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On Localization of Injective Modules

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1. INTRODUCTION

It is not true, in general, that the localization $T^{-1}E$ of an injective R-module E over a commutative ring R with respect to a multiplicatively closed subset T of R is an injective $T^{-1}R$ module. Counter-examples are given in [2, 3, 5].

However, if R is Noetherian, $T^{-1}E$ is indeed injective, as was pointed out in [2]. To see this, we tensor the exact sequence

$$\operatorname{Hom}(R, E) \xrightarrow{\phi} \operatorname{Hom}(I, E) \longrightarrow 0$$

to obtain the commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}(R, E)\otimes T^{-1}R & \xrightarrow{\phi\otimes 1} & \operatorname{Hom}(I, E)\otimes T^{-1}R \to 0 \\ {\scriptstyle\gamma_R}\downarrow & & \downarrow{\scriptstyle\gamma_I} \\ \operatorname{Hom}(R, E\otimes T^{-1}R) & \xrightarrow{\psi} & \operatorname{Hom}(I, E\otimes T^{-1}R) \end{array}$$

for any ideal I of R. Here ψ is defined in the obvious way and $\gamma_R: f\otimes x \to f_x$, where $f_x(r) = f(r)\otimes x$, etc.

The map γ_R is an isomorphism, and γ_I is also an isomorphism provided that I is finitely related, [6, Lemma 3.83]. Thus, if R is Noetherian, γ_I is an isomorphism for all ideals I of R so that ψ is an epimorphism and $T^{-1}E = E\otimes T^{-1}R$ is R-injective and therefore also $T^{-1}R$-injective (see Proposition 2.2 of [2]).

In the case of localization with respect to maximal ideals, more positive results exists. E. Matlis [4, Proof of Theorem 24] proves that if R is an h-local domain and E an R-module, then E is an injective R-module if and only if E_M is an injective R_M-module for every maximal ideal M of R. Furthermore he proves that $\operatorname{inj\,dim}_R E = \sup_{M\in \operatorname{mspec} R} \operatorname{inj\,dim}_{R_M} E_M$, where mspec R is the set of all maximal ideals of R.

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Let T be any multiplicatively closed set and $\bar{T}$ the image of T in R_M. In this paper we prove that if E is T-torsion-free, then E is T-injective if and only if E_M is a $\bar{T}$-injective R_M-module. From this result it follows that if R is an order in a semi-simple ring, then E is an injective R-module if and only if E_M is an injective R_M-module for every maximal ideal M of R.

We obtain similar results for the torsion case with the additional restriction that R be h-local.

Finally, these results are combined to show that for R T-h-local, an arbitrary R-module E is T-injective if and only if E_M is T-injective for every maximal ideal M of R. If, in addition, R is an order in a semi-simple ring, then E is an injective R-module if and only if E_M is an injective R_M-module for every maximal ideal M. Examples of such rings include finite products of valuation domains.

2. Notation and Preliminaries

Throughout this paper, R will denote a commutative ring with identity and R_M will denote the ring R localized at a maximal ideal M.

Let T be any multiplicatively closed subset of R. The *T-torsion* part of an R module E, denoted by $t_T(E)$, is defined to be the set of all elements $e \in E$ such that $te = 0$ for some $t \in T$. The R-module E is called a *T-torsion* module if $t_T(E) = E$ and is *T-torsion-free* if $t_T(E) = 0$. The module $E/t_T(E)$ is always T-torsion-free. A module E is *T-injective* if for each ideal I of R such that $I \cap T \neq \varnothing$ and each homomorphism $\alpha: I \to E$ there exists $x \in E$ such that $\alpha(a) = xa$ for all $a \in I$. E is *T-divisible* if $E = tE$ for all $t \in T$.

We shall denote the set of regular elements of R by S, and we observe that the image of S, denoted by $\bar{S}$, of S in R_M is a multiplicatively closed subset of R_M, although in general this is not the set of regular elements of R_M.

The T-injective R-modules have the following property:

PROPOSITION 2.1. *An R-module E is T-injective if and only if every diagram of R-modules and R-homomorphisms*

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longrightarrow & B \\ & & {\scriptstyle\alpha}\downarrow & \swarrow {\scriptstyle\beta} & \\ & & E & & \end{array}$$

in which B/A is a T-torsion module and the row is exact, can be extended to a commutative diagram with the new R-homomorphism β.

