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Zero-divisor graphs, von Neumann regular rings, and Boolean algebras

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Abstract

For a commutative ring R with set of zero-divisors $Z(R)$, the zero-divisor graph of R is $\Gamma(R) = Z(R) - \{0\}$, with distinct vertices x and y adjacent if and only if $xy = 0$. In this paper, we show that $\Gamma(T(R))$ and $\Gamma(R)$ are isomorphic as graphs, where $T(R)$ is the total quotient ring of R, and that $\Gamma(R)$ is uniquely complemented if and only if either $T(R)$ is von Neumann regular or $\Gamma(R)$ is a star graph. We also investigate which cardinal numbers can arise as orders of equivalence classes (related to annihilator conditions) in a von Neumann regular ring.

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1. Introduction

Let R be a commutative ring with 1, and let $Z(R)$ be its set of zero-divisors. The *zero-divisor graph* of R, denoted by $\Gamma(R)$, is the (undirected) graph with vertices $Z(R)^* = Z(R) - \{0\}$, the set of nonzero zero-divisors of R, and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$. Thus $\Gamma(R)$ is the empty graph if and only if R is an integral domain. Moreover, a nonempty $\Gamma(R)$ is finite if and only if R is finite [4, Theorem 2.2]. The concept of a zero-divisor graph of a commutative ring was introduced by Beck [7]. However, he let all elements of R be vertices of the graph and was mainly interested in colorings. The present definition of $\Gamma(R)$ and the emphasis

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on studying the interplay between graph-theoretic properties of $\Gamma(R)$ and ring-theoretic properties of R are from [4]. For example, in [4, Theorem 2.3] it was proved that $\Gamma(R)$ is connected with $\mathrm{diam}(\Gamma(R)) \leqslant 3$. The zero-divisor graph of a commutative ring has also been studied in [5,3,9,18,19], and the zero-divisor graph concept has recently been extended to noncommutative rings in [22] and commutative semigroups in [8].

In this paper, we continue the investigation begun in [18] of the zero-divisor graph $\Gamma(R)$ of a commutative von Neumann regular ring R. Recall that R is *von Neumann regular* if for each $x \in R$, there is a $y \in R$ such that $x = x^2 y$ or, equivalently, R is reduced and zero-dimensional [14, Theorem 3.1]. In the second section, we show that R and its total quotient ring $T(R)$ have isomorphic zero-divisor graphs. As a corollary, we give necessary and sufficient conditions for two reduced Noetherian rings to have isomorphic zero-divisor graphs. In Section 3, we determine which zero-divisor graphs $\Gamma(R)$ are complemented or uniquely complemented. If R is reduced, then $\Gamma(R)$ is uniquely complemented if and only if R is complemented, if and only if $T(R)$ is von Neumann regular. For a ring R with nonzero nilpotent elements, $\Gamma(R)$ is uniquely complemented if and only if $\Gamma(R)$ is a star graph. (Recall that a graph G is a *star graph* if there is a vertex which is adjacent to every other vertex and these are the only adjacency relations.) We also show that if $\Gamma(R)$ is complemented, then $T(R/\mathrm{nil}(R))$ is von Neumann regular. In fact, $\Gamma(R)$ is complemented, but not uniquely complemented, if and only if $R \cong D \times B$, where D is an integral domain and B is either $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$. In the fourth section, we examine the equivalence relation on a commutative ring given by $a \sim b$ if $\mathrm{ann}(a) = \mathrm{ann}(b)$. If R is von Neumann regular, then each equivalence class contains a unique idempotent. Thus to each element of the Boolean algebra $B(R)$ of idempotents of R is assigned a cardinal number equal to the order of the equivalence class of the element. We investigate the converse of when an assignment of cardinal numbers to each element of a Boolean algebra comes from the equivalence classes of a von Neumann regular ring.

Throughout, R is a commutative ring with $1 \neq 0$, $U(R)$ its group of units, $\mathrm{nil}(R)$ its ideal of nilpotent elements, $Z(R)$ its set of zero-divisors, $Z(R)^* = Z(R) - \{0\}$ its set of nonzero zero-divisors, $\mathrm{Spec}(R)$ its set of prime ideals, $\mathrm{minSpec}(R)$ its set of minimal prime ideals, and $T(R) = R_S$, where $S = R - Z(R)$, its total quotient ring. As usual, an $x \in S$ is called a regular element of R. We let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$, and $GF(p^n)$ denote the nonnegative integers, integers, rationals, $\mathbb{Z}$ modulo n, and the field with p^n elements, respectively. For any undefined ring-theoretic terminology, see [6,11,14,15] or [17].

2. $\Gamma(T(R))$ and $\Gamma(R)$ are isomorphic

In this section, we show that $\Gamma(T(R))$ and $\Gamma(R)$ are isomorphic as graphs. (Recall that two graphs G and G' are *isomorphic*, denoted by $G \cong G'$, if there is a bijection $\varphi : G \to G'$ of vertices such that the vertices x and y are adjacent in G if and only if $\varphi(x)$ and $\varphi(y)$ are adjacent in G'.) Note that $\Gamma(R)$ is an induced subgraph of $\Gamma(T(R))$; so this is not a natural isomorphism, but depends on the cardinality of equivalence classes of vertices of the two graphs.

We first generalize a result from [18] which will be used in the proof of Corollary 2.5. The “in particular” part is [18, Corollary 2.4] (and [3, Theorem 4.1] for finite products of finite fields). Note that if each A_i and B_j in Theorem 2.1 is a field, then A and B are von Neumann regular rings. Also, the "⇐" implication in Theorem 2.1 is a special case of Theorem 2.2 when $J = I$ and each $B_i = T(A_i)$ since $|T(R)| = |R|$ for any commutative ring R.

Theorem 2.1. *Let $\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$ be two families of integral domains, and let $A = \prod_{i\in I} A_i$ and $B = \prod_{j\in J} B_j$. Then $\Gamma(A) \cong \Gamma(B)$ if and only if there is a bijection $\varphi: I \to J$ such that $|A_i| = |B_{\varphi(i)}|$ for each $i \in I$. In particular, if $\Gamma(A) \cong \Gamma(B)$ and each A_i is a finite field, then each B_j is also a finite field and $A_i \cong B_{\varphi(i)}$ for each $i \in I$, and thus $A \cong B$.*

Proof. Since $|T(R)| = |R|$ for any commutative ring R, the proof follows directly from the proof of [18, Theorem 2.3 and Corollary 2.4]. For the "in particular" part, just note that two finite fields are isomorphic if and only if they have the same cardinality. □

Let R be a commutative ring. As in [19, (3.5)], for $x, y \in R$, we define $x \sim y$ if and only if $\mathrm{ann}(x) = \mathrm{ann}(y)$. Clearly $\sim$ is an equivalence relation on R, and restricts to an equivalence relation on $\Gamma(R)$ $(=Z(R)^*)$. We next show that $\Gamma(T(R))$ and $\Gamma(R)$ are isomorphic by showing that there is a bijection between equivalence classes of $\Gamma(T(R))$ and $\Gamma(R)$ such that the corresponding equivalence classes have the same cardinality.

Theorem 2.2. *Let R be a commutative ring with total quotient ring $T(R)$. Then the graphs $\Gamma(T(R))$ and $\Gamma(R)$ are isomorphic.*

Proof. Let $S = R - Z(R)$ and $T = T(R)$. Denote the equivalence relations defined above on $Z(R)^*$ and $Z(T)^*$ by $\sim_R$ and $\sim_T$, respectively, and denote their respective equivalence classes by $[a]_R$ and $[a]_T$. Note that $\mathrm{ann}_T(x/s) = \mathrm{ann}_R(x)_S$ and $\mathrm{ann}_T(x/s) \cap R = \mathrm{ann}_R(x)$; thus $x/s \sim_T x/t, x \sim_R y \Leftrightarrow x/s \sim_T y/s, ([x]_R)_S = [x/1]_T$, and $[x/s]_T \cap R = [x]_R$ for all $x, y \in Z(R)^*$ and $s, t \in S$. Since $Z(T) = Z(R)_S$, by the above comments we have $Z(R)^* = \bigcup_{\alpha\in A} [a_\alpha]_R$ and $Z(T)^* = \bigcup_{\alpha\in A} [a_\alpha/1]_T$ (both disjoint unions) for some $\{a_\alpha\}_{\alpha\in A} \subset R$.

We next show that $|[a]_R| = |[a/1]_T|$ for each $a \in Z(R)^*$. If $[a]_R$ is finite, then $[a]_R = [a/1]_T$. The inclusion "⊂" is clear. For the reverse inclusion, let $x \in [a/1]_T$. Then $x = b/s$ with $b \in [a]_R$ and $s \in S$. Since $\{s^n b \mid n \geqslant 1\} \subset [a]_R$ is finite, $b = s^i b$ for some integer $i > 1$, and hence $b/s = s^i b/s = s^{i-1} b \in [a]_R$. Now suppose that $[a]_R$ is infinite. Clearly $|[a]_R| \leqslant |[a/1]_T|$. Define an equivalence relation $\approx$ on S by $s \approx t$ if and only if $sa = ta$. Then $s \approx t$ if and only if $sb = tb$ for all $b \in [a]_R$. It is easily verified that the map $[a]_R \times S/\approx \to [a/1]_T$, given by $(b, [s]) \to b/s$, is well-defined and surjective. Thus $|[a/1]_T| \leqslant |[a]_R||S/\approx|$. Also, the map $S/\approx \to [a]_R$, given by $s \to sa$, is clearly well-defined and injective. Hence $|S/\approx| \leqslant |[a]_R|$, and thus $|[a/1]_T| \leqslant |[a]_R|^2 = |[a]_R|$ since $|[a]_R|$ is infinite. Hence $|[a]_R| = |[a/1]_T|$. Thus there is a bijection $\varphi_\alpha : [a_\alpha] \to [a_\alpha/1]$ for each $\alpha \in A$.

