Multiple eigenvalues for singular Hammerstein integral equations with applications to boundary value problems

K.Q. Lan∗

Department of Mathematics, Ryerson University, Toronto, Ont., Canada M5B 2K3

Received 2 August 2004; received in revised form 9 February 2005

Abstract

New results on the existence of multiple positive eigenvalues for singular Hammerstein integral equations are obtained. The radius of the spectrum of a linear Hammerstein integral operator is first employed to obtain a better upper bound for the eigenvalues we seek. These results are applied to treat some second-order ordinary differential equations with two point, three point and some periodic boundary value problems.

© 2005 Elsevier B.V. All rights reserved.

MSC: 34B18; 34B15; 34B16; 47H10; 47H30

Keywords: Positive eigenvalue; Singular Hammerstein integral equation; Singular boundary value problem; Cone; Kernel

1. Introduction

We consider the existence of one or several positive eigenvalues for a singular Hammerstein integral equation of the form

\[ \dot{z}(t) = \int_0^1 k(t, s) f(s, z(s)) \, ds \equiv Az(t), \quad t \in [0, 1], \]

where \( k \) is continuous and \( f \) is allowed to have singularities in its first and second variables.

∗ Research was supported in part by the Natural Sciences and Engineering Research Council of Canada.

∗ Tel.: +1 416 979 5000x6962; fax: +1 416 979 5064.

E-mail address: klan@ryerson.ca.
The existence of at least one positive eigenvalue for (1.1) was studied in [5], where \( f(t, u) \equiv h(u) \) and \( h \) is continuous, so \( f \) has no singularities in its second variable, and a well-known abstract result on the existence of positive eigenvalues for compact maps defined on a cone in a Banach space is applied. Lan [9] employs the same abstract result to treat (1.1), where \( f \) is allowed to have singularities in its first variable. Results obtained in [9] are applied to obtain results on the existence of at least one positive eigenvalue for some second-order ordinary differential equations with separated boundary conditions [9] and multiple eigenvalues for conjugate boundary conditions [10]. Infante [6] improves results in [9] and allows \( k \) to take negative values in order to study some three-point boundary value problems. When \( \lambda = 1 \), (1.1) has been widely studied for example, in [1,2,7,11,15]. All of the results mentioned above require \( f \) to have no singularities in its second variable.

In this paper, we shall obtain new results on the existence of one or several multiple positive eigenvalues for (1.1) by using the abstract result mentioned above. We allow \( f \) to have singularities not only in its first variable but also in its second variable. However, we need the kernel \( k \) to satisfy a stronger inequality than those used in [5,6,9,10]. Moreover, we shall provide new conditions on \( f \) by employing the radius of the spectrum of a linear operator of the form

\[
Lz(t) = \int_0^1 k(t, s) \phi(s)z(s) \, ds \quad \text{for } t \in [0, 1].
\]

This provides better upper bound estimates for the eigenvalues we seek.

We shall apply our results to the existence of multiple positive eigenvalues for two point, three point and periodic boundary value problems. We refer to [1–3,13,16–19,29] for the study of some related eigenvalue problems.

As applications of our results, we consider the following boundary value problems

\[
\dot{z}''(t) + g(t)(az(t) + b\dot{z}(t)) = 0 \quad \text{a.e. on } [0, 1]
\]

with two point, three point and some periodic boundary value problems, where \( \beta \) is allowed to take negative values, so (1.3) may be singular. We shall use our results to show that (1.3) has infinitely many positive eigenvalues. Our results are related to some similar boundary value problems arising from fluid dynamics (see, for example, [20,21,25]).

2. Eigenvalues of singular integral equations

In this section we shall consider the existence of positive multiple eigenvalues of (1.1). Let \( 0 \leq r^* < r \). We always assume the following conditions:

(C1) \( f : [0, 1] \times [r^*, r] \rightarrow \mathbb{R}_+ \) satisfies Carathéodory conditions on \([0, 1] \times [r^*, r] \) and there exists \( g_r \in L^1[0, 1] \) such that

\[
f(t, u) \leq g_r(t) \quad \text{for almost all } t \in [0, 1] \quad \text{and all } u \in [r^*, r].
\]

(C2) \( k : [0, 1] \times [0, 1] \rightarrow \mathbb{R}_+ \) is continuous and there exist \( c \in (0, 1] \) and \( \Phi \in L^1_{+}[0, 1] \) such that

\[
c\Phi(s) \leq k(t, s) \leq \Phi(s) \quad \text{for all } t, s \in [0, 1].
\]
The above hypotheses are special cases of more general conditions given in [12]. Here, we require $k$ to be continuous since the kernels arising from the boundary value problems which we shall discuss in this paper are continuous.

