A Stochastic Model of Data Access and Communication

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Stochastic models are an important technique for predicting the performance of computer systems and communication networks. Although much work has been done to develop analytic and simulation models, these models usually assume that data access is uniformly distributed, that data is static, and that data access and communication occur according to the Poisson process. In practice, data access is highly skewed, does not occur at Poissonian times, and data items are constantly being created and deleted. A new stochastic model of data access is developed that includes all of these observed phenomena. The model displays a surprising richness of behavior and yet has a small number of independent parameters, is analytically tractable, and is easy to simulate. An axiomatic framework for a general class of continuous models is introduced, and a specific discrete approximation of such a model is developed in detail. The extent to which the model fits empirical observations is also discussed. © 1989 Academic Press, Inc.

Tempora mutantur, nos et mutamur in illis.
Times change, and we change with them too.
—Anonymous

1. INTRODUCTION

Performance modeling is an important tool for predicting the performance of computer systems and communication networks. Much work has been done to develop and study different performance models, using both analytic and simulation methods. Unfortunately most of the existing models are based on uniform access to static data: it is assumed that each data item is as likely to be accessed as any other and that data items are never created or deleted. Moreover, access in time to the data is usually assumed to occur according to the Poisson process.

In this paper, a new stochastic model of data access is introduced that reflects the observed behavior of real computer systems and communication networks better than the existing models. This new model exhibits skewed

*The author was supported in part by NSF Grant #CCR-8716485.
access to data similar to the Zipf distribution. The model also exhibits the burst behavior that has been observed in communication networks. Both the Zipfian and burst behaviors are consequences of the same "renormalization" or "fractal" property of the model. Despite the richness of the phenomena that the model exhibits, it is specified by only three independent parameters, is easy to simulate, and is analytically tractable.

Section 2 offers a brief introduction to computer systems and communication networks, and gives an intuitive overview of the new stochastic model. A continuous version of the stochastic process is axiomatized in Section 3. The various axioms are motivated by the observed behavior of real systems. In Section 4, a discrete approximation of the continuous model is introduced and shown to approximate the continuous case. The extent to which the discrete model fits empirical observations is discussed in Section 5. Section 6 discusses variations on the themes introduced earlier. This section also considers some mathematical questions raised by these models.

2. COMPUTER SYSTEMS AND COMMUNICATION NETWORKS

Computer systems and communication networks store and refer to large amounts of data. The units that access the data are called "programs" or "processes." (The distinction between these two is not important here.) The term "access" includes both the reading and the modifying of data.

Communication networks consist of a collection of computer systems linked together by communication paths. The systems in the network communicate by sending messages to one another. Like individual computer systems, communication networks contain large amounts of data, and this data is accessed by processes. The distinction between a computer system and a network is being blurred, since some of the new computer systems are actually made up of several processing units that communicate with one another over internal communication links.

Data stored by a computer system or more generally by a communication network is divided into subunits, such as files, records, or pages depending on how one is viewing the system. These data units will be called data granules. The collection of all data will be called "the database," although strictly speaking this term is normally used in a much more restricted sense in computer science.

It should be obvious that access to the database will not be completely random. That is, one cannot expect each data granule to be as likely as any other to be accessed. In fact, it has been observed that the amount of skew in data access is substantial: in some cases, 10% or fewer of the data granules account for 90% or more of all accesses. Yet most performance
models of data access rather naively assume that access is uniform throughout the database. Furthermore, such performance models assume that the requests for data will occur at completely random times (i.e., in a Poissonian manner). This has not been observed in practice. There will instead be occasional bursts of high activity superimposed on a background that is approximately Poissonian.

A more realistic model of data access and communication should reflect the observed phenomena discussed above. In particular, a small fraction of all data granules should account for a disproportionate share of all data accesses. One way to model this is to postulate the existence of a hot spot or subset of the database that accounts for a large fraction of all data accesses. Hot spots have been a recognized problem for a long time in computer science, but attempts to deal with it have only recently begun to appear. See Gawlick [5], Gawlick and Kinkade [6], Reuter [11], and O'Neill [10].

