Robin and Neumann Problems for a Class of Singularly Perturbed Semilinear Elliptic Equations

F. A. Howes*

Department of Mathematics, University of California-Davis, Davis, California 95616

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1. INTRODUCTION

We consider in this paper the existence and the asymptotic behavior of solutions of the singularly perturbed problem

\[ \varepsilon \Delta u = \mathcal{F}(x, u, \nabla u), \quad x \text{ in } \Omega, \]

\[ \mu(x) u(x, \varepsilon) + \partial u/\partial n(x, \varepsilon) = \varphi, \quad x \text{ on } \partial \Omega, \]

for three types of functions \( \mathcal{F} \). Here \( \varepsilon \) is a small positive parameter, \( \Omega \subset \mathbb{R}^N \) \((N \geq 2)\) is a bounded region whose boundary \( \partial \Omega \) is a smooth \((N - 1)\) dimensional manifold, and \( x = (x_1, \ldots, x_N) \) is a generic point in \( \mathbb{R}^N \). In addition, \( \Delta = \sum_{i=1}^{N} \partial^2/\partial x_i^2 \) is the \( N \)-dimensional Laplace operator, \( \nabla = (\partial x_1, \ldots, \partial x_N) \) is the \( N \)-dimensional gradient and \( \partial u/\partial n \) is the derivative of \( u \) in the direction of the exterior unit normal at the boundary of \( \Omega \). The function \( \mu \) is assumed to be nonnegative. In order to study how solutions of the problem \((\mathcal{N})\) behave as \( \varepsilon \to 0^+ \) we assume that the region \( \Omega \) is described by a smooth real-valued function \( F \) in the sense that \( D = \{x: F(x) < 0\} \) and \( \partial \Omega = F^{-1}(0) \). We assume further that \( \nabla F(x) \neq 0 \) for \( x \) on \( \partial \Omega \) and then normalize \( F \) by requiring that \( ||\nabla F(x)|| = 1 \) for such \( x \) so that the normal derivative \( \partial u/\partial n \) can be expressed as \( \nabla F \cdot \nabla u \) since the exterior unit normal to the boundary of \( \Omega \) is \( \nabla F(x)||\nabla F(x)|| = \nabla F(x) \). (Here and throughout the paper || \cdot || denotes the usual \( N \)-dimensional Euclidean norm.) Under additional assumptions which involve stability properties of certain solutions \( u = u_\varepsilon(x) \) of the corresponding reduced equation \( \mathcal{F}(x, u, \nabla u) = 0 \) we are able to prove that the problem \((\mathcal{N})\) has a smooth solution \( u = u(x, \varepsilon) \) for each \( \varepsilon > 0 \) sufficiently small such that \( u(x, \varepsilon) = u_\varepsilon(x) + \mathcal{O}(\rho(\varepsilon)) \) in \( \Omega \) where \( \rho(\varepsilon) \to 0^+ \) as \( \varepsilon \to 0^+ \). The essential restriction which must be placed on the function \( \mathcal{F} \) is that \( \mathcal{F}(x, u, z) = \mathcal{O}(||z||^2) \) as \( ||z|| \to \infty \) for \((x, u)\) in compact subsets of \( \Omega \times \mathbb{R} \). Before proceeding to a precise statement of

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our assumptions and theorems we discuss some of the literature on these problems.

Surprisingly enough very little work appears to have been done on Robin and Neumann problems for second-order singularly perturbed elliptic differential equations, even in the linear case. Among the published references we can give are a paper and two abstracts by Oleinik [13-15] and more recent papers by Freidlin [7] and Holland [8, 10]. Oleinik and Holland considered the problem ($\mathcal{M}$) in two dimensions with $\mathcal{F}(x, u, \nabla u) = -A(x, y)u_x - B(x, y)u_y - C(x, y)u + f(x, y)$ for $C(x, y) < 0$. Oleinik's discussion is reminiscent of Levinson's treatment [12] of the corresponding Dirichlet problem for this linear differential equation, while Holland used probabilistic methods related to the ones in [6]. Freidlin also used probabilistic methods to study the $N$-dimensional problem $\epsilon \sum_{i,j=1}^{N} a_{ij}(x) \partial^2 u / \partial x_i \partial x_j + \sum_{j=1}^{N} b_j(x) \partial u / \partial x_j = 0$ in $\Omega$ where $u$ is prescribed on a subset $\Gamma$ of $\partial \Omega$ and $\partial u / \partial n$ is prescribed on $\partial \Omega \setminus \Gamma$. Levinson's paper [12] and those of Vishik and Liusternik [17] and Eckhaus and de Jager [3] treat in considerable detail many interesting questions regarding Dirichlet problems for linear elliptic differential equations, especially in two space dimensions. The reader is advised to consult these papers for further references to the mathematical and scientific literature. A corresponding theory for nonlinear Dirichlet problems is essentially nonexistent; however, some results have been given by Fife [4], Fleming [6], Fife and Tang [5], van Harten [16], Holland [9] and the author [11]. The present paper is in the spirit of [11].

2. The Problem ($\mathcal{N}_1$)

In this section we consider the problem

$$\epsilon \Delta u = h(x, u), \quad x \in \Omega,$$

$$\mu(x) u(x, \epsilon) + \nabla F(x) \cdot \nabla u(x, \epsilon) = \varphi(x), \quad x \in \partial \Omega, \quad (\mathcal{N}_1)$$

where $\Omega = \{x: F(x) < 0\}$ and $\partial \Omega = F^{-1}(0)$. The function $F: \mathbb{R}^N \to \mathbb{R}$ is assumed to be of class $C^{2,\alpha}(\mathbb{R}^N)$ ($0 < \alpha < 1$) with $\|\nabla F(x)\| = 1$ for $x$ on $\partial \Omega$ and $\max_{\partial \Omega} \|\nabla F(x)\|^2 = K$. The function $\mu = \mu(x)$ is assumed to be nonnegative and of class $C^{2,\alpha}(\partial \Omega)$, and the boundary data $\varphi = \varphi(x)$ is also assumed to be of class $C^{2,\alpha}(\partial \Omega)$.

It is natural to associate with the problem ($\mathcal{N}_1$) the corresponding reduced equation

$$h(x, u) = 0, \quad x \in \Omega, \quad (\mathcal{R}_1)$$

and to use various solutions of ($\mathcal{R}_1$) to describe the asymptotic behavior of solutions of ($\mathcal{N}_1$) for small values of $\epsilon > 0$. Before we can define the types of solutions
of \((\mathcal{R}_1)\) that will enable us to do this we make several preliminary smoothness assumptions.

Assume then that \((\mathcal{R}_1)\) has a solution \(u = u_0(x)\) of class \(C^{12,\alpha}(\Omega)\) and define the domain \(\mathcal{D}(u_0) \subset \Omega \times \mathbb{R}\) by

\[
\mathcal{D}(u_0) = \Omega \times \{ u : |u - u_0(x)| \leq \delta \}
\]

for a small positive constant \(\delta\). Assume also that the function \(h(\cdot, u)\) is of class \(C^{10,\alpha}(\Omega)\) for each \(u\) in \(\mathcal{D}(u_0)\), the function \(h(x, \cdot)\) is of class \(C^{11}(\{ u : |u - u_0(x)| \leq \delta \})\) for each \(x\) in \(\Omega\), and in the following definitions that \(h\) possesses the stated number of continuous derivatives with respect to \(u\).