We denote by $C[0, 1]$ the Banach space of all continuous functions from $[0, 1]$ into $\mathbb{R}$ with the usual maximal norm $\|x\| = \max\{|x(t): t \in [0, 1]\}$. Let $P = \{x \in C[0, 1]: x(t) \geq 0 \text{ for all } t \in [0, 1]\}$ denote the standard cone of nonnegative continuous functions. We need a smaller cone $K$ which is defined by

$$K = \{x \in P : q(x) \geq c\|x\|\},$$

where $q : P \to \mathbb{R}^+$ is defined by $q(x) = \min\{|x(t): t \in [0, 1]\}$ and $c$ is given in (C2). Such a cone has been employed in [12,13].

Let $K_r = \{x \in K : \|x\| < r\}, \overline{K}_r = \{x \in K : \|x\| \leq r\}, \partial K_r = \{x \in K : \|x\| = r\}$ and $\overline{K}_{\rho,r} = \{x \in K : \rho \leq \|x\| \leq r\}$.

Since when $r^* > 0$, $f$ is not defined on $[0, 1] \times [0, r^*)$, $A$ is not defined on $\overline{K}_r$. However, it is known that $A$ is well defined on the following closed convex subset of $K$:

$$C(r^*, r) = \{x \in K : r^* \leq x(t) \leq r \text{ for } t \in [0, 1]\}.$$

The above set was first given in [12] and also was employed in [13]. It is shown in [12] that if $r^* > 0$, then $K_{r^*} \cap C(r^*, r) = \emptyset$ and if $r^* < cr$, then $\overline{K}_{r^*/c,r} \subset C(r^*, r) \subset \overline{K}_r$.

The following well-known compactness result is a special case of Lemma 2.2 in [12].

**Lemma 2.1.** Under the hypotheses (C1)–(C2) the map $A$ defined in (1.1) maps $C(r^*, r)$ into $K$ and is compact.

We need the following result on the existence of eigenvalues for a compact map defined on a cone $K$ in a Banach space (see [5, Theorem 2.3.6]).

**Lemma 2.2.** Let $\rho > 0$. Assume that $A : \partial K_\rho \to K$ is compact and satisfies

$$\inf\{\|Ax\| : x \in \partial K_\rho\} > 0.$$ 

Then there exist $\lambda > 0$ and $z \in \partial K_\rho$ such that $\lambda z = Az$.

The above result was employed in [9] to study the existence of eigenvalues for (1.1), where $f$ is defined on $[0, 1] \times [0, r]$ and must be continuous at 0 in its second variable. Here, we employ the above result to treat for (1.1), where $f$ need not be defined on $[0, 1] \times \{0\}$.

Let

$$\frac{1}{m^*} = \max_{t \in [0,1]} \int_0^1 k(t, s)\phi(s) \, ds \quad \text{and} \quad \frac{1}{M^*} = \min_{t \in [0,1]} \int_0^1 k(t, s)\phi(s) \, ds.$$ 

Now, we are in a position to give our results on the existence of eigenvalues and eigenfunctions for (1.1).
Theorem 2.1. Assume that there exist \( \rho \in (\frac{t^*}{\epsilon}, r] \) and \( \psi_\rho \in L^1_+[0, 1] \) such that the following conditions hold.

(i) \( \int_0^1 \phi(s)\psi_\rho(s) \, ds > 0 \).
(ii) \( f(s, u) \geq \psi_\rho(s) \) for all \( u \in [c\rho, \rho] \) and almost all \( s \in [0, 1] \).

Then there exist \( \lambda \geq \frac{1}{\rho m_{\psi_\rho}} \) and \( z \in \partial K_\rho \) such that (1.1) holds.

