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On the irreducible Specht modules for Iwahori–Hecke algebras of type A with $q = -1$

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ABSTRACT

Let p be a prime and $\mathbb{F}$ a field of characteristic p, and let $\mathcal{H}_n$ denote the Iwahori–Hecke algebra of the symmetric group $\mathfrak{S}_n$ over $\mathbb{F}$ at $q = -1$. We prove that there are only finitely many partitions λ such that both λ and λ' are 2-singular and the Specht module S^λ for $\mathcal{H}_{|\lambda|}$ is irreducible.

1. Introduction

Suppose $\mathbb{F}$ is a field and q a non-zero element of $\mathbb{F}$. The *Iwahori–Hecke algebra* $\mathcal{H}_n$ of type A over $\mathbb{F}$ with parameter q is a finite-dimensional algebra which arises in various contexts, and whose representation theory closely resembles the representation theory of the symmetric group in prime characteristic. An important class of modules for $\mathcal{H}_n$ is the class of *Specht modules*, and an interesting problem is to determine exactly which Specht modules are irreducible. This problem has been solved in all cases except when $q = -1$ and the characteristic of $\mathbb{F}$ is not 2. Various partial results are known for this case, together with a conjectured solution [FL, Conjecture 2.2] for the case where $\mathbb{F}$ has infinite characteristic (we adopt the convention that the *characteristic* of a field is the order of its prime subfield). In this note, we concentrate on the case of finite characteristic p. Since the reducibility or not of Specht modules labelled by 2-regular partitions and their conjugates is known, we can concentrate on partitions λ such that neither λ nor λ' is 2-regular. Our main result is that for each p there are only finitely many such partitions which label irreducible Specht modules. Our approach is to use a decomposition map to relate the problem to the (known) classification of Specht modules at a $2p$th root of unity in infinite characteristic, and then to apply a recent result of the author and Lyle which proves the reducibility of a large class of Specht modules. We complete the proof by employing some simple combinatorics of partitions.

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The next section contains the necessary background material, and a statement and proof of the main result. In Section 3, we try to examine more precisely the reducibility of Specht modules in prime characteristic, and give some computational results for small primes.

2. The main result

Suppose $\mathbb{F}$ is a field, and q is a non-zero element of $\mathbb{F}$; we define $e = e(q)$ to be the multiplicative order of q in $\mathbb{F}$ if $q \neq 1$, or $e = \mathrm{char}(\mathbb{F})$ if $q = 1$. For any $n \geqslant 0$, we define $\mathcal{H}_n = \mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ to be the Iwahori–Hecke algebra of the symmetric group $\mathfrak{S}_n$ with parameter q. The essential reference for the representation theory of $\mathcal{H}_n$ is Mathas's book [M].

Many of the important representations of $\mathcal{H}_n$ are labelled by *partitions* of n. Recall that a partition of n is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers which sum to n. When writing partitions, we usually group equal parts with a superscript and omit trailing zeroes.

For every partition λ of n, there is an $\mathcal{H}_n$-module S^λ called the *Specht module*. (Note that we refer to the Specht module defined by Dipper and James [DJ], rather than that used by Mathas.) In the case $e = \infty$, the Specht modules are irreducible and afford all the irreducible representations of $\mathcal{H}_n$. In the case where e is finite, the Specht modules may be reducible, but the irreducible $\mathcal{H}_n$-modules can be obtained from them. Let us say that a partition λ is *e-regular* if there is no i such that $\lambda_i = \cdots = \lambda_{i+e-1} > 0$, and *e-singular* otherwise. When λ is e-regular, the Specht module S^λ has a unique irreducible quotient D^λ, and the modules D^λ afford all the irreducible representations of $\mathcal{H}_n$ as λ ranges over the e-regular partitions of n. The *decomposition matrix* of $\mathcal{H}_n$ has rows indexed by the partitions of n and columns indexed by the e-regular partitions of n, with the (λ, μ)-entry being the composition multiplicity $[S^\lambda : D^\mu]$. In the case where $p = \infty$, we denote this decomposition matrix $D_n^{(e)}$; it is known that (given e) this matrix does not depend on the choice of q.