Define $\varphi : Z(R)^* \to Z(T)^*$ by $\varphi(x) = \varphi_\alpha(x)$ if $x \in [a_\alpha]$. Clearly φ is a bijection. Thus we need only show that x and y are adjacent in $\Gamma(R)$ if and only if $\varphi(x)$ and $\varphi(x)$ are adjacent if $\Gamma(T)$; i.e., $xy = 0$ if and only if $\varphi(x)\varphi(y) = 0$. Let $x \in [a]_R$, $y \in [b]_R$, $w \in$

$[a/1]_T$, and $z \in [b/1]_T$. It is sufficient to show that $xy = 0$ if and only if $wz = 0$. Note that $\text{ann}_T(x) = \text{ann}_T(a) = \text{ann}_T(w)$ and $\text{ann}_T(y) = \text{ann}_T(b) = \text{ann}_T(z)$. Thus $xy = 0 \Leftrightarrow y \in \text{ann}_T(x) = \text{ann}_T(w) \Leftrightarrow yw = 0 \Leftrightarrow w \in \text{ann}_T(y) = \text{ann}_T(z) \Leftrightarrow wz = 0$. Hence $\Gamma(R)$ and $\Gamma(T(R))$ are isomorphic as graphs. □

Corollary 2.3. *Let A and B be commutative rings with isomorphic total quotient rings. Then $\Gamma(A)$ and $\Gamma(B)$ are isomorphic as graphs. In particular, $\Gamma(A) \cong \Gamma(B)$ when A and B have the same total quotient ring.*

Proof. This is clear from Theorem 2.2. □

Corollary 2.4. *Let R be a reduced commutative Noetherian ring such that* minSpec $(R) = \{P_1, \ldots, P_n\}$. *Then $\Gamma(R) \cong \Gamma(K_1 \times \cdots \times K_n)$, where each $K_i = T(R/P_i)$ is a field.*

Proof. This is also clear from Theorem 2.2 since $T(R) \cong K_1 \times \cdots \times K_n$. □

Corollary 2.5. *Let A and B be reduced commutative Noetherian rings which are not integral domains. Then $\Gamma(A) \cong \Gamma(B)$ if and only if there is a bijection φ:* minSpec $(A) \to$ minSpec(B) *such that $|A/P| = |B/\varphi(P)|$ for each $P \in$* minSpec(A).

Proof. Let minSpec$(A) = \{P_1, \ldots, P_m\}$, minSpec$(B) = \{Q_1, \ldots, Q_n\}$, and each $T(A/P_i) = K_i$ and $T(B/Q_j) = L_j$. Thus $\Gamma(A) \cong \Gamma(K_1 \times \cdots \times K_m)$ and $\Gamma(B) \cong \Gamma(L_1 \times \cdots \times L_n)$ by Corollary 2.4.

($\Rightarrow$) Suppose that $\Gamma(A) \cong \Gamma(B)$. Thus by Theorem 2.1, $m = n$ and there is a permutation p of $\{1, \ldots, n\}$ such that $|A/P_i| = |K_i| = |L_{p(i)}| = |B/Q_{p(i)}|$ for each $1 \leqslant i \leqslant n$. Clearly p induces the required bijection φ.

($\Leftarrow$) Suppose that such a bijection φ exists. Then $\Gamma(K_1 \times \cdots \times K_m) \cong \Gamma(L_1 \times \cdots \times L_m)$ by Theorem 2.1, and hence $\Gamma(A) \cong \Gamma(B)$. □

Note that if A and B are both integral domains, then $\Gamma(A) = \emptyset = \Gamma(B)$ with no cardinality conditions needed on A and B.

3. Complemented graphs and von Neumann regular rings

In this section, we first give a graph-theoretic characterization of $\Gamma(R)$ when $T(R)$ is von Neumann regular; namely, for a reduced ring R, $\Gamma(R)$ is uniquely complemented (see definition below) if and only if $T(R)$ is von Neumann regular. We then show that for a ring R with nonzero nilpotent elements, $\Gamma(R)$ is uniquely complemented if and only if $\Gamma(R)$ is a star graph. Moreover, such a $\Gamma(R)$ either has one or two edges or is infinite. Finally, we show that $\Gamma(R)$ is complemented, but not uniquely complemented, if and only if $R \cong D \times B$, where D is an integral domain and B is either $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.

Let G be a (undirected) graph. As in [18], for vertices a and b of G, we define $a \leqslant b$ if a and b are not adjacent and each vertex of G adjacent to b is also adjacent to a; and we define $a \sim b$ if $a \leqslant b$ and $b \leqslant a$. Thus $a \sim b$ if and only if a and b

are adjacent to exactly the same vertices. Clearly $\sim$ is an equivalence relation on G. For $a,b \in Z(R)^*$, we have $a \sim b$ in $\Gamma(R)$ if and only if $\mathrm{ann}(a) - \{a\} = \mathrm{ann}(b) - \{b\}$. (Hence the two equivalence relations defined in Sections 2 and 3 are the same for reduced rings, cf. Remark 3.2.) Also, as in [18], for distinct vertices a and b of G, we say that a and b are *orthogonal*, written $a \perp b$, if a and b are adjacent and there is no vertex c of G which is adjacent to both a and b, i.e., the edge $a - b$ is not part of any triangle of G. Thus for distinct $a,b \in Z(R)^*$, we have $a \perp b$ in $\Gamma(R)$ if and only if $ab = 0$ and $\mathrm{ann}(a) \cap \mathrm{ann}(b) \subset \{0,a,b\}$. We say that G is *complemented* if for each vertex a of G, there is a vertex b of G (called a *complement* of a) such that $a \perp b$, and that G is *uniquely complemented* if G is complemented and whenever $a \perp b$ and $a \perp c$, then $b \sim c$.

Our first two lemmas translate the above graph-theoretic concepts into ring-theoretic terms. Example 3.6(b) shows that the reduced hypothesis is needed in Lemma 3.4.

Lemma 3.1 (cf. [18, Lemma 2.11]). *Consider the following statements for a commutative ring R and $a,b \in Z(R)^*$.*

(1) *$a \sim b$.*
(2) *$aR = bR$.*
(3) *$ann(a) = ann(b)$.*
(a) *If R is reduced, then statements* (1) *and* (3) *are equivalent.*
(b) *If R is von Neumann regular, then all three statements are equivalent.*

Proof. (a) We have already observed that $a \sim b$ if and only if $\mathrm{ann}(a) - \{a\} = \mathrm{ann}(b) - \{b\}$. Thus (1) and (3) are equivalent when R is reduced.

(b) Assume that R is von Neumann regular. Since a von Neumann regular ring is reduced, it is enough to prove the equivalence (2) $\Leftrightarrow$ (3). Clearly (2) $\Rightarrow$ (3) holds for any commutative ring. To show (3) $\Rightarrow$ (2), let $a = a^2c$ for some $c \in R$. Then $1 - ac \in \mathrm{ann}(a) = \mathrm{ann}(b)$, and hence $b(1 - ac) = 0$ yields $b \in aR$. Thus $bR \subset aR$, and similarly $aR \subset bR$. □

Remark 3.2. Observe that when R is reduced, the equivalence relation $\sim$ defined on $Z(R)^*$ can easily be extended to all elements of R by using (3) of the above lemma and thus agrees with the equivalence relation defined in Section 2. Moreover, $[1] = R - Z(R)$ and $[0] = \{0\}$. For von Neumann regular rings, the partial ordering $\leqslant$ also extends to all elements of R with $a \leqslant b$ if and only if $aR \subset bR$, if and only if $\mathrm{ann}(b) \subset \mathrm{ann}(a)$. In this case $[1] = U(R)$. We will use this more general definition in Section 4.

Lemma 3.3. *Let R be a commutative ring and $a,b \in Z(R)^*$. Then the following statements are equivalent.*

(1) *$a \perp b$, $a^2 \neq 0$, and $b^2 \neq 0$.*
(2) *$ab = 0$ and $a + b$ is a regular element of R.*

Proof. (1) $\Rightarrow$ (2) Suppose that (1) holds. Then $ab = 0$ since $a \perp b$. Suppose that $(a + b)c = 0$ for some $c \in R$. Let $y = ac = -bc$; then $ya = yb = 0$. Thus $y \in \{0,a,b\}$ since $a \perp b$. If $y = a$, then $a^2 = ay = 0$, a contradiction. Similarly, $y = b$ yields $b^2 = 0$.

Hence $y=0$. Then $ac=bc=0$, and thus $c\in\{0,a,b\}$ since $a\perp b$. If $c=a$, then $a^2=ac=0$, a contradiction. Similarly, $b^2=0$ if $c=b$. Thus $c=0$, and hence $a+b$ is a regular element of R.