Suppose that the hot spot contains a fraction \( c \) of all granules and that a given access will be to a granule in the hot spot, with probability \( b \). If one supposes that access within the hot spot is uniform and that access outside it is also uniform, then one obtains a model of data access called the \( b-c \) model [13]. However, it is unrealistic to assume that data access would make such a sharp distinction between hot and cool granules. A more realistic assumption is that data access both within and outside the "hot spot" is also skewed, just as it is skewed in the database as a whole.

The hot spot concept is an excellent method for introducing skewness into a data access model. However, one must be careful not to ascribe properties to this concept that are simply accidents of the words being used. For example, the hot spot need not be a localized "spot" in the database, but rather can be a subset spread at random throughout the database.

The actual method used here for introducing skewness to the data access model is to postulate a "renormalizability" or "self-similarity" property. Roughly speaking, this means that the structure of the "hot spot" in the database is probabilistically isomorphic (after renormalizing) to the database as a whole. This is made precise in Section 3. Although this is a stochastic and not a geometric property, we will abuse the language somewhat and refer to it as a "fractal" property, using the term popularized in Mandelbrot [9].

Another important property of the database is that it is dynamic. It evolves or "wanders" in time. Data granules are continually being added and deleted even if the overall size of the database remains approximately constant. Moreover, the lifetime of a single data granule is random.

If one combines the fact that data access is skewed with the fact that the database is evolving, one is forced to conclude that the skewness is also evolving. This will be an important assumption in our model, which is important even for a database that is not evolving rapidly as a whole. The
evolution of the skewness is another renormalizability property of the model. This property is dynamic, while the first renormalizability property mentioned above is static. However, the two properties are dependent on one another: in the discrete case, neither is possible without the other.

3. THE CONTINUOUS MODEL

An axiomatic formulation of a continuous stochastic process is now presented. This process satisfies the requirements described in Section 2 above. A discrete approximation is developed in Section 4 below.

The Space of Data Granules

The set of all data granules is modeled using a measure space.

Axiom 1. (data granules). Let \((\Omega, \mu)\) be a measure space.

The elements of \(\Omega\) are the data granules. The set \(\Omega\) contains all data that ever were, are, or will be accessed. The measure \(\mu\) is the basic or uniform measure on data.

The Database

Since data is continually being inserted and deleted, only a subset of all possible data is accessible at one time.

Axiom 2. (databases). For every \(t \in \mathbb{R}\), \(D_t \subseteq \Omega\) is a \(\mu\)-measurable set such that:

(a) \(0 < \mu(D_t) < \infty\).

(b) for every \(g \in \Omega\), the set \(\{t : g \in D_t\}\) is an interval of the form \([s, d)\).

The set \(D_t\) is the database at time \(t\). If \(\{t : g \in D_t\} = [s, d)\), then \(g\) was inserted at time \(s\) and deleted at time \(d\). When it is not clear from the context which data granule is being considered, \(s\) will be written \(\text{ins}(g)\), and \(d\) will be written \(\text{del}(g)\).

The basic or uniform probability measure at time \(t\) is defined by

\[
\Pr_t(A) = \frac{\mu(A \cap D_t)}{\mu(D_t)},\quad \text{for } A \subseteq \Omega.
\]

This measure is called "uniform" to distinguish it from the activity measure defined below which is systematically skewed with respect to the base
measure. The unsubscripted symbol "Pr" will be used to denote the probability of an event in whatever stochastic process is currently being considered. The context should make it clear what process is intended.

**Database Evolution**

Assume for the moment that $\mu(t)$ is a constant independent of $t$. Now suppose that $D_t$ "accretes" newly inserted data granules at a constant rate. If the measure of the accreted data during a unit of time is $r\mu(D_t)$, then $r$ is called the *rate of evolution* of the database. To remain the same size $D_t$ must lose data granules at the same rate. The simplest way to do this is to delete data granules from $D_t$ uniformly at random. When this occurs, the following property holds. This property is assumed to hold even if $\mu(D_t)$ varies.