**Definition 2.1.** A solution \(u = u_0(x)\) of the reduced equation \((\mathcal{R}_1)\) is said to be \((I_q)\)-stable \((q \geq 0)\) if there exists a positive constant \(m\) such that

\[
e^\iota h(x, u_0(x)) = 0 \quad \text{for} \quad 0 \leq i \leq 2q \quad \text{and} \quad x \in \Omega,
\]

and

\[
e^\iota h(x, u) \geq m > 0 \quad \text{for} \quad (x, u) \in \mathcal{D}(u_0).
\]

**Definition 2.2.** A solution \(u = u_0(x)\) of the reduced equation \((\mathcal{R}_1)\) is said to be \((II_n)\)-stable \((n \geq 2)\) if there exists a positive constant \(m\) such that

\[
e^\iota h(x, u_0(x)) \geq 0 \quad \text{for} \quad 1 \leq i \leq n - 1 \quad \text{and} \quad x \in \Omega,
\]

and

\[
e^\iota h(x, u) \geq m > 0 \quad \text{for} \quad (x, u) \in \mathcal{D}(u_0).
\]

**Definition 2.3.** A solution \(u = u_0(x)\) of the reduced equation \((\mathcal{R}_1)\) is said to be \((III_n)\)-stable \((n \geq 2)\) if there exists a positive constant \(m\) such that

\[
e^\iota h(x, u_0(x)) \geq 0, \quad \text{and} \quad \iota h(x, u_0(x)) \leq 0
\]

for \(1 \leq i_0, i_\iota \leq n - 1\) and \(x \in \Omega\), \((\text{Here } i_0(i_\iota) \text{ denotes an odd (even) integer})\) and

\[
e^\iota h(x, u) \geq m > 0 \quad \text{for} \quad (x, u) \in \mathcal{D}(u_0) \text{ (if } n \text{ is odd)},
\]

\[
e^\iota h(x, u) \leq -m < 0 \quad \text{for} \quad (x, u) \in \mathcal{D}(u_0) \text{ (if } n \text{ is even)}.
\]

We note that the assumption of \((I_q)\)-, \((II_n)\)- or \((III_n)\)-stability constitutes an extension to nonlinear functions of the usual maximum principle assumption that the coefficient of \(u\) is positive in \(\Omega\) (cf. Section 1).

Using these definitions we can now discuss the existence and the asymptotic behavior of solutions of the problem \((\mathcal{A}_1)\).

**Theorem 2.1.** Assume that the reduced equation \((\mathcal{R}_1)\) has an \((I_q)\)-stable solution \(u = u_0(x)\) of class \(C^{12,\alpha}(\Omega)\). Then there exists an \(\varepsilon_0 > 0\) such that the
problem \((\mathcal{M}_1)\) has a solution \(u = u(x, \varepsilon)\) of class \(C^{(2, \alpha)}(\Omega)\) whenever \(0 < \varepsilon \leq \varepsilon_0\).

In addition, for \(x\) in \(\Omega\) we have that

\[
u(x, \varepsilon) = u_0(x) + \mathcal{O}(v(x, \varepsilon)) + \mathcal{O}(\varepsilon^{1/(2\alpha + 1)}),
\]

where

\[
u(x, \varepsilon) = \nu(\varepsilon m_1^{-1/2}) \exp[(\varepsilon m_1^{-1})^{-1/2}F(x)] \quad \text{if} \quad q = 0
\]

and

\[
u(x, \varepsilon) = \nu q m_2^{-1/2} \varepsilon^{1/(2\alpha + 2)}(1 - m_2 \varepsilon^{-1/(2\alpha + 2)}F(x))^{-q-1} \quad \text{if} \quad q \geq 1.
\]

Here \(\nu = \max_{\partial \Omega} |\varphi(x) - \mu(x) u_0(x) - \nabla F(x) \cdot \nabla u_0(x)|\), \(m_1\) is a positive constant such that \(Km_1 < m\) for \(K = \max_{\Omega} \|\nabla F(x)\|^2\), and \(m_2\) is a positive constant such that

\[Km_2^2 < m q^q 2^{q/((q + 1)(2q + 1))}.
\]

**Proof.** The theorems of this paper are proven by using a recent result of Amann [1] which in the context of the general problem \((\mathcal{M})\) can be stated as follows. Suppose that for \(0 < \varepsilon \leq \varepsilon_0\) there are functions \(\omega = \omega(x, \varepsilon)\) and \(\bar{\omega} = \bar{\omega}(x, \varepsilon)\) of class \(C^{(2, \alpha)}(\Omega)\) such that \(\omega < \bar{\omega}, \mu(x) \omega(x, \varepsilon) + \nabla F(x)\) for \(x\) on \(\partial \Omega\), and

\[
\mu(x) \omega(x, \varepsilon) + \nabla F(x) \cdot \nabla \omega(x, \varepsilon) = \varphi(x)
\]

and for \(x\) in \(\Omega\) \(\omega(x, \varepsilon) \leq \bar{\omega}(x, \varepsilon)\). The smoothness requirement on \(\mathcal{F}\) is simply that \(\mathcal{F}(\cdot, u, z)\) is of class \(C^{(10, \alpha)}(\Omega)\), \(\mathcal{F}(x, \cdot, z)\) is of class \(C^{(1)}\), and \(\mathcal{F}(x, u, \cdot)\) is of class \(C^{(1)}\).

We turn now to the proof of Theorem 2.1. Define for \(\varepsilon > 0\) and \(x\) in \(\Omega\)

\[
\omega(x, \varepsilon) = u_0(x) - \varphi(x) - \Gamma(\varepsilon)
\]

and

\[
\bar{\omega}(x, \varepsilon) = u_0(x) + \varphi(x) + \Gamma(\varepsilon),
\]

where \(\Gamma(\varepsilon) = (\varepsilon m_1^{-1})^{1/(2\alpha + 1)}\) for \(\gamma\) a positive constant to be determined below. It is clear that for \(x\) on \(\partial \Omega\) \((= F^{-1}(0))\)

\[
\mu(x) \omega(x, \varepsilon) + \nabla F(x) \cdot \nabla \omega(x, \varepsilon) \leq \varphi(x) \leq \mu(x) \bar{\omega}(x, \varepsilon) + \nabla F(x) \cdot \nabla \bar{\omega}(x, \varepsilon).
\]
(Recall that $\|\nabla F(x)\|^2 = 1$ for such $x$.) We only verify that $\epsilon \Delta \omega \geq h(x, \omega)$ for $x$ in $\Omega$ since the validity of the opposite inequality for $\bar{\omega}$ then follows by symmetry. Differentiating and substituting we have that

$$\epsilon \Delta \omega - h(x, \omega) = \epsilon \Delta u_0 - \epsilon \Delta v - h(x, u_0)$$

$$- \sum_{i=1}^{2q} \frac{1}{i!} \partial_i h(x, u_0)(\omega - u_0)^i$$

$$- \frac{1}{(2q + 1)!} \partial^2 h(x, \xi)(\omega - u_0)^{2q+1},$$

where $(x, \xi)$ in $\mathcal{O}(u_0)$ is the appropriate intermediate point. Using the $(I_\tau)$-stability of $u_0$ we continue with the inequality

$$\epsilon \Delta \omega - h(x, \omega) \geq -\epsilon M - \epsilon \Delta v$$

$$+ \frac{m}{(2q + 1)!} \epsilon^{2q+1} + \frac{\epsilon \gamma}{(2q + 1)!}$$

$$\geq 0$$

if $\epsilon$ is sufficiently small (say, $0 < \epsilon \ll \epsilon_0$) and $\gamma \geq (2q + 1)! M$. Here $M = \max_{\Omega} |\Delta u_0(x)|$ and we have used the fact that for $\epsilon$ sufficiently small $\epsilon \Delta v < (m/(2q + 1))\epsilon^{2q+1}$ in $\Omega$ if $m_1$ and $m_2$ are defined as above.

It only remains to construct a function $\bar{u} = \bar{u}(x, \epsilon)$ of class $C^{(2,q)}(\Omega)$ such that $\omega(x, \epsilon) \leq \bar{u}(x, \epsilon) \leq \bar{\omega}(x, \epsilon)$ in $\mathcal{O}$ and $\mu(x) \bar{u}(x, \epsilon) + \nabla F(x) \cdot \nabla \bar{u}(x, \epsilon) = \varphi(x)$ on $\partial \Omega$. Define for $0 < \epsilon \leq \epsilon_0$ and $x$ in $\mathcal{O}$

$$\bar{u}(x, \epsilon) = u_0(x) + \tau(\epsilon) g(x)(\exp[\tau^{-1}(\epsilon)F(x)] - 1)$$

where $\tau(\epsilon) > 0$ is a transcendentally small term (that is, $\tau(\epsilon) = O(\epsilon^k)$ for all $k \geq 1$) and $g(x) = \tilde{\varphi}(x) - \tilde{\mu}(x) u_0(x) - \nabla F(x) \cdot \nabla u_0(x)$ for $C^{(2,q)}$-extensions $\tilde{\varphi}, \tilde{\mu}$ of $\varphi, \mu$, respectively, to $\tilde{\Omega}$ (that is, $\tilde{\varphi} |_{\partial \Omega} = \varphi$ and $\tilde{\mu} |_{\partial \Omega} = \mu$). Then clearly $\omega(x, \epsilon) \leq \bar{u}(x, \epsilon) \leq \bar{\omega}(x, \epsilon)$ in $\bar{\Omega}$ and for $x$ on $\partial \Omega$

$$\mu(x) \bar{u}(x, \epsilon) + \nabla F(x) \cdot \nabla \bar{u}(x, \epsilon) = \mu(x) u_0(x) + \nabla F(x) \cdot \nabla u_0(x) + g(x) |_{\partial \Omega} = \varphi(x).$$

Thus all of the hypotheses of Amann's theorem are satisfied and the conclusion of Theorem 2.1 follows.