Proof. Since \( \partial K_\rho \subset C(r^*, r) \), it follows from Lemma 2.1 that \( A : \partial K_\rho \rightarrow K \) is compact. Let \( x \in \partial K_\rho \). Then \( r^* < c\rho \leq x(t) \leq \rho \) for \( t \in [0, 1] \). By hypotheses (i) and (ii), we have for \( t \in [0, 1] \),

\[
\|Ax\| \geq \int_0^1 k(t, s)f(s, x(s)) \, ds \geq c \int_0^1 \phi(s)\psi_\rho(s) \, ds
\]

and \( \inf\{\|Ax\| : x \in \partial K_\rho\} \geq c \int_0^1 \phi(s)\psi_\rho(s) \, ds > 0 \). It follows from Lemma 2.2 that there exist \( \lambda > 0 \) and \( z \in \partial K_\rho \) such that (1.1) holds. This, together with (ii), implies

\[
\lambda z(t) \geq \int_0^1 k(t, s)\psi_\rho(s) \, ds \quad \text{for } t \in [0, 1].
\]

Taking maximum gives \( \lambda \rho \geq 1/m_{\psi_\rho} \). \( \square \)

Remark 2.1. Since (i) is equivalent to \( \max\{\int_0^1 k(t, s)\psi_\rho(s) \, ds : 0 \leq t \leq 1\} > 0 \), Theorem 2.1 generalizes Theorem 1.2 with \( G_0 = [0, 1] \) in [9].

In Theorem 2.1, condition (ii) allows one to obtain a lower bound for the eigenvalue \( \lambda \). The following result shows that if a suitable upper bound is imposed on \( f \), one can obtain an upper bound for \( \lambda \).

Theorem 2.2. In Theorem 2.1, assume further that there exists \( \phi_\rho \in L^1_+[0, 1] \) such that the following condition holds:

(iii) \( f(s, u) \leq \phi_\rho(s) \) for all \( u \in [c\rho, \rho] \) and almost all \( s \in [0, 1] \).

Then there exist \( \lambda \in \left[ \frac{1}{\rho m_{\psi_\rho}}, \frac{1}{\rho m_{\phi_\rho}} \right] \) and \( z \in \partial K_\rho \) such that (1.1) holds.

Proof. By Theorem 2.1, there exist \( \lambda \geq \frac{1}{\rho m_{\phi_\rho}} \) and \( z \in \partial K_\rho \) such that (1.1). By (iii), we have \( \lambda z(t) \leq \int_0^1 k(t, s)\phi_\rho(s) \, ds \) for \( t \in [0, 1] \). Taking maximum gives \( \lambda \rho \leq \frac{1}{m_{\phi_\rho}} \) and this result follows. \( \square \)

In Theorem 2.2, the upper bound imposed on \( f \) is independent of the variable \( u \). However, if the upper bound imposed on \( f \) depends on the variable \( u \), one can obtain a smaller upper bound for \( \lambda \).

To obtain this, we need the radius of the spectrum of the linear operator \( L_\phi : C[0, 1] \rightarrow C[0, 1] \) defined by

\[
L_\phi z(t) = \int_0^1 k(t, s)\phi(s)z(s) \, ds \quad \text{for } t \in [0, 1], \tag{2.3}
\]
The radius of the spectrum of $L_\phi$, denoted $r(L_\phi)$, is given by the well-known spectral radius formula $r(L_\phi) = \lim_{n \to \infty} \|L^n_\phi\|^{1/n}$. We recall that $\lambda \in \mathbb{R}$ is an eigenvalue of $L_\phi$ with corresponding eigenfunction $\varphi \in C[0, 1]$ if $\varphi \neq 0$ and $\lambda \varphi = L_\phi \varphi$. By Theorems 2.6 and 2.8 in [28], $r(L_\phi)$ is an eigenvalue of $L_\phi$ with eigenfunctions in $K$ and $m_\phi \leq r(L_\phi) \leq M_\phi$.

**Theorem 2.3.** In Theorem 2.1, assume further that there exists $\phi_\rho \in L^1_+(0, 1)$ such that the following condition holds:

(iii’) $f(s, u) \leq \phi_\rho(s)u$ for all $u \in [\rho, \rho]$ and almost all $s \in [0, 1]$.

Then there exist $\lambda \in \left[\frac{1}{\rho m_\phi_\rho}, r(L_\phi_\rho)\right]$ and $\varphi \in \partial K_\rho$ such that (1.1) holds.