**Axiom 3. (database evolution).** There is a positive constant $r$ such that if $t \leq t'$, then $Pr_{t'}(D_t) = e^{-2r(t'-t)}$.

The exponential term of Axiom 3 has a factor of 2 in the exponent because data granules are being both added and removed. As a result $D_t$ diverges from $D_t$ at the rate $2r$.

**The Activity Measures**

The probability $Pr_t$ measures the size of a subset of the database relative to the whole database at time $t$. Since the access pattern is skewed, $Pr_t$ does not measure the *access* probability. To measure this, a second probability measure $\overline{Pr}_t$ is needed. This new probability measure is called the *activity measure*. For an event $A \subseteq \Omega$, $\overline{Pr}_t(A)$ should be interpreted as the probability that a data granule is accessed from $A$ at time $t$, given that some granule is accessed at time $t$. Although $\overline{Pr}_t$ is an important part of the model, it will later be seen that it can be computed from another structure that is more fundamental: the hot spots.

Fix a time $t$. The skewness of access to data is modeled by assuming that there is a "hot spot" $H_1 \subseteq D_t$ which is more frequently accessed than the complement $D_t \setminus H_1$. Let $c$ be the "size" of $H_1$, i.e., $Pr_t(H_1) = c$, and suppose that $b$ is the probability that one of the granules in $H_1$ is accessed, i.e., $Pr_t(H_1) = b$. $H_1$ is then said to satisfy the $b/c$-activity rule. This is not the same as the $b/c$ model mentioned above, which assumes uniform access within the hot spot. Quite the contrary, the access pattern within $H_1$ is assumed to be skewed: $H_1$ has its own hot spot $H_2$, which has properties inside $H_1$ that are similar to the properties of $H_1$ inside the entire database. More precisely, this means that $Pr_t(H_2|H_1) = Pr_t(H_1)$ or $Pr_t(H_2) = c^2$ and that $\overline{Pr}_t(H_2|H_1) = \overline{Pr}_t(H_1)$ or $\overline{Pr}_t(H_2) = b^2$. Of course, $H_2$ must also have
a hot spot $H_3$, and so on. $D_i$ itself is then $H_0$. In other words, the hot spot concept is assumed to be infinitely recursive.

Even this is not quite sufficient since each event $H_i \setminus H_{i+1}$ should also be skewed. Accordingly, assume that there is a hot spot $H_\alpha$ for each level $\alpha \geq 0$, such that the following renormalizability property holds for both $\Pr_i$ and $\overline{\Pr}_i$:

**Static Renormalizability.** A sequence of events $\{H_\alpha | \alpha \geq 0\}$ in a probability space $H_0$ is **statically renormalizable** if the following hold:

(a) for every $\alpha \leq \beta$, $H_\alpha \supseteq H_\beta$,

(b) for every $\alpha \leq \beta$ and $\gamma \geq 0$, $\Pr(H_\beta | H_\alpha) = \Pr(H_{\beta+\gamma} | H_{\alpha+\gamma})$.

It is easy to check that static renormalizability implies that $\Pr(H_\alpha) = \Pr(H_1)^\alpha$, for every $\alpha \geq 0$. In particular, if $H_1$ satisfies the $b/c$-activity rule, then $\Pr_i(H_\alpha) = c^\alpha$ and $\overline{\Pr}_i(H_\alpha) = b^\alpha$.

A convenient way to define a nested sequence of events is to use the concept of a random variable. In this case, let $V_i$ be the random variable on $D_i$ defined by

$$V_i(g) = \sup\{\alpha : \alpha = 0 \text{ or } g \in H_\alpha\}, \quad \text{for } g \in \Omega. \quad (2)$$

Note that for convenience $V_i$ has been extended to all of $\Omega$ by defining it to be zero outside $D_i$. Except for a set of measure zero, $H_\alpha$ is the same as the event $(V_i > \alpha)$. In particular, $D_i$ is essentially the same as $(V_i > 0)$. Since $\Pr_i(H_\alpha) = c^\alpha$ and $\overline{\Pr}_i(H_\alpha) = b^\alpha$, it follows that $V_i$ is an exponential random variable with respect to both $\Pr_i$ and $\overline{\Pr}_i$.