**Theorem 2.2.** Assume that the reduced equation $(\mathcal{R}_1)$ has a subharmonic $(II_u)$-stable solution $u = u_0(x)$ of class $C^{(2,q)}(\Omega)$ such that $\mu(x) u_0(x) + \nabla F(x) \cdot \nabla u_0(x) \leq \varphi(x)$ for $x$ on $\partial \Omega$. Then there exists an $\epsilon_0 > 0$ such that the problem $(\mathcal{A}_1)$ has a
solution \( u = u(x, \epsilon) \) of class \( C^{(2, \alpha)}(\Omega) \) whenever \( 0 < \epsilon \leq \epsilon_0 \). In addition, for \( x \) in \( \overline{\Omega} \) we have that

\[
u_0(x) \leq u(x, \epsilon) \leq u_0(x) + v(x, \epsilon) + ce^{1/n},
\]

where

\[
v(x, \epsilon) = n(n - 1) m_3^{1/(n+1)}(1 - m_3 \epsilon^{1/(n+1)} F(x))^{-2/(n-1)}
\]

and \( c \) is a known positive constant depending on \( u_0 \). Here \( \nu = \max_{\partial \Omega} \left| \varphi(x) - \mu(x) u_0(x) - \nabla F(x) \cdot \nabla u_0(x) \right| \) and \( m_3 \) is a positive constant such that \( K m_3^2 < m(n - 1)^{n-1}/(n + 1)! \) for \( K = \max_{\Omega} ||F(x)||^2 \).

**Proof.** Define for \( \epsilon > 0 \) and \( x \) in \( \Omega \)

\[
\omega(x, \epsilon) := u_0(x)
\]

and

\[
\bar{\omega}(x, \epsilon) = u_0(x) + w(x, \epsilon) + (\gamma m^{-1})^{1/n}
\]

where \( \gamma \) is a positive constant to be determined below. Clearly \( \mu(x) \omega(x, \epsilon) + \nabla F(x) \cdot \nabla \omega(x, \epsilon) \leq \varphi(x) \leq \mu(x) \bar{\omega}(x, \epsilon) + \nabla F(x) \cdot \nabla \bar{\omega}(x, \epsilon) \) for \( x \) on \( \partial \Omega \) and \( \epsilon \Delta \omega \geq h(x, \omega) \) for \( x \) in \( \Omega \) since \( u_0 \) is subharmonic (that is, \( \Delta u_0 \geq 0 \)). It is just as easy to see that \( \epsilon \Delta \bar{\omega} \leq h(x, \bar{\omega}) \) since

\[
h(x, \bar{\omega}) - \epsilon \Delta \bar{\omega} = h(x, u_0) + \sum_{i=1}^{n-1} \frac{1}{i!} \partial_{u_i} h(x, u_0)(\bar{\omega} - u_0)^i
\]

\[
+ \frac{1}{n!} \partial_{u_n} h(x, \xi)(\bar{\omega} - u_0)^n - \epsilon \Delta u_0 - \epsilon \Delta w
\]

for \( (x, \xi) \) in \( \partial\Omega(u_0) \). By the \( (\Pi_n) \)-stability of \( u_0 \) we then have the inequality

\[
h(x, \bar{\omega}) - \epsilon \Delta \bar{\omega} \geq \frac{m}{n!} \omega^n + \frac{\epsilon \gamma}{n!} - \epsilon M - \epsilon \Delta w
\]

\[
\geq 0 \text{ if } \epsilon \text{ is sufficiently small}
\]

(e.g., \( 0 < \epsilon \leq \epsilon_0 \)) and \( \gamma \geq n! M \) since \( \epsilon \Delta w \leq (m/n!) \omega^n \) in \( \Omega \) for such \( \epsilon \) if \( m_3 \) is as defined above. Here \( M = \max_{\partial \Omega} |\Delta u_0(x)| \).

The proof will be concluded if we can construct a function \( \tilde{u} = \tilde{u}(x, \epsilon) \) of class \( C^{(\alpha, \alpha)}(\overline{\Omega}) \) lying between \( \omega \) and \( \bar{\omega} \) and satisfying the boundary condition. Define for \( 0 < \epsilon \leq \epsilon_0 \) and \( x \) in \( \overline{\Omega} \)

\[
\tilde{u}(x, \epsilon) = u_0(x) + \tau(\epsilon) g(x, \epsilon)(\exp[\tau^{-1}(\epsilon) F(x)] - 1) + \tau(\epsilon)L
\]
where \( \tau(\epsilon) > 0 \) is transcendentally small, \( L = \max_{\Omega} |\tilde{\varphi}(x) - \tilde{\mu}(x) u_0(x) - \nabla F(x) \cdot \nabla u_0(x)| \) for \( C^{(2,\alpha)} \) - extensions \( \tilde{\varphi}, \tilde{\mu} \) of \( \varphi, \mu \), respectively, to \( \tilde{\Omega} \), and

\[
g(x, \epsilon) = \varphi(x) - \mu(x) u_0(x) - \nabla F(x) \cdot \nabla u_0(x) - \mu(x) \tau(\epsilon) L.
\]

Then \( o < \tilde{u} \leq \tilde{m} \) in \( \tilde{\Omega} \) and for \( x \) on \( \partial \tilde{\Omega} \)

\[
\mu(x) \tilde{u}(x, \epsilon) + \nabla F(x) \cdot \nabla \tilde{u}(x, \epsilon) = \mu(x) u_0(x) + \mu(x) \tau(\epsilon) L + \nabla F(x) \cdot \nabla u_0(x) + g(x, \epsilon) |\tilde{\Omega}|
\]

Thus Amann's theorem applies and the conclusion of Theorem 2.2 follows.

**Theorem 2.3.** Assume that the reduced equation \((E_1)\) has a superharmonic \((\mathcal{I}_n)\)-stable solution \( u = u_0(x) \) of class \( C^{(2,\alpha)}(\Omega) \) such that \( \mu(x) u_0(x) + \nabla F(x) \cdot \nabla u_0(x) \geq \varphi(x) \) for \( x \) on \( \partial \Omega \). Then there exists an \( \epsilon_0 > 0 \) such that the problem \((\mathcal{N}_i)\) has a solution \( u = u(x, \epsilon) \) of class \( C^{(2,\alpha)}(\Omega) \) whenever \( 0 < \epsilon \leq \epsilon_0 \).

In addition, for \( x \) in \( \Omega \) we have that

\[
u(x) - w(x, \epsilon) - c\epsilon^{1/n} \leq u(x, \epsilon) \leq u_0(x),
\]

where the function \( w \) and the positive constant \( c \) are defined in the conclusion of Theorem 2.2.

**Proof.** Make the change of variable \( u \rightarrow -u \) and apply Theorem 2.2 to the transformed problem.

We give now some examples of this theory.