The subject of this paper is the problem of classifying the irreducible Specht modules. Let us say that a partition λ is *(e, p)-reducible* if the Specht module S^λ is reducible when $\mathbb{F}$ has characteristic p, or *(e, p)-irreducible* otherwise; this condition is known to depend only on e and p, not on the particular choice of $\mathbb{F}$ and q. In the case where $e > 2$, the classification of (e, p)-reducible partitions has been completed in a series of papers [JM2,L1,F1,F2,L2,JLM], but the case $e = 2$ remains open. This paper is a small contribution towards completing this case.

If λ is a partition, let λ' denote the conjugate partition, defined by

$$\lambda'_j = \bigl|\{i \mid \lambda_i \geqslant j\}\bigr|.$$

It is known that λ is (e, p)-reducible if and only if λ' is; this is because $S^{\lambda'}$ is essentially the dual of S^λ [M, Exercise 3.14(iii)]. Furthermore, if λ is e-regular, it is known whether λ is (e, p)-reducible [JM2, Theorem 4.15]. So in order to complete the classification of irreducible Specht modules, it suffices to consider the case $e = 2$, and to consider only partitions λ such that both λ and λ' are 2-singular. Let us say that λ is *doubly-singular* if this is the case. Now we can state our main result.

Theorem 2.1. *Suppose p is a prime. Then there are only finitely many doubly-singular $(2, p)$-irreducible partitions.*

We remark that this theorem is certainly not true in the case $p = \infty$. For example, any partition of the form (a^b) is $(2, \infty)$-irreducible. This was observed by Mathas, using [JM1, Theorem 4.7].

To prove Theorem 2.1, we use the classification of $(2p, \infty)$-partitions and the theory of *decomposition maps*. An excellent introduction to decomposition maps can be found in Geck's article [G]. Using the set-up in Section 3 of [G], one can obtain the following (recall that $D_n^{(e)}$ denotes the decomposition matrix for an Iwahori–Hecke algebra at an eth root of unity in a field of infinite characteristic).

Theorem 2.2. *Suppose $p = \mathrm{char}(\mathbb{F})$ is a prime and $e = 2$, and let D be the decomposition matrix of $\mathcal{H}_n$. Then for any non-negative integer i there is a matrix A with rows indexed by $(2p^i)$-regular partitions of n and columns indexed by 2-regular partitions of n, with the following properties:*

- *the entries of A are non-negative integers;*
- *there is at least one non-zero entry in each row of A;*
- *$D = D_n^{(2p^i)} A$.*

This result arises from a decomposition map between an Iwahori–Hecke algebra at a $(2p^i)$th root of unity in a field of infinite characteristic, and $\mathcal{H}_n$. The matrix A is simply the decomposition matrix associated to this map.

As a consequence, we get the following.

Corollary 2.3. *Suppose λ is a partition of n, and suppose that λ is $(2p^i, \infty)$-reducible for some i. Then λ is $(2, p)$-reducible.*

Proof. Since λ is $(2p^i, \infty)$-reducible, the sum of the entries in the λ-row of $D_n^{(2p^i)}$ is at least 2. Now the properties of A guarantee that the sum of the entries in the λ-row of D is at least 2, so that λ is $(2, p)$-reducible. $\square$

We shall use this result mainly in the case $i = 1$, employing the known classification of $(2p, \infty)$-reducible partitions. This is most easily stated in terms of hook lengths in the Young diagram. Given a partition λ, recall that the *Young diagram* $[\lambda]$ is the set

$$\{(i, j) \in \mathbb{N}^2 \mid j \leqslant \lambda_i\},$$

whose elements we call the *nodes* of λ. Given such a node (i, j), define the *hook length* $h_\lambda(i, j)$ to be the integer $1 + \lambda_i - j + \lambda'_j - i$. If the Young diagram is drawn with the English convention, this is the number of nodes of λ directly below or directly to the right of (i, j), including (i, j) itself.

Now given $e > 2$, say that λ is an *e-JM partition* if the following condition holds: for every $(i, j) \in [\lambda]$ for which e divides $h_\lambda(i, j)$, we have either

- e divides $h_\lambda(i, k)$ for all $1 \leqslant k \leqslant \lambda_i$, or
- e divides $h_\lambda(k, j)$ for all $1 \leqslant k \leqslant \lambda'_j$.