The value of $V_i$ can be interpreted as an "activity level" of data access. The actual value of this level is not significant by itself. If one makes a linear change of units, say $Y_i = kV_i$, then $Y_i$ defines exactly the same collection of hot spots as $V_i$ but with the constant $c$ replaced by $c^{1/k}$ and the probability of access $b$ replaced by $b^{1/k}$. It is a reasonable convention to choose the unit of level so that $c + b = 1$. When this is done, one writes $q$ and $p$ instead of $b$ and $c$, respectively. For example, a common choice of activity rule is the 80% / 20% rule, for which $p = 0.2$ and $q = 0.8$. The probability $p$ will be called the hot fraction.

If $\Pr_i(V_i > \alpha)$ and $\overline{\Pr}_i(V_i > \alpha)$ are differentiated with respect to $\alpha$, then we find that

$$d\overline{\Pr}_i(V_i > \alpha) = \frac{\ln(q)}{\ln(p)} q^{\alpha_p^\alpha} d\Pr_i(V_i > \alpha).$$
It follows that if $A$ is an event expressible in terms of the events $(V_i > \alpha)$, then

$$F_t(A) = \frac{\ln(q)}{\ln(p)} \int_A q^V p^{-V} \, d\Pr_t.$$  

Formula (3) will be assumed to valid for arbitrary events. As a result the random variables $V_i$ can be taken as a fundamental concept, from which the activity measures can be derived. In other words, the hot spots determine the access probabilities. The next two axioms summarize the discussion above.

**Axiom 4.** (static renormalizability). There is a family of identically distributed, exponential random variables $\{V_i : t \in \mathbb{R}\}$, where $V_i$ is supported on the probability space $(D, \Pr_t)$.

**Axiom 5.** (activity measure). For positive constants $p$ and $q$, such that $p + q = 1$, if a data granule is accessed at time $t$, then the probability that this granule is in $A \subseteq \Omega$ is

$$F_t(A) = \frac{\int_{D_t \cap A} \left(\frac{\ln(q)}{\ln(p)}(q/p)^V\right) \, d\mu}{\int_{D_t} d\mu} = \int_A \frac{\ln(q)}{\ln(p)} \left(\frac{q}{p}\right)^V \, d\Pr_t.$$  

The quantity $\ln(p)/\ln(q)$ will occur frequently. By analogy with fractal geometry, it will be called the fractal dimension of the model. When $p = q = 0.5$, the dimension is 1, and access to the database is uniform at Poissonian access times. When $p < 0.5$, the dimension will be larger than 1, and access to the database will be skewed. The letter $Q$ will be used for $\ln(p)/\ln(q)$ (to understand why just pronounce the word “skew”). The reciprocal $\ln(q)/\ln(p)$ will be written $R$.

**The Evolution of Skewness**

So far the activity rule refers to a fixed time $t$, and the random variables $V_i$ have no connection with one another. A second renormalization property links them together by requiring that they evolve in much the same way as the database $D_t$ evolves, except that the hot spots evolve more rapidly than the database. More precisely, the rate of evolution of a hot spot should be inversely proportional to its size. This renormalization of a rate of evolution will be called dynamic renormalization.
Axiom 6. (dynamic renormalizability). The family \( \{ V_t \} \) of random variables are dependent on one another such that if \( \alpha > 0 \) and \( t \leq t' \), then

\[
\Pr_r(V_t > \alpha | V_{t'} > \alpha) = \exp \left( \frac{-2r(t' - t)}{\Pr_r(V_{t'} > \alpha)} \right).
\]

Note that Axiom 3 is a special case of Axiom 6.