**Example 2.1.** Consider first the problem

\[
\epsilon \Delta u = (u - u_0(x))^{2^{2^2-1}}, \quad x \in \Omega,
\]

\[
\mu(x) u(x, \epsilon) + \nabla F(x) \cdot \nabla u(x, \epsilon) = \varphi(x), \quad x \in \partial \Omega,
\]

where \( u_0 \) is of class \( C^{(2,\alpha)}(\Omega) \). Clearly \( u = u_0(x) \) is \((I_0)\)-stable and we conclude from Theorem 2.1 that the problem \((E_1)\) has a solution \( u = u(x, \epsilon) \) such that

\[
\lim_{\epsilon \rightarrow 0^+} u(x, \epsilon) = u_0(x) \quad \text{for} \quad x \in \Omega.
\]

**Example 2.2.** Consider next the problem

\[
\epsilon \Delta u = u^3 - u = h(u), \quad x \in \Omega,
\]

\[
\mu(x) u(x, \epsilon) + \nabla F(x) \cdot \nabla u(x, \epsilon) = \varphi(x), \quad x \in \partial \Omega.
\]

The reduced equation \( h(u) - 0 \) has the three solutions \( u_0 = 0, u_1 = 1 \) and
\( u_2 = -1 \), and since \( h'(u) = 3u^2 - 1 \) it follows that \( u_1 \) and \( u_2 \) are \((I_0)\)-stable. (Simply choose \( \delta < 1/3^{1/2} \) in Definition 2.1.) We conclude from Theorem 2.1 that the problem \((E_2)\) has two solutions \( u = u_1(x, \epsilon) \) and \( u = u_2(x, \epsilon) \) such that

\[
\lim_{\epsilon \to 0^+} u_1(x, \epsilon) = 1 \quad \text{and} \quad \lim_{\epsilon \to 0^+} u_2(x, \epsilon) = -1
\]

for \( x \) in \( \overline{\Omega} \).

**Example 2.3.** Consider finally the problem

\[
\epsilon \Delta u = (u - u_0(x))^2, \quad x \text{ in } \Omega, \quad (E_3)
\]

\[
u(x, \epsilon) + VF(x) \cdot \nabla u(x, \epsilon) = \varphi(x), \quad x \text{ on } \partial \Omega,
\]

where \( u_0 \) is a subharmonic function of class \( C^{(\alpha, \alpha)}(\overline{\Omega}) \) satisfying \( u_0(x) + VF(x) \cdot \nabla u_0(x) \leq \varphi(x) \) for \( x \) on \( \partial \Omega \). Then from Theorem 2.2 we conclude that \((E_3)\) has a solution \( u = u(x, \epsilon) \geq u_0(x) \) such that

\[
\lim_{\epsilon \to 0^+} u(x, \epsilon) = u_0(x) \quad \text{for } x \in \overline{\Omega}.
\]

3. The Problem \((\mathcal{M}_2')\)

We consider now the problem

\[
\epsilon \Delta u = A(x, u) \cdot \nabla u + h(x, u), \quad x \text{ in } \Omega, \quad (\mathcal{M}_2')
\]

\[
u(x) u(x, \epsilon) + VF(x) \cdot \nabla u(x, \epsilon) = \varphi(x), \quad x \text{ on } \partial \Omega,
\]

where \( \Omega = \{ x : F(x) < 0 \} \) and \( \partial \Omega = F^{-1}(0) \) for functions \( h, \mu, F \) and \( \varphi \) with the same properties as in Section 2. The vector-valued function \( A = A(x, u) = (a_1(x, u), \ldots, a_N(x, u)) \) is assumed to be of class \( C^{(\alpha, \alpha)}(\overline{\Omega}) \) for each \( u \) in \( \mathcal{D}(u_0) \) and of class \( C^{(\alpha)}(\{ u : | u - u_0(x) | \leq \delta \}) \) for each \( x \) in \( \overline{\Omega} \). Here of course \( u = u_0(x) \) is a solution of the reduced equation corresponding to \((\mathcal{M}_2')\), namely

\[
A(x, u) \cdot \nabla u + h(x, u) = 0, \quad x \text{ in } \Omega. \quad (\mathcal{R}_2)
\]

As a matter of notational convenience, we define

\[
A_u(x, u) = (a_1, u(x, u), \ldots, a_N, u(x, u)) \quad \text{and} \quad \mathcal{D}_\delta(u_0) = \mathcal{D}(u_0) \cap \{ x : \text{dist}(x, \partial \Omega) < \delta \},
\]

where the domain \( \mathcal{D}(u_0) \) is as defined in Section 2.

To study the behavior of solutions of the problem \((\mathcal{M}_2')\) we single out certain solutions of \((\mathcal{R}_2)\) which are stable in the following sense.
DEFINITION 3.1. A solution \( u = u_0(x) \) of the reduced equation (\( \mathcal{R}_2 \)) is said to be (I,)-, (II,)- or (III,)-stable if the function \( h(x, u) = A(x, u) \cdot \nabla u_0(x) + h(x, u) \) is (I,)-, (II,)- or (III,)-stable in the sense of Definitions 2.1, 2.2 or 2.3, respectively.

Using this definition of stability we can now discuss the analogs of Theorems 2.1-2.3 for the problem (\( \mathcal{A}_2 \)).

**Theorem 3.1.** Assume that the reduced equation (\( \mathcal{R}_2 \)) has an (I,)-stable solution \( u = u_0(x) \) of class \( C^{(2,\alpha)}(\Omega) \) such that \( A(x, u) \cdot \nabla F(x) \geq 0 \) for \( (x, u) \) in \( \mathcal{D}(u_0) \). Then there exists an \( \varepsilon_0 > 0 \) such that the problem (\( \mathcal{A}_2 \)) has a solution \( u = u(x, \varepsilon) \) of class \( C^{(2,\alpha)}(\Omega) \) whenever \( 0 < \varepsilon \leq \varepsilon_0 \). In addition, for \( x \) in \( \Omega \) we have that

\[
 u(x, \varepsilon) = u_0(x) + \mathcal{C}(v(x, \varepsilon)) + \mathcal{O}(\rho(\varepsilon)),
\]

where the function \( v \) is defined in the conclusion of Theorem 2.1 and

\[
 \rho(\varepsilon) = \begin{cases} 
 \varepsilon & \text{if } q = 0, \\
 \varepsilon^{1/(2q+2)(2q+1-1)} & \text{if } q \geq 1.
\end{cases}
\]

**Proof.** The proof of this theorem is only a repetition of the proof of Theorem 2.1 once we observe that for \( \mathcal{F}(x, u, \nabla u) = A(x, u) \cdot \nabla u + h(x, u) \) and \( \omega = \omega \) or \( \bar{\omega} \),

\[
 \mathcal{F}(x, \omega, \nabla \omega) = \mathcal{F}(x, u_0, \nabla u_0) + \{ \mathcal{F}(x, \omega, \nabla u_0) - \mathcal{F}(x, u_0, \nabla u_0) \}
 + \{ \mathcal{F}(x, \omega, \nabla u_0) - \mathcal{F}(x, m, \nabla u_0) \}
 = \sum_{i=1}^{2q} \frac{1}{i!} \partial_u^i h(x, u_0)(\omega - u_0)^i
 + \frac{1}{(2q+1)!} \partial_u^{2q+1} h(x, \xi)(\omega - u_0)^{2q+1}
 + A(x, \omega) \cdot \nabla(\omega - u_0)
\]

with \( (x, \xi) \) in \( \mathcal{D}(u_0) \).

Define \( \omega, \bar{\omega} \) and \( \bar{u} \) as in the proof of Theorem 2.1 with the exception that \( \Gamma(\varepsilon) = (\varepsilon^{\gamma+1})^{1/(2q+2)(2q+1)} \) if \( q \geq 1 \). Then it is only necessary to verify that \( \varepsilon \Delta \omega \geq A(x, \omega) \cdot \nabla \omega + h(x, \omega) \). (The validity of the opposite inequality for \( \bar{\omega} \) follows by symmetry.) By our opening remark we have that

\[
 \varepsilon \Delta \omega - A(x, \omega) \cdot \nabla \omega - h(x, \omega)
 = \varepsilon \Delta u_0 - \varepsilon \Delta v + A(x, \omega) \cdot \nabla v
 + \frac{1}{(2q+1)!} \partial_u^{2q+1} h(x, \xi)(\varepsilon + \Gamma(\varepsilon))^{2q+1}
 \geq -\varepsilon M - \varepsilon A \cdot v + A(x, \omega) \cdot \nabla v + \frac{m}{(2q+1)!} v^{2q+1}
 + \frac{\varepsilon^{\gamma}}{(2q+1)!}.
\]
Here $M = \max_{\partial \Omega} |\Delta u_0(x)|$ and

$$
\sigma = \begin{cases} 
1 & \text{if } q = 0, \\
(2q + 2)^{-1} & \text{if } q \geq 1.
\end{cases}
$$

