta)\circ_\alpha f_{\gamma,\alpha}(a)$ and $\beta > \alpha$. Thus, $(f_{\beta,\alpha}(x_\beta), y_\alpha)$ is an arc in $\mathrm{Cay}(G_\alpha, A_\alpha)$.

($\Leftarrow$) Suppose $\beta > \alpha$ and $(f_{\beta,\alpha}(x_\beta), y_\alpha)$ is an arc in $\mathrm{Cay}(G_\alpha, A_\alpha)$, i.e. $y_\alpha = f_{\beta,\alpha}(x_\beta)\circ_\alpha f_{\gamma,\alpha}(a)$ for the given $a \in G_\gamma$. Since $\beta > \alpha$ and $\gamma > \alpha$ $\beta\wedge\gamma = \alpha$. Thus $y_\alpha = f_{\beta,\beta\wedge\gamma}(x_\beta)\circ_{\beta\wedge\gamma} f_{\gamma,\beta\wedge\gamma}(a) = x_\beta * a$. Therefore, (x_β, y_α) is an arc in $\mathrm{Cay}\left(\bigcup_{\alpha\in Y} G_\alpha, \{a\}\right)$. $\square$

Corollary 4.2. *Let Y be a finite chain, $\left(\bigcup_{\alpha\in Y} G_\alpha; *\right)$ a strong chain of groups $(G_\alpha; \circ_\alpha)$, $\alpha \in Y$. Take $a \in G_\gamma \subseteq \bigcup_{\alpha\in Y} G_\alpha$. Then*

(1) *the Cayley graph* $\mathrm{Cay}\left(\bigcup_{\alpha\in Y} G_\alpha, \{a\}\right)$ *contains $|Y|$ disjoint strong group subdigraphs (G_α, E_α), $\alpha \in Y$, where $(G_\alpha, E_\alpha) := \mathrm{Cay}(G_\alpha, A_\alpha)$ with $A_\alpha := \{f_{\gamma,\alpha}(a)\}$ if $\gamma \geqslant \alpha$, $A_\alpha := \emptyset$ otherwise, in particular, $A_\gamma = \{a\}$, and*
(2) *if $\beta \neq \alpha$, $x_\beta \in G_\beta$, $y_\alpha \in G_\alpha$, then (x_β, y_α) is an arc in* $\mathrm{Cay}\left(\bigcup_{\alpha\in Y} G_\alpha, \{a\}\right)$ *if and only if $\beta > \alpha$ and $(f_{\beta,\alpha}(x_\beta), y_\alpha)$ is an arc in* $\mathrm{Cay}(G_\alpha, A_\alpha)$.

Proof. By Theorem 4.1(1), we get (1). For (2) we get that $a \in A_\alpha$ if $\beta\wedge\gamma=\alpha$. Indeed, if $\alpha \neq \beta$, and $\beta\wedge\gamma=\alpha$ for $\gamma \in Y$ we get $\alpha=\gamma$ since Y is a chain, and thus $f_{\gamma,\gamma}(a)=a$. Now Theorem 4.1(2) is (2). $\square$

The following theorem takes an approach opposite to Theorem 4.1. We describe when a graph, whose vertex set can be associated with the elements of a strong semilattice of groups, is a strong semilattice of groups digraph, by constructing the appropriate connection set. Corollary 4.4 does the same for strong chains of groups digraphs. Here, it is necessary to consider connection sets A with more than one element.

Theorem 4.3. *Let Y be a finite semilattice, $\left(\bigcup_{\alpha\in Y} G_\alpha; *\right)$ a strong semilattice of groups $(G_\alpha; \circ_\alpha)$, $\alpha \in Y$, e_α denotes the identity element of the group $(G_\alpha; \circ_\alpha)$, and let $\left(\bigcup_{\alpha\in Y} G_\alpha, E\right)$ be a digraph such that:*

(1) *the graph $\left(\bigcup_{\alpha\in Y} G_\alpha, E\right)$ contains $|Y|$ disjoint strong group subdigraphs (G_α, E_α) where $(G_\alpha, E_\alpha)=\mathrm{Cay}(G_\alpha, A_\alpha)$ and $A_\alpha := \{a_\alpha \in G_\alpha | (e_\alpha, a_\alpha) \in E\}$ for $\alpha \in Y$, and*
(2) *if $B_\alpha := \{a_\alpha \in A_\alpha | (e_\xi, a_\alpha) \in E\ \ for\ all\ \xi \geqslant \alpha\}$, then $(x_\beta, y_\alpha) \in E$, $x_\beta \in G_\beta$, $y_\alpha \in G_\alpha$, if and only if $\alpha \leqslant \beta$ and there exist $\gamma \in Y$, $\beta\wedge\gamma=\alpha$ and $a_\gamma \in B_\gamma$ such that $y_\alpha = f_{\beta,\alpha}(x_\beta)\circ_\alpha f_{\gamma,\alpha}(a_\gamma)$. Moreover, if $\beta > \alpha$ and one has one $(x_\beta, y_\alpha) \in E$ then for all $x_\beta \in G_\beta$ there exists $y_\alpha \in G_\alpha$ such that $(x_\beta, y_\alpha) \in E$.*

Then $\left(\bigcup_{\alpha\in Y} G_\alpha, E\right)$ is a strong semilattice of groups digraph.