Then the following is a special case of the results in [JM2,F2,L2].

Theorem 2.4. *Suppose $e > 2$. Then a partition λ is (e, ∞)-irreducible if and only if λ is an e-JM partition.*

So in trying to classify $(2, p)$-irreducible partitions, we can restrict attention to $(2p)$-JM partitions. We can also make another strong restriction, thanks to a recent result of the author and Lyle. Say that a partition λ is *broken* if there exist $1 < c < d$ such that $\lambda_{c-1} - \lambda_c > 1$ and $\lambda_{d-1} = \lambda_d > 0$, and *unbroken* otherwise. Note that if λ is broken, then so is λ'.

Theorem 2.5. *(See [FL, Theorem 2.1].) Suppose λ is a broken partition. Then λ is $(2, p)$-reducible.*

Applying Corollary 2.3, we find that any doubly-singular partition λ which is $(2, p)$-irreducible must be an unbroken $(2p)$-JM partition. So in order to prove Theorem 2.1, it suffices to show that there are only finitely many such partitions, for any p. This follows from a few simple combinatorial results.

Given a partition λ and an integer $e > 1$, say that λ is an *e-core* if there is no $(i, j) \in [\lambda]$ for which e divides $h_\lambda(i, j)$. Obviously, if λ is an e-core, then λ is an e-JM partition.

Lemma 2.6. *Suppose λ is an unbroken doubly-singular partition, and is a $(2p)$-JM partition. Then λ is a $2p$-core.*

Proof. If not, then we have $2p \mid h_\lambda(i,j)$ for some $(i,j) \in [\lambda]$. Since λ is a $(2p)$-JM partition, we have either $2p \mid h_\lambda(i,k)$ for all $1 \leqslant k \leqslant \lambda_i$, or $2p \mid h_\lambda(k,j)$ for all $1 \leqslant k \leqslant \lambda'_j$. By replacing λ with λ' if necessary, we can assume the latter case.

Claim. $\lambda_1 - \lambda_2 > 1$.

Proof. First suppose $\lambda_2 < j$. Then $\lambda'_j = 1$, so $\lambda_1 = h_\lambda(1,j) + j - 1$. Since $h_\lambda(1,j)$ is divisible by $2p$, it is at least $2p$, and hence $\lambda_1 \geqslant j + 2p - 1$. So $\lambda_1 - \lambda_2 \geqslant 2p$.

On the other hand, suppose $\lambda_2 \geqslant j$. Then $h_\lambda(1,j)$ and $h_\lambda(2,j)$ are both divisible by $2p$, and $h_\lambda(1,j) - h_\lambda(2,j) = \lambda_1 - \lambda_2 + 1$. So $\lambda_1 - \lambda_2$ is congruent to -1 modulo $2p$, and in particular is at least $2p - 1$. $\square$

But if λ is a 2-singular partition with $\lambda_1 - \lambda_2 > 1$, then λ is broken; contradiction. $\square$

Lemma 2.7. *Suppose λ is a 2-singular partition, and let a be maximal such that $\lambda_{a-1} = \lambda_a > 0$. If λ is a $2p$-core, then $\lambda_a \leqslant 2p - 2$.*

Proof. Suppose for a contradiction that $\lambda_a \geqslant 2p - 1$, and consider the hook lengths $h_\lambda(a-1,j)$ and $h_\lambda(a,j)$ for $1 \leqslant j \leqslant \lambda_a$. Note that for any j we have $h_\lambda(a-1,j) = h_\lambda(a,j) + 1$. Furthermore, since a is maximal such that $\lambda_{a-1} = \lambda_a$, we have $\lambda'_j - \lambda'_{j+1} = 0$ or 1 for any $1 \leqslant j \leqslant \lambda_a - 1$, and hence either $h_\lambda(a,j) = h_\lambda(a,j+1) + 1$ or $h_\lambda(a,j) = h_\lambda(a-1,j+1) + 1$. This implies that for any $1 \leqslant k \leqslant \lambda_a$, the set

$$\{h_\lambda(a-1,j) \mid k \leqslant j \leqslant \lambda_a\} \cup \{h_\lambda(a,j) \mid k \leqslant j \leqslant \lambda_a\}$$

equals the interval $\{1,2,\dots,l\}$ for some $l \geqslant \lambda_a - k + 2$. Taking $k = 1$, we find that the hook lengths of λ include $2p$, a contradiction. $\square$