Now by assumption $A(x, \omega) \cdot \nabla \varphi(x, \epsilon) \geq 0$ for $(x, \omega)$ in $\mathcal{D}_\delta(u_0)$ and so for $x$ in $\Omega$ such that $\text{dist}(x, \partial \Omega) < \delta$ we have the desired inequality $\epsilon A_\omega - A(x, \omega) \cdot \nabla \omega - h(x, \omega) \geq 0$ if $\gamma \geq (2q + 1)! M$ and $\epsilon$ is sufficiently small (say, $0 < \epsilon \leq \epsilon_0$) since for such $\epsilon \epsilon A_\omega < (m/(2q + 1))^{\sigma q+1}$ in $\Omega$. Finally for $x$ in $\Omega$ such that $\text{dist}(x, \partial \Omega) \geq \delta$

$$
A(x, \omega) \cdot \nabla \varphi(x, \epsilon) = O(\rho_1(\epsilon)) \text{ where } \rho_1(\epsilon) \text{ is transcendentally small}
$$

if $q = 0$ and $\rho_1(\epsilon) = O((l+q)\cdot(l+q))$ if $q \geq 1$. Since $\rho_1(\epsilon) = o(\epsilon)$

we also have that for such $x \epsilon A_\omega - A(x, \omega) \cdot \nabla \omega - h(x, \omega) \geq 0$

for $0 < \epsilon \leq \epsilon_0$.

The conclusion of the theorem now follows from Amann's theorem.

It is now an easy matter to prove the analog of Theorem 2.2 and so we only state the result as Theorem 3.2.

**Theorem 3.2.** Assume that the reduced equation $(\mathcal{R}_3)$ has a subharmonic $(\mathcal{H}_3)$-stable solution $u = u_0(x)$ of class $C^{(\omega, \alpha)}(\Omega)$ such that $\mu(x)u_0(x) + \nabla F(x) \cdot u_0(x) \leq \varphi(x)$ for $x$ on $\partial \Omega$ and $A(x, \omega) \cdot \nabla F(x) \geq 0$ for $(x, \omega)$ in $\mathcal{D}_\delta(u_0)$. Then there exists an $\epsilon_0 > 0$ such that the problem $(\mathcal{N}_3)$ has a solution $u = u(x, \epsilon)$ of class $C^{(\omega, \alpha)}(\Omega)$ whenever $0 < \epsilon \leq \epsilon_0$. In addition, for $x$ in $\Omega$ we have that

$$
u_0(x) \leq u(x, \epsilon) \leq u_0(x) + w(x, \epsilon) + \epsilon \tilde{p}(\epsilon),$$

where the function $w$ and the positive constant $c$ are defined in the conclusion of Theorem 2.2, and $\tilde{p}(\epsilon) = \epsilon^{1/n^2}$.

If the function $u_0$ is superharmonic, $(\mathcal{H}_3)$-stable and satisfies $\mu(x)u_0(x) + \nabla F(x) \cdot \nabla u_0(x) \geq \varphi(x)$ for $x$ on $\partial \Omega$ and $A(x, \omega) \cdot \nabla F(x) \geq 0$ for $(x, \omega)$ in $\mathcal{D}_\delta(u_0)$ then the result for the problem $(\mathcal{N}_3)$ corresponding to Theorem 2.3 is clearly valid. We leave its precise formulation to the reader.

Before giving some examples of the theory of this section we make a few remarks concerning the "boundary inequality" $A(x, \omega) \cdot \nabla F(x) \geq 0$ for $(x, \omega)$ in $\mathcal{D}_\delta(u_0)$. It can be viewed geometrically as the requirement that the characteristic curves of the first-order differential equation $(\mathcal{R}_3)$ must be outgoing everywhere along the boundary of $\Omega$. (The degenerate case in which $A(x, \omega) \cdot \nabla F(x) \equiv 0$ implies of course that $\partial \Omega$ is itself a characteristic curve.) In light of this interpretation the conclusions of the theorems of this section are not surprising since solutions of $(\mathcal{R}_3)$ with outgoing characteristics reach the boundary of $\Omega$ with
predetermined values which in general are different from the boundary conditions imposed by \( \mathcal{M}_2 \). If such a function is to approximate a solution of \( \mathcal{M}_2 \) in \( \Omega \) then it must be supplemented by a boundary layer corrector term (that is, a function of the form \( v \) or \( w \)) near \( \partial \Omega \).

The layer terms \( v, w \) and the error terms \( \rho(\epsilon), \tilde{\rho}(\epsilon) \) in the conclusions of Theorems 3.1 and 3.2, respectively, can be sharpened under additional assumptions on the functions \( A \) and \( F \). First of all, if the boundary inequality is satisfied in the strong sense that \( A(x, u) \cdot VF(x) \geq k > 0 \) in \( \mathcal{D}(u_0) \) for a positive constant \( k \) then the terms \( v \) and \( w \) can be replaced by \( \chi(x, \epsilon) = \nu k_1^{-1} \exp[(\nu k_1^{-1})^{-1}F(x)] \) where \( k_1 < k \) is a positive constant. This follows because for \( \epsilon \) sufficiently small \( \epsilon A\chi(x, \epsilon) < k \nabla \chi(x, \epsilon) \cdot \nabla F(x) \) for \( x \) in \( \Omega \) such that \( \text{dist}(x, \partial \Omega) < \delta \) and \( |\epsilon A\chi(x, \epsilon) - k \nabla \chi(x, \epsilon) \cdot \nabla F(x)| \) is transcendentally small for \( x \) in \( \Omega \) such that \( \text{dist}(x, \partial \Omega) > \delta \). In addition, the term \( \rho(\epsilon) \) \( (\rho(\epsilon)) \) can be replaced by \( \rho(\epsilon) = \epsilon^{1/(2q+1)}(\rho(\epsilon) - \epsilon^{1/n}) \). Secondly if \( A(x, \omega(x, \epsilon)) \cdot \nabla \omega(x, \epsilon) = \mathcal{O}(\epsilon^{1/(2q+1)}) \) for \( \omega - \omega \) or \( \omega \) as in the proof of Theorem 2.1 and \( x \) in \( \Omega \) then the conclusion of Theorem 3.1 is valid with \( \rho(\epsilon) = \epsilon^{1/(2q+1)} \). And if \( A(x, \omega(x, \epsilon)) \cdot \nabla \omega(x, \epsilon) = \mathcal{O}(\epsilon^{1/n}) \) for \( \omega \) as in the proof of Theorem 2.2 and \( x \) in \( \Omega \) then the conclusion of Theorem 3.2 is valid with \( \rho(\epsilon) = \epsilon^{1/n} \).

We conclude this section with some examples.

**Example 3.1.** Consider first the problem

\[
\epsilon A u = x \cdot \nabla u + u^{2q+1}, \quad x \text{ in } \Omega,
\]

\[
\mu(x) u(x, \epsilon) + \nabla F(x) \cdot \nabla u(x, \epsilon) = \varphi(x), \quad x \text{ on } \partial \Omega,
\]

(\( E_4 \))

where \( \Omega \) is the unit ball in \( \mathbb{R}^N \) centered at 0, that is, \( \Omega = \{x: F(x) < 0\} \) for \( F(x) = \frac{1}{2}(\|x\|^2 - 1) \). Clearly \( u = u_0(x) \equiv 0 \) is an \( (I_q) \)-stable solution of the reduced equation and in order to apply Theorem 3.1 we only have to verify that the boundary inequality \( x \cdot \nabla F(x) \geq 0 \) holds for \( x \) in \( \Omega \) such that \( \text{dist}(x, \partial \Omega) < \delta \). However \( \nabla F(x) = x \) and in fact, \( x \cdot \nabla F(x) \geq 1 - \delta \) for such \( x \). We conclude that the problem \( (E_4) \) has a solution \( u = u(x, \epsilon) \) such that

\[
\lim_{\epsilon \to 0^+} u(x, \epsilon) = 0 \quad \text{for } x \text{ in } \overline{\Omega}.
\]

**Example 3.2.** Consider next the problems

\[
\epsilon A u = \pm u^2 x \cdot \nabla u + u - u^3, \quad x \text{ in } \Omega,
\]

\[
\mu(x) u(x, \epsilon) + \nabla F(x) \cdot \nabla u(x, \epsilon) = \varphi(x), \quad x \text{ on } \partial \Omega,
\]