(2) $\Rightarrow$ (1) Suppose that (2) holds. First observe that $a\neq b$ since $a+b$ is a regular element of R. If $a^2=0$, then $a(a+b)=a^2+ab=0$, a contradiction. Thus $a^2\neq 0$, and similarly $b^2\neq 0$. Suppose that $ca=cb=0$ for some $c\in R$. Then $c(a+b)=0$, and hence $c=0$ since $a+b$ is a regular element of R. Since $ab=0$, we have $a\perp b$. □

Lemma 3.4. *Let R be a reduced commutative ring and $a,b,c\in Z(R)^*$. If $a\perp b$ and $a\perp c$, then $b\sim c$. Thus $\Gamma(R)$ is uniquely complemented if and only if $\Gamma(R)$ is complemented.*

Proof. Note that $ab=ac=0$ since $a\perp b$ and $a\perp c$. We first show that $bc\neq 0$; so b and c are not adjacent. If $bc=0$, then either $c=a$ or $c=b$ since $a\perp b$ and $a\perp c$. Either choice contradicts that R is reduced; so $bc\neq 0$. Now suppose that $db=0$ for some $d\in Z(R)^*$. Then $(dc)a=d(ac)=0$ and $(dc)b=(db)c=0$. Thus $dc\neq 0$ implies that the vertex dc is adjacent to both a and b (and $dc\neq a$ or b since R is reduced). This is a contradiction since $a\perp b$; so $dc=0$. Hence $c\leqslant b$. Similarly, $b\leqslant c$, and thus $b\sim c$. The "last" statement is clear. □

Clearly star graphs are uniquely complemented. Suppose that R is von Neumann regular. Then for each $a\in Z(R)^*$, we have $a=ue$, where $u\in U(R)$ and $e\in R$ is idempotent [14, Corollary 3.3]. Clearly $ue\perp(1-e)$. Thus a von Neumann regular ring R has $\Gamma(R)$ complemented. In fact, $\Gamma(R)$ is uniquely complemented by Lemma 3.4 since a von Neumann regular ring R is reduced. Hence by Theorem 2.2, $\Gamma(R)\cong\Gamma(T(R))$ is uniquely complemented if $T(R)$ is von Neumann regular. We next show in Theorem 3.5 that the converse holds when R is a reduced commutative ring. The key fact is that $T(R)$ is von Neumann regular if and only if for each $x\in R$, there is a $y\in R$ such that $xy=0$ and $x+y$ is a regular element of R [2, Theorem 2.3]. In Theorem 3.9, we will show that if R has nonzero nilpotent elements, then $\Gamma(R)$ is uniquely complemented precisely when it is a star graph.

Theorem 3.5. *The following statements are equivalent for a reduced commutative ring R.*
(1) $T(R)$ is von Neumann regular.
(2) $\Gamma(R)$ is uniquely complemented.
(3) $\Gamma(R)$ is complemented.

Moreover, a nonempty $\Gamma(R)$ is a star graph if and only if $R\cong D\times\mathbb{Z}_2$ for some integral domain D.

Proof. (1) $\Rightarrow$ (2) We give a proof that does not depend on Theorem 2.2. Suppose that $T(R)$ is von Neumann regular. Let $a\in Z(R)^*$. By the comments preceding Theorem 3.5, there exists $b\in T(R)$, necessarily nonzero, such that $a\perp b$ in $\Gamma(T(R))$. Choose $s\in R-Z(R)$ such that $sb\in R$. One can then easily verify that $a\perp sb$ in $\Gamma(R)$. Hence

$\Gamma(R)$ is complemented, and thus uniquely complemented by Lemma 3.4 since R is reduced.

(2) $\Rightarrow$ (3) This is true for any graph.

(3) $\Rightarrow$ (1) Let $x \in R$; we may assume that $x \in Z(R)^*$. Then there is a $y \in Z(R)^*$ such that $x \perp y$. By Lemma 3.3, $xy = 0$ and $x + y$ is a regular element of R. Hence $T(R)$ is von Neumann regular by [2, Theorem 2.31].

The "moreover" statement follows from the remarks after [4, Theorem 2.5]. □

We next consider the case when R has nonzero nilpotent elements. In this case, we show in Theorem 3.9 that $\Gamma(R)$ is uniquely complemented if and only if it is a star graph. Moreover, either $\Gamma(R)$ has one or two edges, or $\Gamma(R)$ is an infinite star graph with center x, where $\mathrm{nil}(R) = \{0, x\}$. Recall that a vertex of a graph is called an *end* if there is only one other vertex adjacent to it. First some examples and a key lemma.

Example 3.6. (a) Let $R = \mathbb{Z}[x]/(2x, x^2) = \mathbb{Z}[\bar{x}]$. Then $\mathrm{nil}(R) = \{0, \bar{x}\}$ and $\Gamma(R)$ is an infinite star graph with center $\bar{x}$.

(b) Let D be any integral domain and $R = D \times \mathbb{Z}_4$. Then $\mathrm{nil}(R) = \{(0, \bar{0}), (0, \bar{2})\}$, and it is easily verified that $\Gamma(R)$ is complemented. However, $\Gamma(R)$ is not uniquely complemented since $(0, \bar{2}) \perp (1, \bar{0})$ and $(0, \bar{2}) \perp (1, \bar{2})$, but $(1, \bar{0}) \nsim (1, \bar{2})$. In a similar fashion, if $R = D \times \mathbb{Z}_2[x]/(x^2)$, then $\Gamma(R)$ is complemented, but not uniquely complemented. □

Lemma 3.7. *Let R be a commutative ring with* $\mathrm{nil}(R)$ *nonzero.*

(a) *If $\Gamma(R)$ is complemented, then either $|R| = 8$, $|R| = 9$, or $|R| > 9$ and* $\mathrm{nil}(R) = \{0, x\}$ *for some $0 \neq x \in R$.*

(b) *If $\Gamma(R)$ is uniquely complemented and $|R| > 9$, then any complement of the nonzero nilpotent element of R is an end.*

Proof. (a) Suppose that $\Gamma(R)$ is complemented, and let $a \in \mathrm{nil}(R)$ have index of nilpotence $n \geqslant 3$. Let $y \in Z(R)^*$ be a complement of a. Then $a^{n-1}y = 0 = a^{n-1}a$; so $y = a^{n-1}$ since $a \perp y$. Thus $\mathrm{ann}(a) = \{0, a^{n-1}\}$, since if $za = 0$, then $za^{n-1} = 0$, and $a \perp a^{n-1}$. Similarly $a^i \perp a^{n-1}$ for each $1 \leqslant i \leqslant n-2$. Suppose that $n > 3$. Then $a^{n-2} + a^{n-1}$ kills both a^{n-2} and a^{n-1}, a contradiction since $a^{n-2} \perp a^{n-1}$ and $a^{n-2} + a^{n-1} \notin \{0, a^{n-2}, a^{n-1}\}$. Thus if R has a nilpotent element with index $n \geqslant 3$, then $n = 3$. In this case, $Ra^2 = \{0, a^2\}$ since each $z \in Ra^2$ kills both a and a^2, and $a \perp a^2$. Also, $\mathrm{ann}(a^2) = \{0, a, a^2, a + a^2\}$. (If $za^2 = 0$, then $za \in \mathrm{ann}(a) = \{0, a^2\}$; so either $za = 0$ or $za = a^2$. If $za = 0$, then $z = 0$ or $z = a^2$, while if $za = a^2$, then $(z - a)a = 0$, and hence either $z = a$ or $z = a + a^2$.) Thus R is local with $|R| = 8$, $\mathrm{nil}(R) = Z(R) = \mathrm{ann}(a^2)$ its maximal ideal, and $\Gamma(R)$ is a star graph with center a^2 and two edges.

Now suppose that each nonzero nilpotent element of R has index of nilpotence 2. Let $0 \neq y \in \mathrm{nil}(R)$ have complement $z \in Z(R)^*$. Note that $(ry)y = 0 = (ry)z$ for all $r \in R$. Thus $Ry \subset \{0, y, z\}$. First suppose that $2y \neq 0$. Then necessarily $z = 2y$ since $2y \in Ry \subset \{0, y, z\}$. Also, $\mathrm{ann}(y) = \{0, y, 2y\}$ since $y \perp 2y$. Thus $Ry = \{0, y, 2y\}$; so we have $|R| = 9$. In this case, R is local with maximal ideal $Z(R) = \mathrm{nil}(R) = \mathrm{ann}(y)$ and $\Gamma(R)$ is a star graph with one edge.

Next, suppose that each nonzero nilpotent element of R has index of nilpotence 2 and $|R| \neq 9$. By above, we must have $2y = 0$. We show that $\mathrm{nil}(R) = \{0, y\}$. Suppose that z is another nonzero nilpotent element of R; so $z^2 = 0$. Then $y + z$ is nilpotent of index 2. First, observe as above that $Ry \subset \{0, y, y'\}$ and $Rz \subset \{0, z, z'\}$, where y' and z' are complements of y and z, respectively. Next, observe that $yz = 0$. For if $yz \neq 0$, then $yz = y' = z'$. Thus $Ry = \mathrm{ann}(y) = \{0, y, yz\}$, and hence $|R| = 9$, a contradiction. Let w be a complement of $y + z$. Clearly w is neither y nor z. Then either $wy = y$ or $wy = y'$, since otherwise y kills both w and $y + z$. However, if y' is a multiple of y, then as above we must have $|R| = 9$, a contradiction. Thus $wy = y$, and similarly $wz = z$. But then $w(y + z) = wy + wz = y + z$, which contradicts $w \perp (y + z)$. Hence R has a unique nonzero nilpotent element.