Corollary 2.8. *Suppose λ is an unbroken doubly-singular partition, and is a $2p$-core. Then $\lambda_1, \lambda'_1 \leqslant 4p - 6$. Hence there are only finitely many unbroken doubly-singular $2p$-cores.*

Proof. Let a be maximal such that $\lambda_{a-1} = \lambda_a > 0$, and let b be maximal such that $\lambda'_{b-1} = \lambda'_b > 0$. Applying Lemma 2.7 to λ and λ', we have $\lambda_a, \lambda'_b \leqslant 2p - 2$.

The fact that λ is unbroken implies that $\lambda_i - \lambda_{i+1} \leqslant 1$ for $i = 1, \dots, a-1$; in particular, since we have $\lambda_{\lambda'_b} - \lambda_{\lambda'_b+1} > 1$, we must have $a \leqslant \lambda'_b$. We also deduce that $\lambda_1 \leqslant \lambda_a + a - 2$.

So

$$\begin{aligned}
\lambda_1 &\leqslant \lambda_a + a - 2 \\
&\leqslant 2p - 2 + \lambda'_b - 2 \\
&\leqslant 2p - 2 + 2p - 2 - 2 \\
&= 4p - 6;
\end{aligned}$$

replacing λ with λ', we also get $\lambda'_1 \leqslant 4p - 6$. It is clear that bounding λ_1 and λ'_1 above leaves only finitely many partitions, so there are only finitely many unbroken doubly-singular $2p$-cores. $\square$

This completes the proof of Theorem 2.1.

3. Some small values of *p*

We have shown that every doubly-singular $(2, p)$-irreducible partition must be an unbroken $2p$-core, and that there are only finitely many such partitions. But not every such partition is $(2, p)$-irreducible: there are doubly-singular unbroken partitions which are $(2, \infty)$-reducible (for example, the partition $(4^2, 1)$, which is a $2p$-core provided $p > 3$), and these partitions are $(2, p)$-reducible by Corollary 2.3 with $i = 0$. But it seems reasonable to conjecture that every doubly-singular unbroken $2p$-core which is $(2, \infty)$-irreducible is also $(2, p)$-irreducible. However, the author has very little evidence for this. We end this paper by summarising the information we have for small values of p.

3.1. $p = 2$

In the case $e = p = 2$, $\mathcal{H}_n$ is actually the group algebra of the symmetric group, and the irreducible Specht modules in this case have been classified by James and Mathas [JM3]. And our results verify this classification: there is only one unbroken doubly-singular 4-core, namely the partition (2^2), and this is indeed the only doubly-singular $(2, 2)$-irreducible partition.

3.2. $p = 3$

There are ten unbroken doubly-singular 6-cores, namely the partitions

$$(2^2),\ (3^2),\ (2^3),\ (3^2,1),\ (3,2^2),\ (4^2),\ (3^2,2),\ (2^4),\ (3^3),\ (4,3^2,1).$$

It is not difficult to verify that these partitions are all $(2, 3)$-irreducible; the tables in [M, Appendix B] deal with all except $(4, 3^2, 1)$, for which one may use the fact that (3^3) is $(2, 3)$-irreducible together with the Branching Rule. So we have completed the classification of $(2, 3)$-irreducible partitions.

3.3. $p = 5$

There are 227 unbroken doubly-singular 10-cores, of which 115 are $(2, \infty)$-irreducible. (Note that even though a general classification of $(2, \infty)$-partitions is still unknown, any single partition can be checked using the LLT algorithm [LLT] and Ariki's Theorem [A].) The author has been unable to determine whether these partitions are all $(2, 5)$-irreducible. The first case where the $(2, 5)$-irreducibility is difficult to determine is the partition $(6^2, 5, 4)$.

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