**Proof.** By Theorem 2.1, there exist $\lambda \geq \frac{1}{\rho m_\phi_\rho}$ and $\varphi \in \partial K_\rho$ such that $\lambda \varphi(t) = \int_0^1 k(t, s) f(s, \varphi(s)) \, ds$. By (iii’), we have for $t \in [0, 1]$,

$$\lambda \varphi(t) \leq \int_0^1 k(t, s) \phi_\rho(s) \varphi(s) \, ds = L_\phi_\rho \varphi(t).$$

This implies $\lambda^2 \varphi(t) \leq L_\phi_\rho \varphi(t) \leq L^2_\phi_\rho \varphi(t)$ for $\varphi \in [0, 1]$ and iterating gives

$$\lambda^n \varphi(t) \leq L^n_{\phi_\rho} \varphi(t) \text{ for } t \in [0, 1].$$

Hence, we have $\lambda^n \leq \|L^n_{\phi_\rho}\|$ and $\lambda \leq \|L^n_{\phi_\rho}\|^{1/n}$. This implies $\lambda \leq \lim_{n \to \infty} \|L^n_{\phi_\rho}\| = r(L_{\phi_\rho})$ and the result follows. □

**Remark 2.2.** (iii’) in Theorem 2.3 implies that $f$ satisfies $f(s, u) \leq \phi_\rho(s)u$ for all $u \in [\rho, \rho]$ and almost all $s \in [0, 1]$. By Theorem 2.2, we obtain $\lambda \leq \frac{1}{m_{\phi_\rho}}$. Note that $r(L_{\phi_\rho}) \leq \frac{1}{m_{\phi_\rho}}$. Therefore, the interval for the eigenvalue $\lambda$ in Theorem 2.3 is smaller than that of Theorem 2.2.

In (1.1), if the function $f$ is continuous on $[0, 1] \times [r, r^*]$ or has the form $g(s) f(u)$, then some simple conditions on $f$ can be given. This leads one to consider the following Hammerstein integral equation

$$\lambda \varphi(t) = \int_0^1 k(t, s) g(s) f(s, \varphi(s)) \, ds \equiv \varphi(t), \quad t \in [0, 1].$$

Such equations have been considered, for example in [13,28].

We always assume $g \in L^1_+(0, 1)$ is such that $\int_0^1 \Phi(s) g(s) \, ds > 0$ and $f : [0, 1] \times (0, \infty) \to \mathbb{R}_+$ is continuous.
Notation. We make the following definitions.

\[ \overline{f}(u) := \max_{t \in [0, 1]} f(t, u), \quad \underline{f}(u) := \min_{t \in [0, 1]} f(t, u), \]

\[ \overline{f}_c \rho := \sup \{ \overline{f}(u) : u \in [c \rho, \rho] \}, \quad \underline{f}_c \rho := \inf \{ \underline{f}(u) : u \in [c \rho, \rho] \}, \]

\[ f^0 := \lim_{u \to 0^+} \overline{f}(u)/u, \quad f_0 := \lim_{u \to 0^+} \underline{f}(u)/u, \]

\[ f_{\infty} := \lim_{u \to \infty} \overline{f}(u)/u, \quad f_{\infty} := \lim_{u \to \infty} \underline{f}(u)/u. \]

By Theorem 2.2, we obtain

**Theorem 2.4.** Assume that there exists \( \rho > 0 \) such that the following condition holds:

\[ (H) \quad f(t, u) > 0 \text{ for } t \in [0, 1] \text{ and } u \in [c \rho, \rho]. \]

Then there exist \( \lambda \in [\overline{f}_c \rho / m\bar{g}, \overline{f}_c \rho / m\bar{g}] \) and \( z \in \partial K_\rho \) such that (2.4) holds.

**Proof.** By (H), \( \overline{f}_c \rho > 0 \). Let \( \psi_\rho(s) = \rho g(s) \overline{f}_c \rho \) and \( \phi_\rho(s) = \rho g(s) \overline{f}_c \rho \). Then

\[ \int_0^1 \Phi(s) \psi_\rho(s) \, ds = \rho \overline{f}_c \rho \int_0^1 \Phi(s) g(s) \, ds > 0 \]

and

\[ \psi_\rho(s) \leq g(s) f(s, u) \leq \phi_\rho(s) \text{ for } u \in [c \rho, \rho] \text{ and almost all } s \in [0, 1]. \]

The result follows from Theorem 2.2. \( \square \)

By Theorem 2.3, we have the following new result.