(\( E_{5 \pm} \))

where again \( \partial \Omega = F^{-1}(0) \) for \( F(x) = \frac{1}{2}(\|x\|^2 - 1) \). The corresponding reduced equations have the constant solutions \( u_0 \equiv 0, u_1 \equiv 1 \) and \( u_2 \equiv -1 \) and it is easy to see that of these only \( u_0 \) is stable; in fact, it is \( (I_0) \)-stable (cf. Example 2.2)
for both \((E_5^+)\) and \((E_5^-)\). In the case of \((E_5^+)\) the existence of a solution \(u = u(x, \epsilon)\) such that
\[
\lim_{\epsilon \to 0^+} u(x, \epsilon) = 0 \quad \text{for } x \in \Omega
\]
follows immediately from Theorem 3.1 because \(A(x, u) \cdot \nabla F(x) = u^2(x \cdot x) \geq 0\) for all \(x\). However, in the case of \((E_5^-)\) we can reach the same conclusion by noting that
\[
A(x, u) \cdot \nabla F(x) = -u^2(x \cdot x) = O(\epsilon \exp[2\epsilon^{-1/2}F(x)] + \epsilon^2)
\]
for \(u = O(\epsilon^{1/2} \exp[\epsilon^{-1/2}F(x)] + \epsilon)\) and referring to our remarks preceding the examples.

4. The Problem \((\mathcal{N}_3)\)

We consider finally the problem
\[
\epsilon \Delta u = \sum_{j=1}^{N} b_j(x, u) u^2_j + A(x, u) \cdot \nabla u + h(x, u), \quad x \in \Omega,
\]
\[
\mu(x) u(x, \epsilon) + \nabla F(x) \cdot \nabla u(x, \epsilon) = \varphi(x), \quad x \in \partial \Omega,
\]
where as usual \(\Omega = \{x: F(x) < 0\}\) and \(\partial \Omega = F^{-1}(0)\). The functions \(A, h, \mu, F\) and \(\varphi\) have the same properties as in Sections 2 and 3, and for \(j = 1, \ldots, N\) the functions \(b_j = b_j(x, u)\) are assumed to be as smooth as the functions \(a_j = a_j(x, u)\).

We associate with \((\mathcal{N}_3)\) the corresponding reduced equation
\[
\sum_{j=1}^{N} b_j(x, u) u^2_j + A(x, u) \cdot \nabla u + h(x, u) = 0, \quad x \in \Omega, \quad (\mathcal{R}_3)
\]
and study the behavior of solutions of \((\mathcal{N}_3)\) by means of solutions of \((\mathcal{R}_3)\) which are stable in the following sense.

**Definition 4.1.** A solution \(u = u_0(x)\) of the reduced equation \((\mathcal{R}_3)\) is said to be \((I)\), \((II)\), or \((III)\)-stable if the function \(\bar{h}(x, u) = \sum_{j=1}^{N} b_j(x, u) u^2_j(x) + A(x, u) \cdot \nabla u_0(x) + h(x, u)\) is \((I)\), \((II)\)- or \((III)\)-stable in the sense of Definitions 2.1, 2.2 or 2.3, respectively.

Using this definition we can now study the analogs of Theorems 2.1–2.3 for the problem \((\mathcal{N}_3)\).

**Theorem 4.1.** Assume that the reduced equation \((\mathcal{R}_3)\) has an \((I)\)-stable solution \(u = u_0(x)\) of class \(C^{[2,0]}(\bar{\Omega})\) such that
\[
|\varphi(x) - \mu(x) u_0(x) - \nabla F(x) \cdot \nabla u_0(x)| \leq \epsilon \Omega \sum_{j=1}^{N} b_j(x, u) F^2_j(x) \geq 0
\]
and

$$2 \sum_{j=1}^{N} b_j(x, u) u_0 \varepsilon^j(x) F_{j,x}(x) + A(x, u) \cdot \nabla F(x) \geq 0$$

for \( (x, u) \) in \( \mathcal{D}(u_0) = \mathcal{D}(u_0) \cap \{ x : \text{dist}(x, \partial \Omega) < \delta \} \). Then there exists an \( \epsilon_0 > 0 \) such that the problem \((M_3)\) has a solution \( u = u(x, \epsilon) \) of class \( C^{(2, \alpha)}(\Omega) \) whenever \( 0 < \epsilon \leq \epsilon_0 \). In addition, for \( x \) in \( \Omega \) we have that

$$u_0(x) - c_1 \epsilon^{1/(2q+1)} \leq u(x, \epsilon) \leq u_0(x) + v(x, \epsilon) + c_2 \rho(\epsilon) \text{ if }$$

$$\mu(x) u_0(x) + \nabla F(x) \cdot \nabla u_0(x) \leq \phi(x) \text{ and }$$

$$u_0(x) - v(x, \epsilon) - c_2 \rho(\epsilon) \leq u(x, \epsilon) \leq u_0(x) + c_1 \epsilon^{1/(2q+1)} \text{ if }$$

$$\mu(x) u_0(x) + \nabla F(x) \cdot \nabla u_0(x) \geq \phi(x),$$

where the function \( v \) is defined in the conclusion of Theorem 2.1,

$$\rho(\epsilon) = \begin{cases} \epsilon & \text{if } q = 0, \\ \epsilon^{1/(2q+2)(2q-1)} & \text{if } q \geq 1, \end{cases}$$

and \( c_1, c_2 \) are known positive constants depending on \( u_0 \).

**Proof.** Suppose for definiteness that \( \mu(x) u_0(x) + \nabla F(x) \cdot \nabla u_0(x) \leq \phi(x) \) for \( x \) on \( \partial \Omega \).

Define for \( \epsilon > 0 \) and \( x \) in \( \Omega \)

$$\omega(x, \epsilon) = u_0(x) - (\epsilon m^{-1})^{1/(2q+1)}$$

and

$$\tilde{\omega}(x, \epsilon) = u_0 + v(x, \epsilon) + (\epsilon m^{-1})^{1/(2q+1)},$$

where

$$\sigma = \begin{cases} 1 & \text{if } q = 0, \\ 1/(2q+2) & \text{if } q \geq 1, \end{cases}$$

and \( \gamma \) is a positive constant to be determined below. Clearly \( \omega \) and \( \tilde{\omega} \) satisfy the correct boundary inequalities. (Recall that \( \mu \geq 0 \).) In order to verify that \( \epsilon \Delta \omega \geq \mathcal{F}(x, \omega, \nabla \omega) \) and \( \epsilon \Delta \tilde{\omega} \leq \mathcal{F}(x, \tilde{\omega}, \nabla \tilde{\omega}) \) in \( \Omega \) for

$$\mathcal{F}(x, u, \nabla u) = \sum_{j=1}^{N} b_j(x, u) u_0 \varepsilon^j(x) F_{j,x}(x) + A(x, u) \cdot \nabla u + h(x, u)$$

we note that for \( \omega = \omega \text{ or } \tilde{\omega} \)

$$\mathcal{F}(x, \omega, \nabla \omega) = \mathcal{F}(x, u_0, \nabla u_0) + \{ \mathcal{F}(x, \omega, \nabla u) - \mathcal{F}(x, u_0, \nabla u_0) \}$$

$$+ \{ \mathcal{F}(x, \omega, \nabla \omega) - \mathcal{F}(x, \omega, \nabla u_0) \}$$
\[ = \sum_{i=1}^{2q} \frac{1}{i!} \partial_u^i h(x, u_0)(\omega - u_0)^i \]
\[ + \frac{1}{(2q + 1)!} \partial_u^{2q+1} h(x, \xi)(\omega - u_0)^{2q+1} \]
\[ + \sum_{j=1}^N b_j(x, \omega)(\omega - u_0)^2 \]
\[ + 2 \sum_{i=1}^N b_j(x, \omega) u_{0,x_i}(\omega - u_0)x_i \]
\[ + A(x, \omega) \cdot \nabla(\omega - u_0) \]

with \((x, \xi) \in \mathcal{D}(u_0)\).