(b) Suppose that $\Gamma(R)$ is uniquely complemented and $|R| > 9$. Let x be the unique nonzero nilpotent element of R from part (a) above, and let y be a complement of x. We first show that $x + y$ is also a complement of x. Clearly $x(x + y) = 0$ since $x^2 = 0$ and $x \perp y$. Note that $x = -x$, and thus $x + y \in Z(R)^*$. Suppose that $wx = 0 = w(x + y)$ for some $w \in Z(R)^*$. Then $wy = 0$, and hence either $w = y$ or $w = x$ since $x \perp y$. If $w = y$, then $y^2 = 0$, a contradiction. Thus $w = x$, and hence $x \perp (x + y)$. By uniqueness of complements, we have $x + y \sim y$. Suppose that $zy = 0$ for some $z \in Z(R)^* - \{x\}$. Then $z \neq y$ since $y^2 \neq 0$, and thus $z(x + y) = 0$ since $x + y \sim y$. Hence $zx = 0$ since $zy = 0$. This contradicts that $x \perp y$. Thus no such z can exist; so y is an end. □

Remark 3.8. (a) The proof of Lemma 3.7(a) shows that if $\Gamma(R)$ is complemented and $|\mathrm{nil}(R)| > 2$, then either $|R| = 8$ (with $|\mathrm{nil}(R)| = 4$) or $|R| = 9$ (with $|\mathrm{nil}(R)| = 3$). In either case, $\Gamma(R)$ is uniquely complemented; see Remark 3.12(a) for the commutative rings R in Lemma 3.7(a) which have $\Gamma(R)$ uniquely complemented and $|R| = 8$ or 9. In all other cases, it follows that if $\Gamma(R)$ is complemented and $\mathrm{nil}(R)$ is nonzero, then $|\mathrm{nil}(R)| = 2$.

(b) The rings in Example 3.6(b) have $\Gamma(R)$ complemented and $|\mathrm{nil}(R)| = 2$, but $\Gamma(R)$ is not uniquely complemented. In particular if $R = \mathbb{Z}_2 \times B$, where B is either $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$, then $|R| = 8$, $|\mathrm{nil}(R)| = 2$, and $\Gamma(R)$ is complemented, but not uniquely complemented. It is easily shown that (up to isomorphism) these are the only two commutative rings with $|R| = 8$ and $|\mathrm{nil}(R)| = 2$.

(c) Let R be a commutative ring with $\mathrm{nil}(R) = \{0, x\}$. Then $\mathrm{ann}(x)$ is a maximal ideal of R. In particular, if $ab \in \mathrm{nil}(R)$ for $a, b \in R$, then either $ax = 0$ or $bx = 0$. In graph-theoretic terms, this says that if the vertices a and b are adjacent in $\Gamma(R)$, then either a or b is adjacent to x. Moreover, if $\mathrm{nil}(R) = \{0, x\}$ is a prime ideal of R, then $Z(R) = \mathrm{ann}(x)$. □

Theorem 3.9. *Let R be a commutative ring with* $\mathrm{nil}(R)$ *nonzero. If $\Gamma(R)$ is uniquely complemented, then either $\Gamma(R)$ is a star graph with at most two edges or $\Gamma(R)$ is an infinite star graph with center x, where* $\mathrm{nil}(R) = \{0, x\}$.

Proof. Suppose that $\Gamma(R)$ is uniquely complemented and $\mathrm{nil}(R)$ is nonzero. If $|R| \leqslant 9$, then either $|R| = 8$ or $|R| = 9$ by Lemma 3.7(a) and Remark 3.8(a) and (b). In either case, $\Gamma(R)$ is a star graph with one or two edges, respectively (see the proof of Lemma 3.7(a)). Thus we may assume that $|R| > 9$. Hence $\mathrm{nil}(R) = \{0, x\}$ for some $0 \neq x \in R$ by Lemma 3.7(a).

We first show that $\Gamma(R)$ is infinite. Let c be a complement for x; thus $\mathrm{ann}(c) = \{0, x\}$ by Lemma 3.7(b). We first show that c^2 is also a complement for x. If not, then there is a $y \in Z(R)^* - \{c^2, x\}$ such that $xy = 0 = c^2y$. Thus $(cy)^2 = c^2y^2 = 0$; so either $cy = 0$ or $cy = x$. If $cy = 0$, then $y \in \mathrm{ann}(c) = \{0, x\}$, a contradiction. Therefore $cy = x$. But then $cy^2 = xy = 0$. Hence $y^2 \in \mathrm{ann}(c) = \mathrm{nil}(R)$. Then $y \in \mathrm{nil}(R) = \{0, x\}$, again a contradiction. Thus $c^2 \perp x$, and hence $\mathrm{ann}(c^2) = \{0, x\}$. Thus $c^{2^n} \perp x$ for each integer $n \geqslant 1$. Since $\mathrm{ann}(c^i) \subset \mathrm{ann}(c^j)$ for all integers $1 \leqslant i \leqslant j$, we have $\mathrm{ann}(c^n) = \{0, x\}$ for all integers $n \geqslant 1$. Hence each c^n is an end. Next, note that the c^n's are all distinct. For suppose that $c^i = c^j$ for some $1 \leqslant i < j$. Then $c^i(c^{j-i} - 1) = 0$, and thus $c^{j-i} - 1 = x$ since $\mathrm{ann}(c^i) = \mathrm{ann}(c)$ and $c \in Z(R)^*$. But then $c^{j-i} = 1 + x \in U(R)$ since $x \in \mathrm{nil}(R)$, and hence $c \in U(R)$, a contradiction. Thus $\Gamma(R)$ is infinite.

We next show that $\Gamma(R)$ is a (infinite) star graph with center x. Let c be a complement of x; thus $\mathrm{ann}(c) = \{0, x\}$ since c is an end by Lemma 3.7(b). By way of contradiction, suppose that $\Gamma(R)$ is not a star graph. Then by Lemma 3.7(b), there is an $a \in Z(R)^* - \{c, x\}$ with $ax = 0$ such that a is not a complement of x; thus $\{0, x\}$ is properly contained in $\mathrm{ann}(a)$. By hypothesis, $a \perp y$ for some $y \in Z(R)^*$. Note that a, x, y, and c are all distinct. One can easily show that $cy \in Z(R) - \{0, a, c, x, y\}$ (use $\mathrm{ann}(c^2) = \mathrm{ann}(c)$ to show that $cy \neq x$) and $a(cy) = (ay)c = 0$ and $x(cy) = (xc)y = 0$. By hypothesis, there is a $z \in Z(R)^*$ such that $z \perp cy$. One can then also verify that $z \notin \{0, a, c, x, y, cy\}$. We show that $zx = 0$. Then $zx = 0 = (cy)x$, and hence z and cy are not orthogonal, a contradiction. So suppose that $zx \neq 0$. Then $zx = x$ since $zx \in \mathrm{nil}(R) = \{0, x\}$. Hence $(zy)c = z(cy) = 0$ since $z \perp cy$, and thus either $zy = 0$ or $zy = x$ since $\mathrm{ann}(c) = \{0, x\}$. If $zy = 0$, then $xy = (zx)y = (zy)x = 0$. But this is a contradiction since $ax = 0$ and $a \perp y$. Similarly, $zy = x$ implies $(zx)y = (zy)x = x^2 = 0$, and hence $xy = (zx)y = 0$, which again contradicts $a \perp y$. Thus $\Gamma(R)$ is an infinite star graph with center x. □

Corollary 3.10. *Let R be a commutative ring. Then $\Gamma(R)$ is uniquely complemented if and only if either $T(R)$ is von Neumann regular or $\Gamma(R)$ is a star graph.*

Proof. This follows from Theorem 3.5 for reduced rings and Theorem 3.9 for rings with nonzero nilpotent elements. □

Corollary 3.11. *Let R be a finite commutative ring which is not a field (i.e., $\emptyset \neq \Gamma(R)$ is finite). Then $\Gamma(R)$ is uniquely complemented if and only if R is isomorphic to either $\mathbb{Z}_8$, $\mathbb{Z}_9$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_3[x]/(x^2)$, $\mathbb{Z}_4[x]/(2x,\ x^2-2)$, or $F_1 \times \cdots \times F_n$, where each F_i is a finite field and $n \geqslant 2$. Moreover, $\Gamma(R)$ is a star graph if and only if R is isomorphic to either $\mathbb{Z}_8$, $\mathbb{Z}_9$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_3[x]/(x^2)$, $\mathbb{Z}_4[x]/(2x,\ x^2-2)$, or $F \times \mathbb{Z}_2$ for F a finite field.*

Proof. This follows from Theorems 3.5 and 3.9 and the list of rings in Remark 3.12(a) below. □

Remark 3.12. (a) The commutative rings for which $\Gamma(R)$ is a star graph can be described as follows: If $\Gamma(R)$ has one edge, then R is isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_9$, or $\mathbb{Z}_3[x]/(x^2)$; if $\Gamma(R)$ has two edges, then R is isomorphic to either $\mathbb{Z}_6, \mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x,\ x^2-2)$ [3, Example 2.1(a)]. If $\Gamma(R)$ is a finite star graph with three or more edges, then $R \cong F \times \mathbb{Z}_2$ for F a finite field [4, Theorem 2.13]. (Hence a finite star graph has p^n vertices for some prime number p and integer $n \geqslant 1$ [4, Corollary 2.7].) If $\Gamma(R)$ is an infinite star graph, then either $R \cong D \times \mathbb{Z}_2$ for D an integral domain or $\text{nil}(R) = \{0, x\}$ is a prime ideal of R and $Z(R) = \text{ann}(x)$ [9, Theorem 1.12; 19, (2.1)].

(b) Note that if $\Gamma(R)$ is uniquely complemented, then the total quotient ring of $R/\text{nil}(R)$ is von Neumann regular. In fact, $R/\text{nil}(R)$ is an integral domain when $\text{nil}(R)$ is nonzero by the comments in part (a) above. However, the converse is false even for finite rings. For an infinite example, let T be any von Neumann regular ring and $R = T[x]/(x^2)$. Then $\text{nil}(R) = Z(R) = (x)/(x^2)$ and $R/\text{nil}(R) = T$ is von Neumann regular. But $\Gamma(R)$ is the complete graph on $|T| - 1$ vertices, which is uniquely complemented if and only if $T = \mathbb{Z}/3\mathbb{Z}$.