Clustering

The formulation described above omits an important aspect of a real system: the clustering of data. There are different possibilities for the choice of the unit of data. To the user or an application program, the unit of data might be a record. To the low-level operating system software, the unit of data is a page which contains a fixed number of bytes of data but a variable number of records. To the high-level operating system software, the unit of data is a file, which can have a variable number of pages of data. In each case a number of data granules of one type are clustered to form a single granule of another type.

This can be expressed mathematically by postulating that there is an equivalence relation \( \sim \) on \( \Omega \), such that \( g \sim h \) means that the data granules \( g \) and \( h \) are in the same cluster. These clusters can be randomly allocated or they can be allocated in a more systematic way.

Clustering can arise in still other ways. The organization of data granules into pages is just one possibility. In virtual memory, for example, it is more appropriate to regard the elements of \( \Omega \) as being "episodes" in which a page is more frequently accessed, while the quotient \( \Omega / \sim \) is the actual set of pages. In general \( \Omega \) could be infinite and might not even be discrete, so long as the quotient is a discrete set. Although equivalence relations will not be considered in the rest of this paper, they represent an important extension of the theory presented here.

4. THE DISCRETE APPROXIMATION

The axioms of the continuous model axiomatize "activity" without explicitly considering "access." This is no surprise since access is a discrete concept. A model is now introduced which is discrete, includes access to the database explicitly, and approximates the continuous case. The discrete model approximates the continuous one by replacing equalities involving measures by equalities of expectations. For example, the database size at time \( t \) will now be determined only on the average. One can do no better
than this because data granules will be inserted and deleted in discrete units having positive measure.

The particular discrete approximation was chosen for its tractability and the ease with which it can be simulated. The output of the discrete approximation is a sequence of accesses to data granules, occurring at times determined by a process that satisfies two of the three axioms of the Poisson process. The omission of the third axiom is dictated by several considerations. The full Poisson process is not possible for a discrete model, even approximately. More importantly, real systems do not exhibit pure Poissonian behavior. Those properties of the Poisson process that seem to hold were included, and the property that has not been observed was dropped. The resulting model exhibits the kind of "burst" behavior that is observed in real systems.

To avoid confusion with the continuous case, the discrete process will use $W$ instead of $V$, but the other terminology introduced in the previous section will be maintained.

The Space of Data Granules

The space $\Omega$ is approximated using the set of integers. To avoid a confusion of terminology, the elements of this approximation will be regarded as being the identifiers of the data granules. Identifiers are normally assigned successively to data granules in the order in which they are inserted into the database. It is convenient to choose the origin of the time axis (i.e., time $t = 0$) to be the time when the data granule with identifier 0 is inserted. The measure $\mu$ is defined by assigning mass 1 to every identifier. In other words, $\mu$ is truly the uniform measure. Axiom 1 is therefore satisfied.

The Database

The database $D$, at time $t$ is assumed to have, on the average, a fixed number of identifiers. This parameter will be called the database size, abbreviated $\sigma$. Each data granule has an insertion time and a deletion time. Once deleted, a data granule is never again accessed, and its identifier is never reused. As a result, the insertion and deletion times satisfy Axiom 2.

Database Evolution

The evolution of the database is determined by the two functions ins and del. Insertion is relatively easy. The values ins($i$) are obtained from a Poisson process whose intensity is $\rho = r\sigma$, where $r$ is the rate of evolution occurring in Axiom 3.
Deletion seems to be more difficult. It appears at first that one can obtain values for del\(i\) using a stochastic process as was done for insertion, but this will not work. This method will not even ensure that del\(i\) > ins\(i\). If one ignores granule identifiers, then the sequence of times when deletions take place forms a Poisson process. However, data granules are not deleted in the same order as they are inserted, and computing the mapping from insertion to deletion times seems to be computationally infeasible, because the database must be scanned for each deletion.