We first have that
\[ (2q + 1)! \partial_u^q h(x, u_0)(\omega - u_0)^q \] 
\[ - \frac{1}{(2q + 1)!} \partial_u^{2q+1} h(x, \xi)(\omega - u_0)^{2q+1} \]
\[ \geq -\epsilon M + \frac{\epsilon \gamma}{(2q + 1)!} \]
\[ \geq 0 \quad \text{if} \quad \gamma \geq (2q + 1) !M \]

for \(M = \max x | \Delta u_0(x) | \). On the other hand we have that
\[ \mathcal{F}(x, \omega, \nabla \omega) - \epsilon \Delta \omega = \frac{1}{(2q + 1)!} \partial_u^{2q+1} h(x, \xi)(\omega - u_0)^{2q+1} \]
\[ + \sum_{j=1}^N b_j(x, \omega) \nabla \omega_{x_j} \]
\[ + 2 \sum_{j=1}^N b_j(x, \omega) u_{0,x_j} \nabla \omega_{x_j} \]
\[ + A(x, \omega) \cdot \nabla \omega \]
\[ - \epsilon \Delta u_0 - \epsilon \Delta \omega \]
\[ \geq \frac{m}{(2q + 1)!} \omega_{x_j}^{2q+1} + \frac{\epsilon \gamma}{(2q + 1)!} \]
\[ + \sum_{j=1}^N b_j(x, \omega) \omega_{x_j}^{2q+1} + 2 \sum_{j=1}^N b_j(x, \omega) u_{0,x_j} \nabla \omega_{x_j} \]
\[ + A(x, \omega) \cdot \nabla \omega - \epsilon M - \epsilon \Delta \omega. \]
By assumption $\sum_{j=1}^{N} b_j(x, \omega) v_{x_j}^2 \geq 0$ and $2 \sum_{j=1}^{N} b_j(x, \omega) u_{0,x} v_{x_j} + A(x, \omega) \cdot \nabla v \geq 0$ for $(x, \omega)$ in $\mathcal{D}_0(u_0)$ and so for $x$ in $\Omega \cap \{x: \text{dist}(x, \partial \Omega) < \delta\}$ we have the desired inequality for $\epsilon$ sufficiently small (say, $0 < \epsilon \leq \epsilon_0$) since $\gamma \geq (2q + 1)! M$ and

$$\epsilon \Delta v < \left(\frac{m}{(2q+1)!}\right) \psi^{2q+1} \quad \text{for} \quad 0 < \epsilon \leq \epsilon_0.$$ 

However, for $x$ in $\Omega \cap \{x: \text{dist}(x, \partial \Omega) \geq \delta\}$

$$\sum_{j=1}^{N} b_j(x, \omega) v_{x_j}^2 + 2u_{0,x} v_{x_j} + A(x, \omega) \cdot \nabla v = \mathcal{O}(\rho_1(\epsilon))$$

where $\rho_1(\epsilon)$ is transcendentally small if $q = 0$ and $\rho_1(\epsilon) = \mathcal{O}(\epsilon^{(1+q-1)/(2q+2)})$ if $q \geq 1$. Since $\rho_1(\epsilon) = o(\epsilon^p)$ and $\epsilon \Delta v < \left(\frac{m}{(2q+1)!}\right) \psi^{2q+1}$ for $0 < \epsilon \leq \epsilon_0$ it follows that also $\epsilon \Delta \omega \leq \mathcal{F}(x, \omega, \nabla \omega)$ for $x$ in $\Omega \cap \{x: \text{dist}(x, \partial \Omega) \geq \delta\}$.

Finally we take $\bar{w} = \bar{u}(x, \epsilon)$ to be the function defined in the proof of Theorem 2.1. Therefore the hypotheses of Amann’s theorem are satisfied and the conclusion of Theorem 4.1 follows. (If $\mu(x) u_0(x) + \nabla u_0(x) \cdot \nabla u_0(x) \leq \varphi(x)$ for $x$ on $\partial \Omega$ then we define $\omega(x, \epsilon) = u_0(x) - \varphi(x, \epsilon) - (\epsilon^2 \gamma m^{-1})^{(2q+1)}/2q+1$ and $\bar{w}(x, \epsilon) = u_0(x) + (\epsilon^2 \gamma m^{-1})^{(2q+1)}/2q+1$ and proceed as above.)

The analog of Theorems 2.2 and 3.2 for the problem $(\mathcal{N}_3)$ can now be stated and proved without difficulty.

**Theorem 4.2.** Assume that the reduced equation $(\mathcal{R}_3)$ has a subharmonic $(\Pi_n)$-stable solution $u = u_0(x)$ of class $C^{0,2}(\Omega)$ such that $\mu(x) u_0(x) + \nabla u_0(x) \cdot \nabla u_0(x) \leq \varphi(x)$ for $x$ on $\partial \Omega$. Assume also that

$$\sum_{j=1}^{N} b_j(x, u) F_{x_j}^2(x) \geq 0$$

and

$$2 \sum_{j=1}^{N} b_j(x, u) u_{0,x}(x) F_{x_j}(x) + A(x, u) \cdot \nabla F(x) \geq 0$$

for $(x, u)$ in $\mathcal{D}_0(u_0)$. Then there exists an $\epsilon_0 > 0$ such that the problem $(\mathcal{N}_3)$ has a solution $u = u(x, \epsilon)$ of class $C^{0,2}(\Omega)$ whenever $0 < \epsilon \leq \epsilon_0$. In addition, for $x$ in $\Omega$ we have that

$$u_0(x) \leq u(x, \epsilon) \leq u_0(x) + \varphi(x, \epsilon) - \epsilon \bar{p}(\epsilon),$$

where the function $\varphi$ and the positive constant $c$ are defined in the conclusion of Theorem 2.2 and $\bar{p}(\epsilon) = \epsilon^{1/2q}$. Finally if $u_0$ is superharmonic and $(\Pi_n)$-stable then the result corresponding to Theorem 4.2 is valid provided that $\mu(x) u_0(x) + \nabla u_0(x) \cdot \nabla u_0(x) \geq \varphi(x)$ for $x$ on $\partial \Omega$, and that

$$\sum_{j=1}^{N} b_j(x, u) F_{x_j}^2(x) \leq 0$$
for \((x, u)\) in \(\mathcal{D}_\delta(u_0)\). We leave its precise formulation to the reader.

Some remarks on the boundary inequalities

\[
2 \sum_{j=1}^{N} b_j(x, u) u_{0,x_j}(x) F_{x_j}(x) + A(x, u) \cdot \nabla F(x) \geq 0
\]

for \((x, u)\) in \(\mathcal{D}_\delta(u_0)\) are now in order. The first inequality is the quantitative formulation of the restriction that the characteristic curves of the first-order equation \((\mathcal{A}_\delta)\) must be outgoing in a neighborhood of the boundary of \(\Omega\). The second inequality also has a heuristic geometric interpretation. Suppose that \(u = u(x, \epsilon)\) is a solution of \((\mathcal{A}_\delta)\) and consider the Hessian matrix \(\mathcal{H}(x, \epsilon) = \frac{\partial^2 u(x, \epsilon)}{\partial x_i \partial x_k}\) for \(i, k = 1, \ldots, N\). Then \(u\) is a convex (concave) function if \(\mathcal{H}(x, \epsilon) \geq 0\) (\(\leq 0\)). Now the trace of \(\mathcal{H}(x, \epsilon)\) is \(\sum_{k=1}^{N} \partial^2 u/\partial x_i^2 = \Delta u = \mathcal{O}(\epsilon^{-1} \sum_{j=1}^{N} b_j(x, u)u_{x_j}^2)\) and so \(u\) is convex (concave) near the boundary of \(\Omega\) if \(\sum_{j=1}^{N} b_j(x, u)u_{x_j}^2 \geq 0\) (\(\leq 0\)) there. Thus if \(u_0(x)\) is to be the uniform limit as \(\epsilon \to 0^+\) of \(u(x, \epsilon)\) in \(\mathcal{O}\) and if \(\mu(x) = \nabla F(x) \cdot \nabla u_0(x) \leq \varphi(x)\) (\(\geq \varphi(x)\)) on \(\partial \Omega\) then it must be the case that \(\sum_{j=1}^{N} b_j(x, u)u_{x_j}^2 > 0\) (\(\leq 0\)) for \(x\) in \(\Omega \cap \{x : \text{dist}(x, \partial \Omega) < \delta\}\).