Proof. The proof is analogous to the proof of the Baer criterion (see, e.g., [6, Theorem 3.20, p. 67]). ▮

A similar result holds for $\bar{T}$-injective R_M-modules.

The following results will be useful:

PROPOSITION 2.2. *An R_M-module is R_M-injective if and only if it is R-injective.*

Proof. [2, p. 417]. ▮

PROPOSITION 2.3. *Let E be any R_M-module and M a maximal ideal of R. Then E is a $\bar{T}$-injective R_M-module if and only if E is a T-injective R-module.*

Proof. Suppose E is a $\bar{T}$-injective R_M-module. Let I be an ideal of R such that $I \cap T \neq \varnothing$ and let $\alpha: I \to E$ be an R-homomorphism. Define an R_M-homomorphism $\alpha': I_M \to E$; $i/u \to 1/u\ \alpha(i)$. Since E is $\bar{T}$-injective, there exists an $x \in E$ such that

$$\alpha'(i/u) = x \cdot i/u \qquad \text{for all} \quad i/u \in I_M.$$

Thus

$$\begin{aligned}\alpha(i) = \alpha'(i/1) &= x \cdot i/1 \\ &= x \cdot i \qquad \text{for all} \quad i \in I\end{aligned}$$

(by the definition of E as R-module), and it follows that E is T-injective. Conversely, let J be an ideal of R_M, $J \cap \bar{T} \neq \varnothing$. Let $r \in T$ such that $r/1 \in J \cap \bar{T}$. Then $r \in J^c \cap T$.

Let $\alpha: J \to E$ be any R_M-homomorphism. Define $\bar{\alpha}: J^c \to E$; $t \to \alpha(t/1)$. As E is T-injective, there exists an $x \in E$ such that $\bar{\alpha}(t) = tx$; therefore $\alpha(r/u) = x \cdot r/u$ for all $r/u \in J$, and thus E is $\bar{T}$-injective. ▮

3. THE TORSION-FREE CASE

We have the following relationship between T-injectivity and T-divisibility for T-torsion-free R-modules.

PROPOSITION 3.1. *The following properties of an R-module C are equivalent:*

(a) *C is T-torsion-free and T-injective;*

(b) *C is T-torsion-free and T-divisible.*

Proof. [7, Proposition 3.7, p. 58]. ▮

It is easy to see that for any T-torsion-free R-module C, C_M is a $\bar{T}$-torsion-free R_M-module, for any $M \in \operatorname{mspec} R$.

PROPOSITION 3.2. *Let C be a T-torsion-free R-module. Then C is a T-injective R-module if and only if C_M is a $\bar{T}$-injective R_M-module for every $M \in \operatorname{mspec} R$.*

Proof. We shall only prove that C is T-divisible if and only if C_M is $\bar{T}$-divisible for every $M \in \operatorname{mspec} R$. The result will then follow from Proposition 3.1 and the above remark.

Suppose C is T-divisible and let c/u be an arbitrary element of C_M and $t/1 \in \bar{T}$. Since C is T-divisible, there exists $d \in C$ such that $c = dt$. Then $d/u \cdot t/1 = c/u$, and thus C_M is $\bar{T}$-divisible.

Conversely, suppose C_M is $\bar{T}$-divisible for every $M \in \operatorname{mspec} R$ and consider the R-homomorphism

$$\theta_t : C \to C; \qquad c \mapsto ct, \quad t \in T.$$

Then $(\theta_t)_M : C_M \to C_M$, $c/u \to t/1 \cdot c/u$ is an epimorphism for every $M \in \operatorname{mspec} R$ since C_M is $\bar{T}$-divisible for every $M \in \operatorname{mspec} R$. Thus θ_t is an epimorphism. Since this hold for all $t \in T$ it follows that C is T-divisible. ∎

COROLLARY 3.3. *Let R be an order in a semi-simple ring and let C be an S-torsion-free R module. Then C is an injective R-module if and only if C_M is an injective R_M-module.*

Proof. If R is an order in a semi-simple ring, then every T-injective R-module is injective [7, Proposition 3.8, p. 58]. The result follows from Propositions 2.2, 2.3, and 3.2. ∎

3.4. Examples of rings which are orders in semi-simple rings include the following:

(a) Finite sums of domains (use [7, Theorem 2.2]).