In fact, there is a way to make the computation feasible. Since the lifetime of a data granule has a known distribution, one can compute the deletion time of each granule by choosing its lifetime and then adding the lifetime to the insertion time. The hot spots are "computed" using a similar method. The lifetime of a data granule has a gamma distribution with parameter 2. This will be shown in Theorem 2 below. One can obtain a random variable having this distribution by adding two independent, equidistributed, exponential random variables. A gamma distribution having a positive integral parameter is also called an *Erlang distribution*. These distributions occur frequently in probabilistic simulations of computer systems and communication networks.

The mean lifetime of a data granule is another parameter of the process. It is called the *lifetime*, and is abbreviated \(\lambda\). This is not an independent parameter, since \(\sigma, r, \rho, \text{ and } \lambda\) are related by the equations \(\rho \lambda = \sigma\) and \(\lambda = 1/r\). Any of these parameters can be allowed to be functions of the time rather than constants, provided these equations hold. If the parameters vary slowly relative to the lifetime of a data granule, then the analysis and simulation for the constant case can be used as a good first approximation to the more general case.

The *Activity Rule and Activity Measure*

As discussed earlier, a direct simulation of the model that generates one access at a time would be very inefficient. It is much more efficient to generate the accesses in a more indirect manner. The method that will be used is to compute the values of ins\(i\), del\(i\), and the sequence of access times to this data granule independently for each identifier. In effect, the entire history of a data granule is generated when it is inserted. All the histories are then merged together to form the full sequence of data accesses. It is remarkable that such a method will produce a model that satisfies both renormalizability properties.

The key to making this approach work is to choose a specific form for the function \(V(t)\) when the identifier \(i\) is fixed while \(t\) varies. There are several
possibilities for this, as discussed in Section 6 below. For the particular process presented here, the following will be used:

**Definition.** Let \( i \in \Omega \) be the identifier of a data granule inserted at time \( \text{ins}(i) = s \). Choose two independent, equidistributed, exponential random variables, whose values are \( u \) and \( v \). Define \( \text{del}(i) = s + u + v \) and

\[
W_t = \frac{1}{\ln(p)} \ln \left( \frac{s + u - t}{u} \right), \quad \text{when } s \leq t \leq s + u
\]

\[
= \frac{1}{\ln(p)} \ln \left( \frac{t - s - u}{v} \right), \quad \text{when } s + u \leq t \leq s + u + v.
\]

The quality \( u \) will be called the *first period*, while \( v \) will be called the *second period*. The mean values of \( u \) and of \( v \) are \( \lambda/2 \). Note that \( W_s(i) = W_{s+u+v}(i) = 0 \) and that \( W_t(i) \) is singular at \( t = s + u \).

There are several properties of this choice of \( W_t \) that must be verified. To satisfy Axiom 4, they must be shown to be exponential random variables. This is shown in Theorem 1 below. To show that Axiom 5 holds, a number of other results must be verified first. As a result this axiom is left for last and is shown by Theorem 6.

Axiom 6 states that the hot spots determined by \( W_t \) must evolve in a certain way. This is shown by Theorem 2 below. The evolution of the database itself as specified by Axiom 3 is a consequence of Axiom 6. Finally, the full sequence of access times produced must be shown to satisfy the first two axioms of the Poisson process. This is done by Theorems 4 and 5 below.

**Theorem 1.** For every \( t \in \mathbb{R} \), \( W_t \) is exponentially distributed, and \( \Pr_t(W_t > 1) = p \).

**Proof.** Fix a time \( t \in \mathbb{R} \). Since the insertion times \( \text{ins}(i) \) form a Poisson process, the points \( x(i) = t - \text{ins}(i) \) also form a Poisson process. If this process is conditioned on the event \( (c \leq x \leq d) \), where \( c \) and \( d \) are constants, then the result is a uniform process on the interval \([c, d]\). It follows that \((d - x)/(d - c)\) is uniform on the interval \([0, 1]\). This property is independent of the endpoints \( c \) and \( d \). Therefore, it will still be true if \( c \) and \( d \) are random variables.