We remark also that if the first inequality is satisfied in the strong sense that

\[
2 \sum_{j=1}^{N} b_j(x, u) u_{0,x_j}(x) F_{x_j}(x) + A(x, u) \cdot \nabla F(x) > k > 0
\]

in \(\mathcal{D}_\delta(u_0)\) for a positive constant \(k\) then the boundary layer terms \(v\) and \(w\) in Theorems 4.1 and 4.2, respectively, can be replaced by \(\chi(x, \epsilon) = \epsilon k_1^{-1} \exp(\epsilon k_1^{-1} F(x))\) where \(k_1 < k\) is a positive constant. In addition, the term \(\rho(\epsilon)(\tilde{\rho}(\epsilon))\) can be replaced by \(\rho(\epsilon) = \epsilon^{1/(2\alpha + 1)}(\tilde{\rho}(\epsilon) = \epsilon^{1/\alpha})\). Finally if

\[
\sum_{j=1}^{N} b_j(x, \omega) v_x [v_{x_j} + 2u_{0,x_j}] + A(x, \omega) \cdot \nabla v = \mathcal{O}(\epsilon^{1/(2\alpha + 1)})
\]

for \(\omega = \omega\) or \(\tilde{\omega}\) as in the proof of Theorem 2.1 and \(x\) in \(\mathcal{O}\) then the conclusion of Theorem 4.1 is valid with \(\rho(\epsilon) = \epsilon^{1/(2\alpha + 1)}\). And if

\[
\sum_{j=1}^{N} b_j(x, \tilde{\omega}) w_x [w_{x_j} + 2u_{0,x_j}] + A(x, \tilde{\omega}) \cdot \nabla w = \mathcal{O}(\epsilon^{1/\alpha})
\]

for \(\tilde{\omega}\) as in the proof of Theorem 2.2 and \(x\) in \(\mathcal{O}\) then the conclusion of Theorem 4.2 is valid with \(\tilde{\rho}(\epsilon) = \epsilon^{1/\alpha}\).

We conclude this section with two examples.
EXAMPLE 4.1. Consider first the problem
\[ \epsilon \Delta u = u - \| \nabla u \|^2, \quad x \in \Omega, \]
\[ \mu(x) u(x, \epsilon) + \nabla F(x) \cdot \nabla u(x, \epsilon) = \varphi(x), \quad x \in \partial \Omega. \]
If \( \varphi(x) \leq 0 \) then the hypotheses of Theorem 4.1 are satisfied with \( u_0(x) \equiv 0 \) and we conclude that the problem \( (E_\epsilon) \) has a nonpositive solution \( u = u(x, \epsilon) \) such that
\[ \lim_{\epsilon \to 0^+} u(x, \epsilon) = 0 \quad \text{for} \quad x \in \overline{\Omega}. \]

EXAMPLE 4.2. Our final example is an application of the theory of this section to a class of boundary value problems from catalytic reaction theory. For background material and further discussion and references the reader is urged to consult the treatise of Aris [2]. The simplified physical problem involves an isothermal, gas-phase reaction \( A \rightarrow \beta B (\beta > 0) \) which is catalyzed on the surface of a porous solid \( \Omega \) in \( \mathbb{R}^3 \). The equation which describes the mass balance between diffusion and reaction inside of \( \Omega \) is then (cf. [2])
\[ \nabla \cdot (D(u) \nabla u) = \Phi^2 R(u), \quad x \in \Omega, \]
where \( u \) is the normalized concentration of the reactant \( A \), \( D(u) \) is the diffusion coefficient, \( R(u) \) is the reaction rate term and \( \Phi^2 \) is the Thiele modulus which measures the effect of diffusion as opposed to reaction. On the surface \( \partial \Omega \) of \( \Omega \) we prescribe the boundary condition
\[ \mu(x) u(x) + (\partial u / \partial n)(x) = \varphi(x) \]
for a non-negative function \( \mu \). We assume that \( D(u) = (1 + \theta u)^{-1} \) and \( R(u) = u^r \) where \( \theta \simeq (\beta - 1) \) is the volume change modulus and \( r \in \mathbb{Z}^+ \) is the reaction order, and finally that the reaction \( A \rightarrow \beta B \) is diffusion-limited, that is, \( \Phi^2 \gg 1 \). Then by introducing the functions \( D \) and \( R \) into equation (*) and setting \( \epsilon = \Phi^{-2} \) we arrive at the boundary value problem
\[ \epsilon \Delta u = \epsilon \theta(1 + \theta u)^{-1} \| \nabla u \|^2 + u^r(1 + \theta u), \quad x \in \Omega, \]
\[ \mu(x) u(x, \epsilon) + \nabla F(x) \cdot \nabla u(x, \epsilon) = \varphi(x), \quad x \in \partial \Omega. \]
Here we assume as usual that \( \partial \Omega = F^{-1}(0) \) and that \( \| \nabla F(x) \| = 1 \) for \( x \) on \( \partial \Omega \). The function \( u = u_0(x) \equiv 0 \) is clearly \( (I_{(r-1)12}) \)-stable (if \( r \) is odd) or \( (II_{12}) \)-stable (if \( r \) is even) and since the coefficient of \( \| \nabla u \|^2 \) is of order \( O(\epsilon) \) we conclude from the theory of this section (cf. our remarks before the examples) that the problem \( (E_\epsilon) \) has a solution \( u = u(x, \epsilon) \) such that
\[ \lim_{\epsilon \to 0^+} u(x, \epsilon) = 0 \quad \text{for} \quad x \in \overline{\Omega}. \]
This result is not surprising in view of our assumption of diffusion-limitation and it has been confirmed experimentally (cf. [2]).
5. CONCLUDING REMARKS

In this final section we make several observations which are pertinent to the three problems (\(\mathcal{N}_1\))-(\(\mathcal{N}_3\)).

With minor modifications the theory for (\(\mathcal{N}_1\))-(\(\mathcal{N}_3\)) developed above could be applied to the problem

\[
\begin{align*}
\epsilon L_\epsilon u &= \mathcal{F}(x, u, \nabla u, \epsilon), \quad x \in \Omega, \\
\mu(x, \epsilon) u(x, \epsilon) + \nabla \mathcal{F}(x) \cdot \nabla u(x, \epsilon) &= \varphi(x, \epsilon), \quad x \text{ on } \partial \Omega
\end{align*}
\]

(\(\mathcal{N}_\epsilon\))

where \(L_\epsilon\) is a general linear second-order uniformly elliptic operator and \(\mathcal{F}(x, u, \nabla u, \cdot), \mu(x, \cdot)\) and \(\varphi(x, \cdot)\) are sufficiently smooth functions of \(\epsilon\). The theorem of Amann quoted in the proof of Theorem 2.1 actually applies to the general equation \(L_\epsilon u = \mathcal{F}(x, u, \nabla u)\) and so the study of (\(\mathcal{N}_\epsilon\)) involves nothing really new.

Finally the interior crossing theory discussed in [11] for Dirichlet problems of the form (\(\mathcal{N}\)) can be applied to the Robin or Neumann problem with little additional difficulty.

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REFERENCES

10. C. J. Holland, The regular expansion in singularly perturbed elliptic equations, in
"Proceedings, Conference on Stochastic Differential Equations and Applications"

11. F. A. Howe, Singularly perturbed semilinear elliptic boundary value problems,

12. N. Levinson, The first boundary value problem for $\varepsilon u + A(x, y)u_x + B(x, y)u_y +
C(x, y)u = D(x, y)$ for small $\varepsilon$, Ann. of Math. 51 (1950), 428–445.

13. O. A. Oleinik, On the second boundary value problem for elliptic equations with a
small parameter in the highest derivatives, Dokl. Akad. Nauk SSSR 79 (1951),
735–737.

14. O. A. Oleinik, On boundary value problems for equations with a small parameter

15. O. A. Oleinik, On equations of elliptic type with a small parameter in the highest
derivatives, Mat. Sb. 31 (1952), 104–117.


17. M. I. Vishik and L. A. Liusternik, Regular degeneration and boundary layer for
linear differential equations with small parameter, Amer. Math. Soc. Transl. Ser. 2 20