**Theorem 2.5.** Assume that there exist \( \rho, \zeta > 0 \) such that the following condition holds.

\[ (H') \quad 0 < f(t, u) \leq \zeta u \text{ for } t \in [0, 1] \text{ and } u \in [c \rho, \rho]. \]

Then there exist \( \lambda \in [\overline{f}_c \rho / m\bar{g}, \zeta r(L\bar{g})] \) and \( z \in \partial K_\rho \) such that (2.4) holds.

**Proof.** By (H'), \( \overline{f}_c \rho > 0 \). Let \( \psi_\rho(s) = \rho g(s) \overline{f}_c \rho \) and \( \phi_\rho(s) = g(s) \zeta \). Then

\[ \int_0^1 \Phi(s) \psi_\rho(s) \, ds = \rho \overline{f}_c \rho \int_0^1 \Phi(s) g(s) \, ds > 0 \]

and

\[ \psi_\rho(s) \leq g(s) f(s, u) \leq g(s) \zeta u \text{ for } u \in [c \rho, \rho] \text{ and almost all } s \in [0, 1]. \]

The result follows from Theorem 2.3. \( \square \)

Now, we turn our attention to the existence of multiple eigenvalues of (2.4). Let \( I_n = \{1, \ldots, n\} \) (\( n \geq 2 \)) and \( \mathbb{N} = \{1, 2, \ldots\} \) denote the set of natural numbers.
Theorem 2.6. For each \( i \in I_n \) (or \( i \in \mathbb{N} \)), assume that there exist \( \rho_i \in (0, \infty) \) such that the following conditions hold.

(i) \( \frac{f^{\rho_i}}{c^\rho_i} > 0 \) for each \( i \in I \) (or \( i \in \mathbb{N} \)).

(ii) \( \frac{\overline{f}^{\rho_i}}{c^\rho_i} < \frac{f^{\rho_{i+1}}}{c^\rho_{i+1}} \) for each \( i \in I_{n-1} \) (or \( i \in \mathbb{N} \)).

Then for each \( i \in I_n \) (or \( i \in \mathbb{N} \)) there exist \( \lambda_i \in \left[ \frac{f^{\rho_i}}{m_g}, \frac{\overline{f}^{\rho_i}}{m_g} \right] \) and \( z_i \in \partial K_{\rho_i} \) such that (2.4) holds and \( \lambda_i < \lambda_{i+1} \) for \( i \in I_{n-1} \) (or \( i \in \mathbb{N} \)).

Proof. By (i) and Theorem 2.4, there exist \( z_i \in \left[ \frac{f^{\rho_i}}{m_g}, \frac{\overline{f}^{\rho_i}}{m_g} \right] \) and \( z_i \in \partial K_{\rho_i} \) such that (2.4) holds. By (ii), we have \( \lambda_i < \lambda_{i+1} \). □

As an application of Theorem 2.6, we obtain the following result on the existence of at least two eigenvalues.

Corollary 2.1. Assume that one of the following conditions holds.

(H1) \( f_0 > 0 \) and \( f^0 < cf_\infty \).
(H2) \( f_\infty > 0 \) and \( f^\infty < cf_0 \).

Then (2.4) has two positive eigenvalues.

Proof. Assume that (H1) holds. Note that \( f_0 > 0 \) implies there exists \( \rho_0 > 0 \) such that \( \frac{f^{\rho_0}}{c^\rho_0} > 0 \). Let \( \eta \in (f^0, c f_\infty) \). Then there exist \( \rho_1, \rho_2 \in (0, \infty) \) with \( \rho_1 < \rho_0 < \rho_2 \) such that

\[
\frac{f(t,u)}{\rho_1} \leq \frac{f(t,u)}{\rho_2} < \eta \quad \text{for } t \in [0, 1] \text{ and } u \in [c \rho_1, \rho_1]
\]

and

\[
\eta < \frac{c f(t,u)}{\rho_2} \leq \frac{f(t,u)}{\rho_2} \quad \text{for } t \in [0, 1] \text{ and } u \in [c \rho_2, \rho_2].
\]

This implies \( \frac{\overline{f}^{\rho_1}}{c^\rho_1} < \frac{f^{\rho_2}}{c^\rho_2} \). The results follows from Theorem 2.6 with \( n = 2 \). The proof is similar when (H2) holds. □