(c) If $\Gamma(R)$ is uniquely complemented and $\text{nil}(R)$ is nonzero, then $R/\text{nil}(R)$ is an integral domain by (b). So in this case, $\Gamma(R/\text{nil}(R))$ is the empty graph.

(d) Note that Lemma 3.7(b) may fail if $\Gamma(R)$ is just assumed to be complemented. For example, let $R = \mathbb{Z}_3 \times Z_4$ (cf. Example 3.6(b)). □

In our next theorem, we characterize those commutative rings R for which $\Gamma(R)$ is complemented, but not uniquely complemented. In particular, such rings are of the form $D \times B$, where D is an integral domain and B is either $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$. We first give a lemma.

Lemma 3.13. *Let $R = A \times B$, where A is a reduced commutative ring and B is a commutative ring with* $\text{nil}(B)$ *nonzero. Then $\Gamma(R)$ is complemented, but not uniquely complemented, if and only if A is an integral domain and B is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.*

Proof. Suppose that $\Gamma(R)$ is complemented, but not uniquely complemented. Since $|\text{nil}(R)| = 2$ by Lemma 3.7(a) and Remark 3.8(a) and $\text{nil}(A \times B) = \text{nil}(A) \times \text{nil}(B)$, we may assume that $\text{nil}(B) = \{0, b\}$. We first show that $|B| = 4$. Thus B is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$. Since $Bb = \{0, b\}$, it is sufficient to show that $\text{ann}(b) = \{0, b\}$. So suppose that there is an $a \in \text{ann}(b) - \{0, b\}$. Then we claim that $y = (1, b)$ has no complement. Any complement of y has the form $z = (0, c)$ for some $0 \neq c \in B$. If $c = b$, then $(0, a)$ is adjacent to both y and z; while, if $c \neq b$, then $(0, b)$ is adjacent to both y and z. Thus the claim is proved, so we must have $|B| = 4$. We next show that A is an integral domain. If not, then since A is reduced, there are nonzero $c, d \in A$ with $cd = 0$ and $c \neq d$. Then we claim that $y = (c, b)$ has no complement. Any complement of y has the form $z = (r, s)$, where $cr = 0$ and $s = 0$ or $s = b$. If $r \neq 0$, then $(0, b)$ is adjacent to both y and z; while if $r = 0$, then $s = b$ and (d, b) is adjacent to both y and z. Thus A must be an integral domain. The converse has already been observed in Example 3.6(b). □

Theorem 3.14. *Let R be a commutative ring. Then $\Gamma(R)$ is complemented, but not uniquely complemented, if and only if R is isomorphic to $D\times B$, where D is an integral domain and B is $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.*

Proof. By Example 3.6(b), if $R=D\times B$, where D and B have the stated forms, then R is complemented, but not uniquely complemented. Conversely, suppose that $\Gamma(R)$ is complemented, but not uniquely complemented. We first show that $R/\mathrm{nil}(R)$ has a nontrivial idempotent. By Lemma 3.4, Lemma 3.7(a), and Remark 3.8(a), $\mathrm{nil}(R)=\{0,b\}$. Since $\Gamma(R)$ is complemented, but not uniquely complemented, there is a vertex a with distinct complements y and z and a vertex w which is adjacent to y, but not z. Thus $wz\neq 0$. Since $y(wz)=0$, $a(wz)=0$, $a\perp y$, and $wz\neq 0$, either $wz=a$ or $wz=y$. Thus either $a^2=0$ or $y^2=0$, and hence either $a=b$ or $y=b$. Suppose that $a=b$. Let $v=w^2-w$. Clearly $vy=0$ since $wy=0$. Since $b\perp y$, we have $wb\neq 0$. Thus $wb=b$, and hence $vb=w^2b-wb=b-b=0$. Since $b\perp y$, we must have $v=0$, $v=y$, or $v=b$. If $v=y$, then $y^2=0$, a contradiction. Thus $w^2-w=v\in\mathrm{nil}(R)$, and hence $w+\mathrm{nil}(R)$ is the desired nontrivial idempotent. The case where $y^2=0$, and thus $y=b$, is similar; just let $v=z^2-z$. Thus $R/\mathrm{nil}(R)$ has a nontrivial idempotent, and hence R has a nontrivial idempotent by [17, Corollary, p. 73]. Therefore, we may assume that $R=A\times B$. Since $|\mathrm{nil}(R)|=2$ and $\mathrm{nil}(A\times B)=\mathrm{nil}(A)\times\mathrm{nil}(B)$, we may assume that A is reduced and $\mathrm{nil}(B)=\{0,b\}$. The result now follows from Lemma 3.13. □

Corollary 3.15. *Let R be a commutative ring such that $\Gamma(R)$ is complemented (either uniquely or not). Then $T(R/\mathrm{nil}(R))$ is von Neumann regular. Moreover, if $\mathrm{nil}(R)$ is nonzero, then $\Gamma(R/\mathrm{nil}(R))$ is either the empty graph or a star graph.*

Proof. If R is uniquely complemented, this is just Remark 3.12(b). If R is not uniquely complemented, then it follows from Theorem 3.14. The "moreover" statement follows from Theorem 3.5. □

Corollary 3.16. *Let R be a finite commutative ring which is not a field (i.e., $\emptyset\neq\Gamma(R)$ is finite). Then $\Gamma(R)$ is complemented, but not uniquely complemented, if and only if R is isomorphic to $F\times B$, where F is a finite field and B is either $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.*

4. Constructing von Neumann regular rings

There are natural conditions which are satisfied by the idempotents of a von Neumann regular ring. These conditions correspond to functions which can be defined for an arbitrary Boolean algebra. In this section, our main objective is to obtain necessary and sufficient conditions so that a Boolean algebra with such a function arises as the idempotents of a von Neumann regular ring.

Given a commutative ring R, the set of idempotents of R forms a Boolean algebra, denoted $B(R)$, under the operations $a\wedge b=ab$ and $a\vee b=a+b-ab$, with largest element 1, smallest element 0, and complement given by $a'=1-a$. For a reference on Boolean algebras and the Boolean algebra of idempotents, see [1] or [17].

Before going into our main topic, we first recall some facts from [18] about $\Gamma(R)$ when R is von Neumann regular. Let the relations $\leqslant$ and $\sim$ on R be defined as in Section 3. Note that for nontrivial idempotents e and f of any commutative ring R, we have $e \leqslant f$ if and only if $ef = e$, if and only if $Re \subset Rf$ (cf. Lemma 3.1 and Remark 3.2). Thus $e \sim f$ if and only if $e = f$. Hence if R is von Neumann regular, then each equivalence class $[a]$ contains a unique idempotent e. In fact, if $a = ue$ with $u \in U(R)$ and $e \in R$ idempotent, then $[a] = [e] = \{ve \mid v \in U(R)\} = \{w \in Re \mid w \text{ a unit of } Re\}$. Also note that a nontrivial idempotent e is minimal with respect to the partial order $\leqslant \Leftrightarrow e$ is an atom in the Boolean algebra $B(R) \Leftrightarrow e$ is a primitive idempotent of $R \Leftrightarrow Re$ is a simple R-module, i.e., Re is a field.

Conversely, let B be a Boolean algebra and define binary operations on B by $a + b = (a \wedge b') \vee (a' \wedge b)$ and $ab = a \wedge b$. Then, as is well known, with these operations B is a commutative von Neumann regular ring. In fact, B is a *Boolean ring*, i.e., $x^2 = x$ for all $x \in B$. However, in general, the von Neumann regular ring R is quite different from the ring $B(R)$. (Note that while multiplication in $B(R)$ is just multiplication in R, the addition in $B(R)$ is given by $e + f - 2ef$ for idempotents e and f of R.) For example, if R is any direct product of fields indexed by a set I, then $B(R)$ is the direct product of copies of $\mathbb{Z}_2$ indexed by I. This difference and the following two basic results are the motivation behind the work in this section.

Theorem 4.1. *Let R and S be commutative von Neumann regular rings. Then $\Gamma(R)$ and $\Gamma(S)$ are isomorphic as graphs if and only if there is a Boolean algebra isomorphism $\varphi : B(R) \to B(S)$ such that $|[e]| = |[\varphi(e)]|$ for each $1 \neq e \in B(R)$.*

Proof. Assume the Boolean algebra isomorphism φ exists. Let $x, y \in Z(R)^*$, and let e_x, e_y denote the unique idempotents of R such that $e_x \in [x]$ and $e_y \in [y]$. Then the vertices x and y of $\Gamma(R)$ are adjacent if and only if $e_x e_y = 0$, i.e., as elements of the Boolean algebra $B(R)$, $e_x \wedge e_y = 0$, if and only if $\varphi(e_x)\varphi(e_y) = \varphi(e_x) \wedge \varphi(e_y) = 0$. Thus, if the cardinalities of the sets $[e_x]$ and $[\varphi(e_x)]$ are equal for all $x \in Z(R)^*$, then the graphs $\Gamma(R)$ and $\Gamma(S)$ are isomorphic.