Note that A_α contains the tails of all arcs in $\left(\bigcup_{\alpha\in Y} G_\alpha, E\right)$ starting in e_α and B_α contains those vertices of A_α which are the tails of all arcs in $\left(\bigcup_{\alpha\in Y} G_\alpha, E\right)$ starting in e_ξ for all $\xi > \alpha$.

Proof. With $B_\alpha = \{a_\alpha \in A_\alpha | (e_\xi, a_\alpha) \in E$ for all $\xi \geqslant \alpha\}$, we set $A := \bigcup_{\alpha\in Y} B_\alpha$. To show that $\left(\bigcup_{\alpha\in Y} G_\alpha, E\right) \cong \mathrm{Cay}\left(\bigcup_{\alpha\in Y} G_\alpha, A\right)$, let $\psi : \left(\bigcup_{\alpha\in Y} G_\alpha, E\right) \to \mathrm{Cay}\left(\bigcup_{\alpha\in Y} G_\alpha, A\right)$ be the identity mapping on $\bigcup_{\alpha\in Y} G_\alpha$. We will show that ψ and ψ^{-1} are digraph homomorphism. For $(x_\beta, y_\alpha) \in E$ we prove that (x_β, y_α) is an arc in $\mathrm{Cay}\left(\bigcup_{\alpha\in Y} G_\alpha, A\right)$.

If $\beta=\alpha$, then by (2), we have $y_\alpha = x_\alpha \circ_\alpha f_{\gamma,\alpha}(a_\gamma)$, for some $a_\gamma \in B_\gamma$ and $\gamma \geqslant \alpha$. Then $y_\alpha = x_\alpha * a_\gamma$, $a_\gamma \in B_\gamma \subseteq A$. Thus (x_α, y_α) is an arc in $\mathrm{Cay}\left(\bigcup_{\alpha\in Y} G_\alpha, A\right)$.

If $\beta > \alpha$, then by (2), we get $y_\alpha = f_{\beta,\alpha}(x_\beta)\circ_\alpha f_{\gamma,\alpha}(a_\gamma)$ for some $\gamma \in Y$ such that $\beta\wedge\gamma=\alpha$ and some $a_\gamma \in B_\gamma$. Hence $y_\alpha = f_{\beta,\beta\wedge\gamma}(x_\beta)\circ_{\beta\wedge\gamma} f_{\gamma,\beta\wedge\gamma}(a_\gamma)$, i.e. $y_\alpha = x_\beta * a_\gamma$, $a_\gamma \in B_\gamma \subseteq A$. Hence (x_β, y_α) is an arc in $\mathrm{Cay}\left(\bigcup_{\alpha\in Y} G_\alpha, A\right)$.

Let now (x_β, y_α) be an arc in $\mathrm{Cay}\left(\bigcup_{\alpha\in Y} G_\alpha, A\right)$. We prove that $(x_\beta, y_\alpha) \in E$. The hypothesis implies $y_\alpha = x_\beta * a$ for some $a \in A = \bigcup_{\alpha\in Y} B_\alpha$. Hence $a \in B_\gamma$ for some $\gamma \in Y$. By Lemma 3.1, we get $\beta \geqslant \alpha$.

If $\alpha=\beta$, then $y_\alpha = f_{\alpha,\alpha\wedge\gamma}(x_\alpha)\circ_{\alpha\wedge\gamma} f_{\gamma,\alpha\wedge\gamma}(a)$, $\alpha = \alpha\wedge\gamma$. Hence $y_\alpha = f_{\alpha,\alpha}(x_\alpha)\circ_\alpha f_{\gamma,\alpha}(a) = x_\alpha \circ_\alpha f_{\gamma,\alpha}(a)$. By (2), we get that $(x_\beta, y_\alpha) \in E$.

If $\beta > \alpha$ then $y_\alpha = f_{\beta,\beta\wedge\gamma}(x_\beta)\circ_{\beta\wedge\gamma} f_{\gamma,\beta\wedge\gamma}(a)$. Thus $\alpha = \beta\wedge\gamma$ and therefore $y_\alpha = f_{\beta,\alpha}(x_\beta)\circ_\alpha f_{\gamma,\alpha}(a)$. By (2), we get $(x_\beta, y_\alpha) \in E$.