3. Multiple positive eigenvalues of boundary value problems

We are interested in the existence of multiple positive eigenvalues for the second-order differential equation of the form

\[
\lambda z''(t) + f(t, z(t)) = 0, \quad \text{a.e on } [0, 1],
\] (3.1)
with one of the following two sets of boundary conditions: the separated boundary condition

\[
\begin{align*}
&xz(0) - z'(0) = 0, \\
&\gamma z(1) + z'(1) = 0,
\end{align*}
\]

(3.2)

where \(x, \gamma > 0\) and \(\Gamma := x + x\gamma + \gamma > 0\) and the three point boundary condition

\[
\begin{align*}
z'(0) = 0, \quad &xz(\eta) = z(1), \quad 0 < \eta < 1 \quad \text{and} \quad 0 < x < 1.
\end{align*}
\]

(3.3)

(3.2) contains boundary conditions (B4), (B4') and (B5) given, for example in [8,9] and (3.3) has been widely studied, for example in [13,26–28].

It is known that (3.1) with (3.2) (or (3.3)) is equivalent to (1.1) with \(k = k_1\) (or \(k = k_2\)), where \(k = k_1\) (or \(k = k_2\)) is the Green’s function to \(-z'' = 0\) subject to (3.2) (or (3.3)). In order to apply the results obtained in Section 2, one has to show that \(k_1\) (or \(k_2\)) satisfies (C2).

It is known (see [8]) that \(k_1 : [0, 1] \times [0, 1] \rightarrow \mathbb{R}_+\) is defined by

\[
k_1(t,s) = \frac{1}{\Gamma} \begin{cases} 
(\gamma + 1 - \gamma t)(1 + xs) & \text{if } 0 \leq s \leq t \leq 1, \\
(1 + xt)(\gamma + 1 - \gamma s) & \text{if } 0 \leq t < s \leq 1.
\end{cases}
\]

(3.4)

and \(k_2 : [0, 1] \times [0, 1] \rightarrow \mathbb{R}_+\) is defined by

\[
k_2(t,s) = \frac{1}{1 - x} \begin{cases} 
1 - x\eta - (1 - x)s & \text{if } t \leq s \leq \eta, \\
1 - x\eta - (1 - x)t & \text{if } s \leq t \text{ and } s \leq \eta, \\
1 - s & \text{if } t \leq s \text{ and } \eta < s, \\
1 - xs - (1 - x)t & \text{if } \eta < s \leq t.
\end{cases}
\]

(3.5)

Let

\[
\Phi_i(s) = k_i(s, s) \quad (i = 1, 2), \quad c_1 = \min \left\{ \frac{1}{1 + x}, \frac{1}{1 + \gamma} \right\} \quad \text{and} \quad c_2 = \frac{x(1 - \eta)}{1 - x\eta}.
\]

It is known that \(k_i\) satisfies (C2) with \(\Phi = \Phi_i\) and \(c = c_i\) (see [8] for \(i = 1\) and [14] for \(i = 2\)). Hence, all results obtained in Section 2 apply.

As applications of our results, we consider the existence of multiple eigenvalues for the following equation

\[
\lambda z''(t) + g(t)(az(t) + bz^\beta(t)) = 0 \quad \text{a.e. on } [0, 1],
\]

(3.6)

with (3.2) or (3.3), where \(a, b > 0, \ g \in L^1_{\#}[0, 1] \) and \(\beta \in (-\infty, \infty)\) with \(\beta \neq 1\).

Similar problems have been studied, for example in [25] where \(\lambda = 1\) and \(a = 0\) in 3.6 and the boundary condition \(z(0) = z(1),\) and [20,21] where \(a = 0\) and the boundary condition \(z'(0) = z(1).\) It is well-known that such boundary layer equations arise from the steady flow of a power-law fluid over an impermeable, semi-infinite flat plane.