Conversely, let $h : \Gamma(R) \to \Gamma(S)$ be an isomorphism of graphs. For each $a \in Z(R)^*$, observe that the image under h of every element in $[a]$ is in the set $[h(a)]$. Hence, we can define a bijection $\varphi : B(R) \to B(S)$ by setting $\varphi(e)$ equal to the unique idempotent in the set $[h(e)]$ when $e \neq 0, 1$, and setting $\varphi(0) = 0$ and $\varphi(1) = 1$. We first note that $\varphi(e') = \varphi(e)'$ for all $e \in B(R)$ since e' is the unique element of $B(R)$ which is adjacent to e and has no element adjacent to both e and itself. Next we claim that φ preserves the ordering on $B(R)$. This follows since $f \leqslant e$ if and only if f is adjacent to e'. Finally, since $e \wedge f$ is the unique largest element which is smaller than both e and f, it follows that $\varphi(e \wedge f) = \varphi(e) \wedge \varphi(f)$. Similarly, φ distributes over $\vee$. Hence φ is an isomorphism of Boolean algebras. Clearly $|[e]| = |\varphi(e)|$ for each $1 \neq e \in B(R)$. □

Corollary 4.2. *Let $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ be two families of integral domains with A_i a subring of B_i and $|A_i| = |B_i|$ for each $i \in I$, and let $A = \prod_{i \in I} A$ and $B = \prod_{i \in I} B$. If R is a subring of B containing A, then $\Gamma(A) \cong \Gamma(R) \cong \Gamma(B)$.*

Proof. Note that $B(A)=B(R)=B(B)$ and $[e]_A \subset [e]_R \subset [e]_B$ for each idempotent $e \in A$. It also follows from our assumptions and the definition that $|[e]_A| = |[e]_B|$. Thus $|[e]_A| = |[e]_R| = |[e]_B|$. The corollary now follows directly from Theorem 4.1. □

Example 4.3. (a) With the same notation as in the corollary, suppose that each integral domain A_i is infinite. Then $|A_i|=|A_i[x]|$ for each $i \in I$, and $A=\prod_{i\in I} A_i \subset (\prod_{i\in I} A_i)[x] \subset \prod_{i\in I}(A_i[x])$. Thus $\Gamma(A[x]) \cong \Gamma(A)$ by Corollary 4.2.

(b) Even though $(\prod_{i\in I} A_i)[[x]]=\prod_{i\in I}(A_i[[x]])$, the result in part (a) need not hold for power series since we may have $|R| < |R[[x]]|$ when R is infinite. However, suppose that $A=\prod_{n\in\mathbb{N}}\mathbb{R}$. Then $|\mathbb{R}[[x]]| = |\mathbb{R}|$, and thus $\Gamma(A) \cong \Gamma(A[x]) \cong \Gamma(A[[x]])$ by Corollary 4.2. □

Remark 4.4 (cf. [18, Example 2.8]). If the rings R and S are each the direct product of finite fields, then $\Gamma(R) \cong \Gamma(S)$ if and only if $R \cong S$ by Theorem 2.1. If R and S are arbitrary von Neumann regular rings, then by Proposition 4.1, $\Gamma(R) \cong \Gamma(S)$ implies that $B(R) \cong B(S)$; however, R and S need not be isomorphic. For example, let $S=\prod_{i\in\mathbb{N}} F_i$, where each $F_i = GF(3^2)$, and let R be its subring $R=\prod_{i\in\mathbb{N}} K_i + \sum_{i\in\mathbb{N}} F_i$, where each $K_i=\mathbb{Z}_3$. Then R and S are each von Neumann regular, and $B(R)=B(S)$ is isomorphic to the direct product of copies of $\mathbb{Z}_2$ (indexed over $\mathbb{N}$). Thus $\Gamma(R) \cong \Gamma(S)$ by Theorem 4.1 (using an easy cardinality argument). However, R and S are not isomorphic rings since S contains a square root of -1, but R does not. □

Recall that the equivalence relation $\sim$ was defined for any undirected graph G. Note that $G/\sim$ is also an undirected graph in the natural way with $[x]$ and $[y]$ adjacent in $G/\sim$ if and only if x and y are adjacent in G. We next show that if R is von Neumann regular, then $\Gamma(R)/\sim$ is also the zero-divisor graph of a von Neumann regular ring, namely $B(R)$.

Proposition 4.5. *Let R be a commutative von Neumann regular ring. Then $\Gamma(R)/\sim$ and $\Gamma(B(R))$ are naturally isomorphic as graphs.*

Proof. Define $\varphi : \Gamma(B(R)) \to \Gamma(R)/\sim$ by $\varphi(e)=[e]$. By the previous comments, φ is bijective and preserves adjacency. □

Let R be a von Neumann regular ring. For each $a \in R$, we denote by $n_R(a)$ the cardinality of $[a]$. It follows that if a is an atom of $B(R)$, then, since Ra is a field, $n_R(a)$ is either an infinite cardinal or $n_R(a) = p^i - 1$ for some prime number p and some integer $i \geqslant 1$. Furthermore, if $ab=0$ for some $b \in R$, then $n_R(a \vee b) = n_R(a+b) = n_R(a)n_R(b)$, since in this case $(a+b)R \cong aR \times bR$. Also note that $n_R(0)=1$ and $n_R(1)=|U(R)|$.

Now let B be a Boolean algebra. Suppose that n is a function which assigns to each element $b \in B$ a nonzero cardinal number $n(b)$ with the following properties:

(i) If b is an atom of B, and $n(b)$ is finite, then $n(b)=p^i-1$ for some prime number p and some integer $i \geqslant 1$.

(ii) If $a, b \in B$ with $a \wedge b = 0$, then $n(a \vee b) = n(a)n(b)$.
(iii) $n(0) = 1$.

Then we call n a *labeling* of B, and we say that (B, n), or B, if n is clear, is a *labeled Boolean algebra*. If R is a von Neumann regular ring, then from the above comments the function n_R is a labeling of $B(R)$. We refer to $(B(R), n_R)$ as the *labeled Boolean algebra associated with* R and say that such a labeled Boolean algebra is *realizable* as a von Neumann regular ring.

Theorem 4.1 may be rephrased as: for commutative von Neumann regular rings R and S, the graphs $\Gamma(R)$ and $\Gamma(S)$ are isomorphic if and only if there is a Boolean algebra isomorphism $\varphi : B(R) \to B(S)$ such that $n_S(\varphi(e)) = n_R(e)$ for each $1 \neq e \in B(R)$. We next give an example that shows that n_R and n_S may agree on the atoms in case $B(R) = B(S)$, and yet be different labelings. This also shows that condition (ii) in the definition of a labeling does not extend to infinite sups even if B is a complete Boolean algebra.

Example 4.6. Let K be a subfield of a field F with $|K| = \beta$ and $|F| = \alpha$, where $\beta^\omega < \alpha < \alpha^\omega$. Such cardinals exists by Koenig's Lemma (see, for example [16]). Let $R = \prod_{n\in\mathbb{N}} K_n + \sum_{n\in\mathbb{N}} F_n$ and $S = \prod_{n\in\mathbb{N}} F_n$, where $F_n = F$ and $K_n = K$ for each $n \in \mathbb{N}$. Then $B(R) = B(S) = \prod_{n\in\mathbb{N}} \mathbb{Z}_2$ is a complete Boolean algebra, and $n_R(e_i) = n_S(e_i) = \alpha$ for each atom $e_i \in B(R)$. For any infinite $I \subset \mathbb{N}$, let e_I be the idempotent with 1 in all coordinates $i \in I$ and zero elsewhere. Then $n_R(e_I) = \beta^\omega + \alpha = \alpha < \alpha^\omega = n_S(e_I)$. Hence, $n_R(e_I) < \prod_{i\in I} n_R(e_i)$; so condition (ii) does not extend to infinite sups. Also note that n_R and n_S agree on atoms, but not on all elements of $B(R)$, and $\Gamma(R)$ and $\Gamma(S)$ are not isomorphic.

For the remainder of this section, we examine which labeled Boolean algebras are realizable as von Neumann regular rings. We will do this by viewing an arbitrary Boolean algebra topologically, by means of the Stone Representation Theorem (for a proof, see [1, p. 207] or [6, p. 14]). Although we could state all of our results directly in terms of Boolean algebras, it is easier and more natural to work with topological spaces.

Theorem 4.7 (Stone Representation Theorem). *If B is a Boolean algebra, then there exists a compact Hausdorff space X having a basis of clopen sets such that B is isomorphic to the Boolean algebra of clopen subsets of X.*

We note that the space X associated to B is Spec(B) with the usual Zariski topology. Furthermore, it follows that B is a subring of $\prod_{x\in X} F_x$, where $F_x \cong \mathbb{Z}_2$ is the factor ring of B modulo the maximal ideal x. We also note that the atoms of B correspond to the isolated points of X.

Let (B, n) be a labeled Boolean algebra, and let X be the topological space associated to B by the Stone Representation Theorem. Then n defines a function on the clopen sets (i.e., sets that are both open and closed) of X, and we will talk about the labeled space (X, n). If $x \in X$ is an isolated point, we write $n(x)$ for $n(\{x\})$. We first obtain some necessary conditions on n, so that (X,n) is realizable; then in Theorem 4.13, we obtain a sufficient condition under the assumption that X is a metric space. We note that a compact Hausdorff space X having a basis of clopen sets is metrizable if and only if X has a countable basis of clopen sets, if and only if B is a countable set. To see this, note that a compact space X is metrizable if and only if it has a countable basis [10, p. 260]. Since a clopen set in a compact Hausdorff space is the union of finitely many elements from a basis, a compact metric space has only countably many clopen subsets. For more on the basic definitions and results in topology, we suggest the reader see [10].