(b) Let R be an integral domain with field of fractions K and let G be a finite group such that the characteristic of K does not divide the order of G. Then $R[G]$ is an order in the semi-simple ring $K[G]$.

4. THE TORSION CASE

In the torsion case, we shall restrict ourselves to T-h-local rings. A ring R is T-h-local if every $t \in T$ is an element of only finitely many maximal ideals of R and every prime ideal P of R, which is not a minimal prime and

$P \cap T \neq \varnothing$, is a subset of only one maximal ideal of $R([1])$. An S-h-local ring is called h-local.

h-Local domains are extensively studied in [4]. We observe that the class of h-local rings is closed under arbitrary sums so that any sum of h-local domains is an h-local ring.

If E is an R-module and M a maximal ideal of R, then define $E(M) = \{0\} \cup \{x \in E \mid M$ is the only maximal ideal such that $\operatorname{ann}_R(x) \subseteq M\}$. The brackets $\langle \cdot \rangle$ will denote a coproduct map.

PROPOSITION 4.1. *Let R be T-h-local and D a T-torsion R-module. D is T-injective if and only if D_M is a $\bar{T}$-injective R_M-module for every $M \in \operatorname{mspec} R$.*

Proof. We need the following lemma, the proof of which is straightforward in view of Theorem 2.6 and Corollary 2.7 in [1], which also hold for arbitrary T-h-local rings.

LEMMA 4.2. *Let A and D be any S-torsion R-modules and $f\colon D \to A$ an R-monomorphism. Then there exist isomorphisms $\theta_D\colon \bigoplus_{N \in \operatorname{mspec} R} D_N \to D$ and $\theta_A\colon \bigoplus_{N \in \operatorname{mspec} R} A_N \to A$ such that $f \circ \theta_D = \theta_A \circ \langle in_A \circ f_N \rangle$, where $in_{A_N}\colon A_N \to \bigoplus_{N \in \operatorname{mspec} R} A_N$ is the injection map.*

Proof of the Theorem. Suppose D_M is $\bar{T}$-injective for all $M \in \operatorname{mspec}(R)$. Consider any exact sequence

$$0 \longrightarrow D \xrightarrow{f} A \xrightarrow{g} R_K/I_K \longrightarrow 0 \tag{4.1}$$

with K a maximal ideal of R and I an ideal of R such that $I \cap T \neq \varnothing$. It follows that A is a T-torsion module.

Since the sequence (4.1) is exact, the sequence

$$0 \longrightarrow D_M \xrightarrow{f_M} A_M \xrightarrow{g_M} (R_K/I_K)_M \longrightarrow 0 \tag{4.2}$$

is exact for every $M \in \operatorname{mspec} R$. But D_M is $\bar{T}$-injective and $\bar{T}$-torsion; thus the sequence (4.2) splits (Proposition 2.1); i.e., for every $f_M\colon D_M \to A_M$ there exists an R_M-homomorphism, $h^M\colon A_M \to D_M$ such that $h^M \circ f_M = 1_{D_M}$.

Consider the diagram

$$\begin{array}{ccc} D_M & \underset{h^M}{\overset{f_M}{\rightleftarrows}} & A_M \\ \downarrow{\scriptstyle in_{D_M}} & & \downarrow{\scriptstyle in_{A_M}} \\ \bigoplus D_N & \underset{\langle in_{D_N} \circ h^N \rangle}{\overset{\langle in_{A_N} \circ f_N \rangle}{\rightleftarrows}} & \bigoplus A_N \end{array}$$

It follows easily that

$$\langle in_{D_N} \circ h^N \rangle \circ \langle in_{A_N} \circ f_N \rangle = 1_{\oplus D_N}$$

and from the lemma that

$$f \circ \theta_D \circ \langle in_{D_N} \circ h^N \rangle \circ \theta_A^{-1} = 1_D.$$

Thus, the exact sequence (4.1) splits, and therefore $\text{Ext}^1_R(R_K/I_K, D) = 0$. But R/I is T-torsion, and therefore

$$R/I = \bigoplus_{M \in \text{mspec } R} R_M/I_M.$$

Thus $\text{Ext}^1_R(R/I, D) = \prod_{M \in \text{mspec } R} \text{Ext}(R_M/I_M, D) = 0$ so that D is T-injective (see the proof of Theorem 9.11 in [6]).