The following result shows that (3.6) with (3.2) or (3.3) has infinitely many positive eigenvalues.
Theorem 3.1. (1) Let $\beta \in [0, \infty)$ with $\beta \neq 1$. Assume that $\{\rho_i\} \subset (0, \infty)$ satisfies the following condition:

\[(h_1) \quad a + b\rho_i^{\beta - 1} < ac + bc\rho_i^{\beta - 1} \text{ for } i \in \mathbb{N}.
\]

Then for each $i \in \mathbb{N}$ there exist $\lambda_i \in \left[\frac{f^{\rho_i}}{m_g}, \frac{\overline{f}^{\rho_i}}{m_g}\right]$ and $z_i \in \partial K_{\rho_i}$ such that (3.6) with (3.2) or (3.3) holds and $\lambda_i < \lambda_{i+1}$. Moreover, if $\beta > 1$, then $\|z_i\| < \|z_{i+1}\|$ for $i \in \mathbb{N}$ and if $0 < \beta < 1$, then $\|z_i\| > \|z_{i+1}\|$ for $i \in \mathbb{N}$.

(2) Let $\beta < 0$. Assume that $\{\rho_i\} \subset (0, \infty)$ satisfies the following condition:

\[(h_2) \quad ac + bc\rho_i^{\beta - 1} < a + b\rho_{i+1}^{\beta - 1} \text{ for } i \in \mathbb{N} \text{ and } \rho_1 \leq u_0 := \left(\frac{-b\rho}{a}\right)^{1/(1-\beta)}.
\]

Then for each $i \in \mathbb{N}$ there exist $\lambda_i \in \left[\frac{f^{\rho_i}}{m_g}, \frac{\overline{f}^{\rho_i}}{m_g}\right]$ and $z_i \in \partial K_{\rho_i}$ such that (3.6) with (3.2) or (3.3) holds and $\lambda_i < \lambda_{i+1}$.

Proof. Let $f(u) = au + bu^\beta$ for $u \in (0, \infty)$. Then $f$ is increasing on $(0, \infty)$. Hence, we have for $u \in [c\rho_i, \rho_i],$

\[
\frac{f(u)}{\rho_i} \leq \frac{f(\rho_i)}{\rho_i} = a + b\rho_i^{\beta - 1} \quad \text{and} \quad \frac{f(u)}{\rho_i} \geq \frac{f(c\rho_i)}{\rho_i} \geq ac + bc\rho_i^{\beta - 1}.
\]

This implies $\overline{f}^{\rho_i} \leq a + b\rho_i^{\beta - 1}$ and $f^{\rho_i} \geq ac + bc\rho_i^{\beta - 1}$ for $i \in \mathbb{N}$. Therefore, we have $f^{\rho_i} > 0$ for $i \in \mathbb{N}$ and by $(h_1)$, $\overline{f}^{\rho_i} < f^{\rho_{i+1}}$. The first result follows from Theorem 2.6. Note that if $\beta > 1$, then $(h_1)$ implies $\{\rho_i\}$ is increasing and if $0 < \beta < 1$, then $(h_1)$ implies $\{\rho_i\}$ is decreasing. The second result follows.

(2) It is easy to verify that $f$ is decreasing on $(0, u_0]$ and $\overline{f}^{\rho_i} \leq ac + bc\rho_i^{\beta - 1}$ and $f^{\rho_i} \geq a + b\rho_i^{\beta - 1} > 0$ for $i \in \mathbb{N}$. It follows from $(h_2)$ that $\overline{f}^{\rho_i} < f^{\rho_{i+1}}$. The result follows from Theorem 2.6. \[\Box\]

Remark 3.1. We show that there exists $\{\rho_i\} \subset (0, \infty)$ which satisfies $(h_1)$ (or $(h_2)$). Let $\rho_1 > 0$. Choose $\rho_2 > \left(\frac{a(1-c)+b\rho_1^{\beta - 1}}{bc}\right)^{1/(\beta - 1)}$. Repeating the process gives

\[
\rho_{i+1} > \left(\frac{a(1-c)+b\rho_i^{\beta - 1}}{bc}\right)^{1/(\beta - 1)} \quad \text{for } i \in \mathbb{N}.
\]