We should note that [12,13,20,21] examine questions related to what we do here. In [20], they define a 'labeled Boolean algebra' though their definition is somewhat different than the one used here. In [13], they ask the following question: Given a fixed field F, a topological space X that comes from a Boolean algebra, and a family $\{K_x\}_{x\in X}$ of field extensions of F, does there exist a von Neumann regular F-algebra R and a homeomorphism $x \mapsto P_x$ from X to $\mathrm{Spec}(R)$ such that $R/P_x \cong K_x$ for each point x in X?

We first start with some examples of labeled spaces which will help motivate our results.

Example 4.8. (i) Let X be a subspace of the real line consisting of a decreasing, convergent sequence $(a_i)_{i\in\mathbb{N}}$ and its limit point a. Let $p>2$ be a fixed prime number, and define $n(a_i)=p^i-1$ and $n(U)=\omega$ for all infinite clopen sets U of X, where the topology on X is inherited from the real line. There is a unique way to extend n to all clopen sets U of X so that n is a labeling on X. We now construct the von Neumann regular ring R whose associated labeled Boolean algebra is the same as (X,n). Let $S=\prod_{x\in X} F_x$, where $F_x = GF(p^i)$ if $x=a_i$, and $F_a = GF(p)$. Let R be the subring of S generated by 1 and all functions in S which are zero on a neighborhood of a. Then the elements of R consist of all elements of S which, except for a finite number of places off the coordinate corresponding to a, are constant and an element of $GF(p)$, i.e., $R=\mathbb{Z}_p\cdot 1+\sum_{x\neq a} F_x$. It is not difficult to check that R is von Neumann regular and realizes the labeled space (X,n).

(ii) Let X be as in (i) and define $n(a_i)=2$ if i is odd and 4 if i is even. Then, regardless of how we define n on the other clopen sets, it is not possible to construct a von Neumann regular ring that realizes (X,n). To see this, notice that every clopen set of X is either finite or cofinite, i.e., contains all but finitely many of the points of X. Therefore for any ring R that realizes (X,n), we must have that for all idempotents $e\in R$, either there are only finitely many primitive idempotents f with $fe\neq 0$ or there are only finitely many primitive idempotents f with $fe=0$. However, let R be such a ring. Let $e\in R$ be the idempotent with $e\sim 3\cdot 1$, and let e_i be the idempotent corresponding to the clopen set $\{a_i\}$. Then $e_ie\neq 0$ if i is even, while $e_ie=0$ if i is odd.

Given a labeled space (X,n) and a prime number p, we say that an isolated point $x\in X$ has *characteristic* p if $n(x)=p^i-1$ for some integer $i\geqslant 1$. For the remainder of the paper, (X,n) will denote a labeled compact Hausdorff space.

Lemma 4.9. *If (X,n) is realizable as a von Neumann regular ring, then for each prime number p there exists a (possibly empty) clopen set $U \subset X$ which contains all the isolated points having characteristic p and the only isolated points with finite characteristic contained in U have characteristic p.*

Proof. Suppose that R is a von Neumann regular ring that realizes (X,n), and let $e \in R$ be the idempotent such that $e \sim p \cdot 1$. Let U be the clopen set corresponding to the idempotent $1-e$. Then U has the desired properties. □

Corollary 4.10. *If (X,n) is realizable as a von Neumann regular ring, then for each prime number p there is a clopen subset $U_p \subset X$ such that U_p contains all the isolated points of characteristic p, and no other isolated points of finite characteristic. Moreover, the sets U_p can be chosen to be pairwise disjoint.*

Proof. The existence of U_p follows immediately from the previous result. To see the last statement, let U_i denote the clopen set corresponding to the ith prime number. Let $V_1 = U_1$, and for each $i > 1$ we can replace U_i with $U_i - (V_1 \cup \cdots \cup V_{i-1})$. □

Let (X,n) be a labeled Boolean algebra. We call a nonempty clopen set $U \subset X$ *uniform* relative to n (or merely uniform, if the n is clear), if $n(V) = n(U)$ for all nonempty clopen sets V contained in U. Clearly isolated points are uniform. Observe that since a set of cardinal numbers is well-ordered, every clopen set contains a uniform clopen set.

We next give a necessary condition for a labeled metrizable space to be realizable.

Theorem 4.11. *Let (X,n) be a labeled metrizable compact Hausdorff space which can be realized as a von Neumann regular ring R. Then $n(U) \leqslant \alpha^{\omega}$ for each clopen set $U \subset X$, where*

$$a = \sup\{n(V) \mid V \text{ a uniform clopen set contained in } U\}.$$

Proof. By our assumption on X, every clopen set U contains a countable collection of uniform clopen subsets $\{V_i\}_{i\in\mathbb{N}}$ such that U is the closure of the set $\bigcup V_i$. For $i \in \mathbb{N}$, let $e_i = e_{V_i}$, where e_V denotes the idempotent of R associated to V. Now let $a \in e_U R$. Suppose that $ae_i = 0$ for all $i \in \mathbb{N}$; then we claim that $a = 0$. For if not, then there exists an idempotent $e_W \in R$ with W a nonempty clopen subset of U and $a \sim e_W$. This implies that $W \cap V_i = \emptyset$ for all $i \in \mathbb{N}$. However, this is impossible since U is the closure of $\bigcup V_i$. Therefore, $a = 0$. Also note that if a is a regular element of $e_U R$, then ae_i is a regular element of $e_i R$. If b is another regular element of $e_U R$, then $ae_i \neq be_i$ for some i. Note that for each $i \in \mathbb{N}$, there are at most α choices for ae_i. Hence the result follows. □

Corollary 4.12. *Let (X,n) be as in Theorem 4.11. If U is a clopen set such that $1 < n(U)$ is finite, then the set I of isolated points $a \in U$ such that $n(a) > 1$ is finite and nonempty. Furthermore, $n(U) = \prod_{a\in I} n(a)$.*

Proof. It follows from the definition of a labeling and the fact that $n(U)$ is finite that I is a finite set (which is necessarily clopen). Thus $W=U-I$ is a clopen set of X. We claim that $n(W)=1$, which proves that I must be nonempty. The last statement then follows from the definition of a labeling. By Theorem 4.11, we can assume that W is uniform. If W consists of a single point, then the result follows from the definition of W. Otherwise, we can write W as the disjoint union of nonempty clopen sets V and V'. (Pick two distinct points in W, then we can find a clopen neighborhood of one that misses the other.) Thus $n(V)n(V')=n(W)$. Hence, if $n(W)>1$, then one of $n(V)$ or $n(V')$ is less than $n(W)$, a contradiction to the fact that W is uniform; so the claim is proved. □

We now give a sufficient condition for (X,n) to be realizable by a von Neumann regular ring.

Theorem 4.13. *Let (X,n) be a metrizable labeled compact Hausdorff space, and let A denote the set of isolated points of X. Suppose that all points $a\in A$ with $n(a)$ finite have been assigned a characteristic as previously described. Furthermore, assume the following conditions hold.*

1. *If U is a uniform clopen set which is not a point, then $n(U)$ is either 1 or an infinite cardinal.*
2. *If U is any clopen set, then $n(U)=\sup\{n(V)\mid V\subset U$ is* either a uniform clopen set or a finite set of isolated points$\}$.
3. *The set $\{p_i\}$ of primes, each of which is assigned as the characteristic to infinitely many isolated points, is finite.*
4. *For each characteristic p_i in condition 3, there exists a clopen set U_{p_i} which contains all the isolated points a such that $char(a)=p_i$, and if $a\in U_{p_i}\cap A$, then either $char(a)=p_i$ or $n(a)$ is an infinite cardinal. Additionally, $U_{p_i}-A$ does not contain a clopen set W with $n(W)=1$.*
5. *There exists a clopen set U_t disjoint from each U_{p_i} such that $U_t\cap A$ contains only finitely many points of any one characteristic and U_t-A does not contain a clopen set W such that $n(W)=1$.*

Then (X,n) is realizable as a von Neumann regular ring.

Proof. To each point x of X we want to assign a characteristic, denoted char(x). For some of the points in X we have already done this. If $x\in U_{p_i}$, we assign char$(x)=p_i$, and if $x\in U_t-A$, then declare char$(x)=0$. To the points of X which have not yet been assigned a characteristic, we assign the characteristic 2. Next, to each $x\in X$ we assign a field F_x which has the same characteristic as x and such that

$$|F_x|=\min\{n(U)\mid U \text{ a clopen neighborhood of } x\}+1.$$

First note that if W is a nonempty clopen set that contains no isolated points, then by conditions 1 and 2, $n(W)$ is either 1 or an infinite cardinal. We next claim that if $x\in U_{p_i}-A$, then $|F_x|$ is either 2 or infinite. If $x\in U_2-A$ is a limit point such that $n(W)=1$ for some clopen neighborhood of x, then clearly $|F_x|=2$. Otherwise, every neighborhood U of x contains either infinitely many isolated points a with $n(a) > 1$ or a nonempty clopen subset W of $U_{p_i} - A$. In the former case, $n(U)$ is infinite by the definition of a labeling. In the latter case, $n(W)$, and hence $n(U)$, is infinite by our assumption on $U_{p_i} - A$ in 4. Thus $|F_x|$ is infinite. A similar argument shows that if $x \in U_t - A$, then $|F_x|$ is infinite; so the claim is proved. Furthermore, we can pick these fields so that for any $x, y \in X$, if $\text{char}(x) = \text{char}(y)$ and $\omega \leqslant |F_x| \leqslant |F_y|$, then $F_x \subset F_y$, with the fields being equal if they have the same cardinality. Since any set of cardinals has a smallest element, any nonempty clopen set U which is disjoint from all the U_{p_i} and U_t contains a point x such that F_x is contained in F_y for all $y \in U$. This is also true if U is contained in either $U_{p_i} - A$ or $U_t - A$.