Conversely, suppose D is T-injective. Since $D = \bigoplus_{N \in \text{mspec } R} D_N$, every D_N is a T-injective R-module. From Proposition 2.2 it follows that every D_N is a $\bar{T}$-injective R_N-module.

COROLLARY 4.3. *Let R be a h-local ring that is an order in a semi-simple ring and let D be a T-torsion R-module. Then D is an injective R-module if and only if D_M is an injective R_M-module.*

Proof. Similar to the proof of Corollary 3.3. ∎

4.4. Examples of rings that are both h-local and an order in a semi-simple ring include finite products of local domains. These are not Noetherian if the local rings are not Noetherian, for example non-discrete valuation domains.

5. THE MAIN THEOREM

We shall now combine Sections 3 and 4 to prove the main theorem, but first we need additional lemmas, which we state without proof:

LEMMA 5.1. *If $t_T(E)$ is the T-torsion part of an R-module E, then $(t_T(E))_M$ is the $\bar{T}$-torsion part of E_M.*

LEMMA 5.2. *Let E be any R-module (R_M-module). If E is T-injective ($\bar{T}$-injective), then $E/t_T E$ ($E/t_{\bar{T}}(E)$) is T-divisible ($\bar{T}$-divisible).*

LEMMA 5.3. *Let*

$$0 \to A \to B \to B/A \to 0$$

be an exact sequence of R-modules with A and B/A T-injective. Then B is T-injective.

We are now ready to prove the main theorem.

THEOREM 5.4. *Let R be a T-h-local ring. Then E is a T-injective R-module if and only if E_M is a $\bar{T}$-injective R_M-module for every $M \in \operatorname{mspec} R$.*

Proof. Suppose E is T-injective. It is easy to prove that $t_T(E)$ is T-injective. Furthermore, $t_T(E) = \bigoplus_{M \in \operatorname{mspec} R} (t_T(E))_M$, and thus every $(t_T(E))_M$ is T-injective. It follows from Proposition 2.2 that $(t_T E)_M$ is a $\bar{T}$-injective R_M-module. $E/t_T(E)$ is T-divisible, and thus $E_M/(t_T E)_M$ is $\bar{T}$-divisible (see the proof of Proposition 3.2). Since it is $\bar{T}$-torsion-free, it follows from Proposition 3.1 that $E_M/(t_T E)_M$ is $\bar{T}$-injective.

Lemma 5.3 now implies that E_M is $\bar{T}$-injective.

Conversely, suppose E_M is $\bar{T}$-injective for every $M \in \operatorname{mspec} R$. Then $(t_T E)_M$ is the $\bar{T}$-torsion part of E_M (from Lemma 5.1) and $(t_T E)_M$ is $\bar{T}$-injective since E_M is $\bar{T}$-injective, and thus $t_T(E)$ is T-injective (Proposition 4.1).

Lemma 5.2 implies that $E_M/(t_T E)_M$ is $\bar{T}$-divisible, and since it is $\bar{T}$-torsion-free, it is $\bar{T}$-injective. Therefore, $E/t_T(E)$ is T-injective, by Proposition 3.1.

The theorem follows from Lemma 5.3. ∎

COROLLARY 5.5. *Let R be an h-local ring that is an order in a semi-simple ring and let E be an R-module. Then E is an injective R-module if and only if E_M is an injective R_M-module.*

Proof. This is similar to the proof of Corollary 3.3. ∎

We can use the above corollary to obtain the following result on the injective dimension of a module:

COROLLARY 5.6. *Let R be an h-local ring that is an order in a semi-simple ring and let E be any R-module. Then*

$$\operatorname{inj\,dim}_R E = \sup_{M \in \operatorname{mspec} R} \operatorname{inj\,dim}_{R_M} E_M.$$

Proof. The proof is a direct extension of the proof of Matlis [4, Theorem 24] in the domain case. ∎

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