It is easy to see that $\{\rho_i\}$ satisfies $(h_1)$. Now, we seek a decreasing sequence $\{\rho_i\} \subset (0, \infty)$ which satisfies $(h_2)$. Let $u_1 = \left(\frac{bc\rho_1^{\beta - 1}}{a-c}\right)^{1/(1-\beta)}$ and $u_2 = \left(\frac{c\rho_1^{\beta - 1}}{a-c}\right)^{1/(1-\beta)}$. Let $\rho_1 < \min\{u_0, u_1\}$. Then $\frac{c\rho_1^{\beta}}{u_1 - \frac{a(1-c)}{b}} > 0$ since $\rho_1 < u_2$ and

\[
\left(\frac{1}{\frac{c\rho_1^{\beta}}{u_1 - \frac{a(1-c)}{b}}}ight)^{1/(1-\beta)} < \rho_1.
\]
We choose $0 < \rho_2 < \left( \frac{e^\beta}{e^\beta \frac{a(1-c)}{b}} \right)^{1/(1-\beta)}$. Then $\rho_2 < \rho_1$ and (h$_2$) holds for $i = 1$. Repeating the process gives

$$
\rho_{i+1} < \left( \frac{1}{e^\beta \frac{a(1-c)}{b}} \right)^{1/(1-\beta)}
$$

and $\rho_{i+1} < \rho_i$. This implies $\{\rho_i\} \setminus (0, \infty)$ satisfies (h$_2$) and is decreasing.

Finally, we mention the results obtained in Section 2 can be applied to the existence of multiple positive eigenvalues of the following boundary value problems.

\[
\begin{align*}
\lambda'z'(t) + az(t) &= f(t, z(t)) \quad \text{a.e. on } [0, 1], \\
z(0) &= z(1). \tag{3.3}_1 \\
\lambda''z''(t) + az(t) &= f(t, z(t)) \quad \text{a.e. on } [0, 1], \\
z(0) &= z(1), \quad z'(0) = z'(1). \tag{3.3}_2 \\
\lambda''z''(t) + az(t) &= f(t, z(t)) \quad \text{a.e. on } [0, 1], \\
z'(0) &= z'(1) = 0. \tag{3.3}_3
\end{align*}
\]

Such equations have been widely studied, for example in [4,12,22–24]. Let $\beta_1 = e^\beta(e^\beta - 1)^{-1}$, $\beta_2 = 2\sqrt{e}e\sqrt{e^\beta - 1}$ and $\beta_3 = \sqrt{e}\sinh \sqrt{e}$. We define three continuous kernels $k_1, k_2, k_3 : [0, 1] \times [0, 1] \to \mathbb{R}_+$ as follows:

\[
k_1(t, s) = \beta_1 \begin{cases} 
\frac{e^{-\sqrt{e}(t-s)}}{e^{-\sqrt{e}(t-s+1)}} & \text{if } 0 \leq s \leq t \leq 1, \\
\frac{e^{-\sqrt{e}(t-s+1)}}{e^{-\sqrt{e}(t-s+1)}} & \text{if } 0 \leq t < s \leq 1.
\end{cases}
\]

\[
k_2(t, s) = \frac{1}{\beta_2} \begin{cases} 
\frac{e^{\sqrt{e}(t-s)}}{e^{\sqrt{e}(s-t)}} + \frac{e^{\sqrt{e}(1-t+s)}}{e^{\sqrt{e}(1-s+t)}} & \text{if } 0 \leq s \leq t \leq 1, \\
\frac{e^{\sqrt{e}(s-t)}}{e^{\sqrt{e}(s-t)}} + \frac{e^{\sqrt{e}(1-s+t)}}{e^{\sqrt{e}(1-s+t)}} & \text{if } 0 \leq t < s \leq 1.
\end{cases}
\]

and

\[
k_3(t, s) = \frac{1}{\beta_3} \begin{cases} 
\cosh \sqrt{e} \cosh \sqrt{e}(1-t) & \text{if } 0 \leq s \leq t \leq 1, \\
\cosh \sqrt{e} \cosh \sqrt{e}(1-s) & \text{if } 0 \leq t < s \leq 1.
\end{cases}
\]

Let $\Phi_1(s) \equiv \beta_1$, $c_1 = e^{-\sqrt{e}}$; $\Phi_2(s) \equiv \beta_2^{-1}(1 + e^{\sqrt{e}})$, $c_2 = 2e^{\sqrt{e}/2}(1 + e^{\sqrt{e}})^{-1}$; $\Phi_3(s) = k_3(s, s)$ and $c_3 = \cosh \sqrt{e}$. It is known that $k_i$ with $\Phi_1$ and $c_i$ satisfies (C$_2$) (see [8, Lemma 3.1]). Therefore, results in Section 2 apply.

References