Let $S = \prod_{x \in X} F_x$, and for each clopen set U, let e_U denote the element of S which is one at all $x \in U$ and zero at all other coordinates. Let R be the subset of S consisting of all elements of the form $\sum a_i e_{V_i} + \sum b_j e_{W_j} + \sum m_k e_{S_k} + f$, where $\{V_i\}$, $\{W_j\}$, and $\{S_k\}$ are clopen sets with the following properties: Each V_i is disjoint from U_{p_i} and U_t; so all the points in each V_i have characteristic 2. Each a_i is from one of the fields F_x, where $x \in V_i$ and $F_x \subset F_y$ for all $y \in V_i$. Each W_j is a subset of either $U_{p_i} - A$ for some i or $U_t - A$, and b_j is chosen like a_i. Each m_k is an integer and S_k contains a nonisolated point and is contained in either U_{p_i} for some i or in U_t. Finally, $f \in \sum_{x \in A} F_x$. We note that every clopen set of X is the union of finitely many of the V_i's, W_j's, S_k's with finitely many isolated points. Then it is not difficult to see that R is a subring of S and the idempotents of R are precisely those elements of the form e_U, where U is a clopen set of X. Thus $B(R)$ is isomorphic to the Boolean algebra of clopen sets of X. As in Example 4.8(i), one checks that R is a von Neumann regular ring. Hence we only have to show that $n(U) = |[e_U]|$ for each clopen set U.

First suppose that U is uniform and disjoint from any U_{p_i} and from U_t. Thus every element in U has characteristic 2 and all the fields F_x, $x \in U$, are the same; we denote this field by F. An element of $e_U R$ has the form $a_1 e_{U_1} + a_2 e_{U_2} + \cdots + a_r e_{U_r}$, where the U_i are clopen sets that partition U and each a_i is an element of the field F. Hence $|F| = n(U) + 1$. Now suppose that U is an arbitrary clopen set disjoint from any U_{p_i} and from U_t. Then by condition 2, it follows that $n(U) = \sup\{\prod_{x \in I}(|F_x| - 1)\}$, where I runs through the finite subsets of U. (We have to word the equation in this awkward manner because of the possibility that $n(U) = 1$, in which case $|F_x| = 2$.) Again, every element of $e_U R$ has the form $a_1 e_{U_1} + a_2 e_{U_2} + \cdots + a_r e_{U_r}$, where the U_i partition U and each a_i is an element of some F_x, where $x \in U_i$ and F_x is contained in each field F_y, $y \in U_i$. Thus it follows that $n(U) = |[e_U]|$. If U is a subset of either U_{p_i} or U_t which contains a nonisolated point, then essentially the same proof will work. Finally, if U is a finite set of isolated points, then it is immediate that $|[e_U]| = n(U)$. Since any clopen subset of X is the finite disjoint union of such sets, the result is proved. □

We next present some examples to show how our results can be applied. For the following, let **C** denote the Cantor set.

Example 4.14. By Theorem 4.13, if α is any infinite cardinal, or if $\alpha = 1$, the labeling of **C** given by $n(U) = \alpha$ for each clopen set U is realizable.

Example 4.15. **C** can be viewed as the interval $I=[0,1]$ with countably many pairwise disjoint open intervals $I_1, I_2 \ldots$, each contained in $(0,1)$, removed in such a way that what is left has no isolated points. For each $k\in\mathbb{N}$, let p_k be an element of I_k, and let $X=\mathbf{C}\cup\{p_k \mid k\in\mathbb{N}\}$. Then X has a dense set of isolated points, namely, $S=\{p_k \mid k\in\mathbb{N}\}$, and the subspace $X\setminus S$ consisting of the set of nonisolated points has no isolated points. If the sequence $(s_k)_{k\in\mathbb{N}}$ is a subset of S which converges to 0, then there is no realizable assignment n such that

$$n(s_k)=\begin{cases}2 & \text{if } k \text{ is even},\\ 4 & \text{if } k \text{ is odd},\end{cases}$$

(see Corollary 4.10). However, if $L=\{s\in S \mid s\leqslant \frac{1}{2}\}$ and $R=\{s\in S \mid s>\frac{1}{2}\}$, then Theorem 4.13 implies that there is a realizable assignment n such that $n(s)=2$ if $s\in L$ and $n(s)=4$ if $s\in R$.

Notice the distinction between the necessary condition in Theorem 4.11 and the sufficient condition in Theorem 4.13(2). Our next example will show that this gap cannot be closed.

Example 4.16. Let X be a convergent decreasing sequence $(a_i)_{i\in\mathbb{N}}$ along with its limit point a as in Example 4.8. Define a labeling on X by $n(U)=\omega$ if U is a finite nonempty clopen subset of X, $n(U)=\mathfrak{c}$, the continuum, if U is an infinite clopen set, and $n(\emptyset)=1$. We claim that this labeled space is realizable as a von Neumann regular ring.

Let B be a transcendence basis for the reals over the rationals. For each $b\in B$, fix a sequence of rationals $\{b_i\}$ that converges to b. Let $Y=X-\{a\}$, and let S be the product of copies of $\mathbb{Q}$ indexed by Y. For each $b\in B$, let $\bar{b}\in S$ be defined by $\bar{b}(a_i)=b_i$. Let R be the (unital) subring generated by $\{\bar{b} \mid b\in B\}\cup\sum_{x\in Y}F_x$, where each $F_x=\mathbb{Q}$. We view elements of R as sequences of rationals. Note that a clopen subset of X is any set which is either finite and does not contain a or cofinite and does contain a. For the purpose of this example, for a finite or cofinite subset U of $X-\{a\}$, let e_U denote the element of S which is 1 at all coordinates $a_i\in U$ and zero elsewhere. Then each such e_U corresponds to a clopen set of X.

Every element $r\in R$ has the form $r=f+g$, where $g\in\sum_{x\in Y}F_x$ and $f=P(\overline{b(1)},\ldots,\overline{b(m)})$ for some integer $m\geqslant 1$ and some polynomial $P(x_1,\ldots,x_m)\in\mathbb{Z}[x_1,\ldots,x_m]$. Furthermore, each element of R converges to a real number. Hence, if such an element r of R had for its zero set an infinite set, it would converge to 0. It would follow that $P(\overline{b(1)},\ldots,\overline{b(m)})=0$. Since B is a transcendence basis, $P(x_1,\ldots,x_m)=0$. Hence, r has cofinite zero set. Therefore, R contains all the elements e_U, U a finite or cofinite subset of $X-\{a\}$, and these are the only idempotents in R. Thus $B(R)$ is isomorphic to the Boolean algebra of clopen sets of X.

Unfortunately, the ring R is not von Neumann regular. However, it is clear that for any $x\in R$, there exists $y\in R$ such that $xy=0$ and $x+y$ is a regular element of R. Hence by [2, Theorem 2.3], $T(R)$ is a von Neumann regular ring. Furthermore, for each $x\in R$, one checks that the set U of coordinates where x is nonzero is either a finite or cofinite subset of Y. Hence the equivalence class $[x]$ (under $x\sim y$ if $\operatorname{ann}(x)=\operatorname{ann}(y)$) contains the idempotent $e_U \in R$. Thus there are no new idempotents in $T(R)$. Also, as we saw in the proof of Theorem 2.2, the equivalence classes $[x]_R \subset R$ and $[x/1]_{T(R)} \subset T(R)$ have the same cardinality. Hence we only have to show that $n(U) = |[e_U]|$ for each clopen set U, and it will follow that (X,n) is realized by the von Neumann regular ring $T(R)$. But this follows as in Theorem 4.13.

We close this section with a couple of questions. Suppose that the compact metric space X is the discrete union of two compact subspaces X_1 and X_2, that is, $X = X_1 \cup X_2$, where $X_1 \cap X_2 = \emptyset$. Let n be a labeling of the clopen subsets of X. Then the restrictions n_1 and n_2 to the clopen subsets of X_1 and X_2 are labelings. It is not hard to show that if (X_1, n_1) and (X_2, n_2) are realizable, then (X,n) is realizable. This fact suggests questions of how the labelings of certain spaces influence the labelings of spaces constructed from them.

For the following question, notice that if X is a compact zero-dimensional metric space that has no isolated points and Y is a subset of X such that Y has an isolated point p, we can define a labeling n of Y by $n(p) = 2$ and $n(U) = \omega$ for each clopen set $U \neq \{p\}$. However, there is no labeling m of X such that $m(U) = n(U \cap Y)$ for all clopen subsets U of X because a labeling of a clopen subset of X assigns an infinite cardinal to each clopen set. However, we do not know the answer to the following question.

Question 4.17. *Suppose that X is a compact zero-dimensional metric space and Y is a closed subset of Y such that Y has no isolated points. Does every labeling of Y extend to a labeling of X? More precisely, if n is a labeling of Y, does there exist a labeling m of X such that $m(U) = n(U \cap Y)$ for each clopen subset U of X?*

We close with a question about a specific kind of labeling of a specific space. Recall that by Example 4.16, it is not necessarily the case that if (X,n) is realizable, then $n(U)$ is the supremum of the cardinals $n(V)$ where V is a uniform subset, of U. However, we do not know the answer to the following.

Question 4.18. *Let X consist of a convergent sequence $(a_k)_{k \in \mathbb{N}}$ along with its limit a. Does there exist a realizable labeling n for X such that $n(a_k)$ is finite for each $k \in \mathbb{N}$, but $n(U) = \mathfrak{c}$ for each infinite clopen subset U of X?*

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