Note that the "Moreover" part in (2) actually follows from the rest of (2) by Lemma 3.1. $\square$

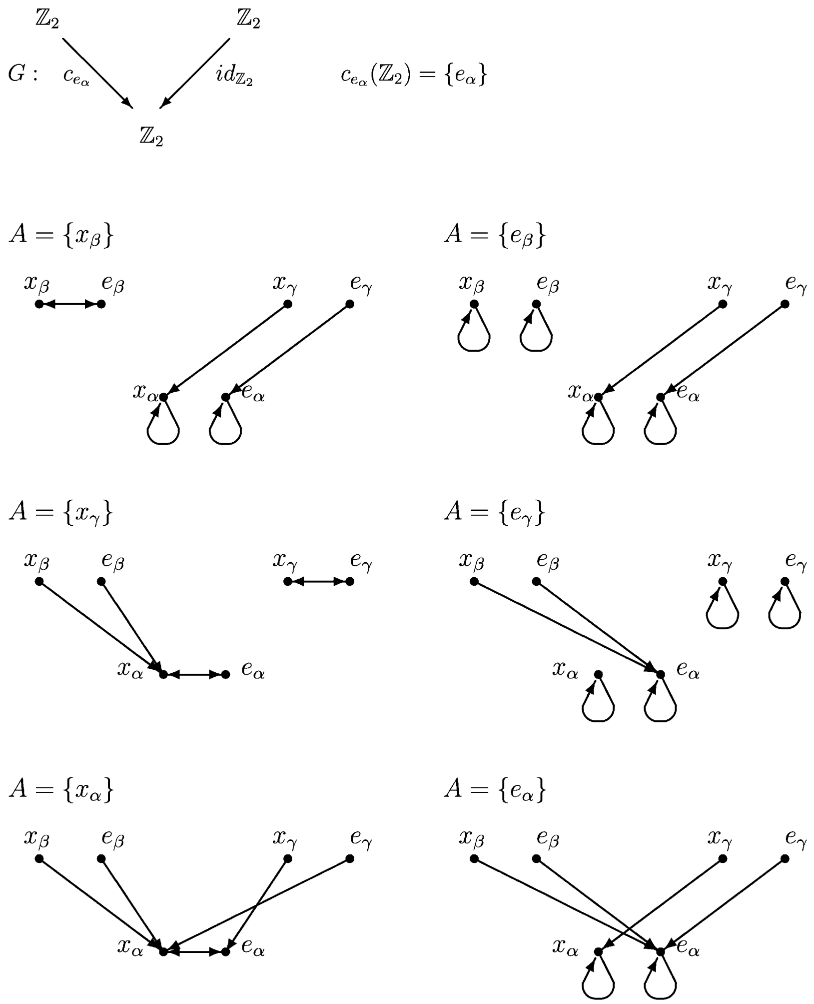
Corollary 4.4. *Let Y be a finite chain, $\left(\bigcup_{\alpha\in Y} G_\alpha; *\right)$ a strong semilattice of groups $(G_\alpha; \circ_\alpha)$, $\alpha \in Y$, e_α denotes the identity element of the group $(G_\alpha; \circ_\alpha)$, and let $\left(\bigcup_{\alpha\in Y} G_\alpha, E\right)$ a digraph such that*:

(1) *the graph $\left(\bigcup_{\alpha\in Y} G_\alpha, E\right)$ contains $|Y|$ disjoint strong group subdigraphs (G_α, E_α) where $(G_\alpha, E_\alpha) = \mathrm{Cay}(G_\alpha, A_\alpha)$ and $A_\alpha := \{a_\alpha \in G_\alpha | (e_\alpha, a_\alpha) \in E\}$ for $\alpha \in Y$, and*
(2) *if $B_\alpha := \{a_\alpha \in A_\alpha | (e_\xi, a_\alpha) \in E$ for all $\xi \geqslant \alpha\}$, then $(x_\beta, y_\alpha) \in E$, $x_\beta \in G_\beta$, $y_\alpha \in G_\alpha$, if and only if $\alpha \leqslant \beta$ and $(f_{\beta,\alpha}(x_\beta), y_\alpha)$ is an arc in $\mathrm{Cay}(G_\alpha, B_\alpha)$. Moreover, if $\beta > \alpha$ and one has one $(x_\beta, y_\alpha) \in E$ then for all $x_\beta \in G_\beta$ there exists $y_\alpha \in G_\alpha$ such that $(x_\beta, y_\alpha) \in E$.*

Then $\left(\bigcup_{\alpha\in Y} G_\alpha, E\right)$ is a strong chain of groups digraph.

For the following examples we recall that for any semigroup G and connection set A we have $\mathrm{Cay}(G, A) = \bigcup_{a\in A} \mathrm{Cay}(G, \{a\})$, cf. Lemma 8.3 in [8].

Example 4.5. We consider the strong semilattice of groups consisting of three copies of $\mathbb{Z}_2$, elements being denoted by e_α, x_α (at the bottom) and similarly with indices β (top left) and γ (top right), with one group homomorphism being the identity mapping (from top right), the other one the constant mapping onto the identity e_α (from top left) as indicated. Then we give the Cayley graphs $\mathrm{Cay}(G, A)$ for all the six different one-element connection sets A, as indicated in the picture.



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