The uniformity property is used in two cases. First let \( c = 0 \) and \( d = u \). Then \((u - (t - \text{ins}(i)))/u\) is uniformly distributed on \([0, 1]\). Therefore,

\[
\Pr_t(((s + u - t)/u < p^\alpha|s \leq t \leq s + u) = p^\alpha,
\]
where \( s = \text{ins}(i) \) and \( \alpha \) is positive. Rearranging terms, this event can be put in the following form:

\[
\left( \frac{s + u - t}{u} < p^\alpha \right) = \left( \frac{\ln\left( \frac{s + u - t}{u} \right)}{\ln(p)} < \alpha \ln(p) \right)
\]

\[
= \left( \frac{1}{\ln(p)} \ln\left( \frac{s + u - t}{u} \right) > \alpha \right)
\]

\[
= (W_i > \alpha).
\]

It follows that,

\[
\Pr_t(W_i > \alpha | s \leq t \leq s + u) = p^\alpha.
\]

In a similar fashion, using \( c = u \) and \( d = u + v \), one can show that

\[
\Pr_t(W_i > \alpha | s + u \leq t \leq s + u + v) = p^\alpha.
\]

Since \( W_i = 0 \) when \( t < s \) or \( t > s + u + v \), it follows that \( \Pr_t(W_i > \alpha) = p^\alpha \) unconditionally. The theorem follows.

The next task is to show that the hot spots and the database itself satisfy the evolution axioms 3 and 6.

**Theorem 2.** If \( t \leq t' \), and \( \alpha > 0 \), then

\[
\Pr_{t'}(W_i > \alpha | W_{t'}, > \alpha) = \exp\left( -\frac{2r(t' - t)}{\Pr_{t'}(W_{t'} > \alpha)} \right).
\]

**Proof.** By Theorem 1, we have that \( \Pr_t(W_i > \alpha) = p^\alpha \). So we want to show that

\[
\Pr_{t'}((W_i > \alpha) \cap (W_{t'} > \alpha)) = p^\alpha \exp(-2r(t' - t)p^{-\alpha}).
\]

Let \( s = \text{ins}(i) \) as usual. When \( (s \leq \tau \leq s + u) \), we have

\[
(W_i > \alpha) = \left( \frac{1}{\ln(p)} \ln\left( \frac{s + u - \tau}{u} \right) > \alpha \right)
\]

\[
= \left( \frac{s + u - \tau}{u} < p^\alpha \right)
\]

\[
= (\tau > s + u - up^\alpha).
\]

There is a similar calculation for the case \( (s + u \leq \tau \leq s + u + v) \). Therefore,

\[
\{ \tau : W_i > \alpha \} = (s + u - up^\alpha, s + u + vp^\alpha).
\]
Hence

\[(W_i > \alpha) \cap (W_i' > \alpha) = ([t, t'] \subseteq (s + u - up^a, s + u + up^a)).\] (4)

Note that since \(\mu(D_i) = \mu(D_i')\), it does not matter whether one uses \(Pr_t\) or \(Pr_t'\), when computing the probability of this event, so we will use \(Pr_t\).

Using the technique in the proof of Theorem 1, we know that \(t - s\) is uniformly distributed on the interval \([0, u + v]\), and hence \((t - s)/(u + v)\) is uniformly distributed on \([0, 1]\). Let \(h = t' - t\). The event in formula (4) above is the same as the event

\([\{t - s, t - s + h\} \subseteq (u - up^a, u + up^a)].\)

The probability of this event is

\[
\frac{\max(0, up^a + vp^a - h)}{u + v}.
\]

Since this depends on \(u\) and \(v\), the required probability is given by

\[
Pr_t'((W_i > \alpha) \cap (W_i' > \alpha)) = \int_0^\infty \int_0^\infty \max\left(0, \frac{(u + v)p^a - h}{u + v}\right) \frac{2}{\lambda} e^{-2u/\lambda} du \frac{2}{\lambda} e^{-2v/\lambda} dv. (5)
\]

Changing variables to \(u\) and \(w = u + v\), formula (5) becomes

\[
\int_{h - p^a}^\infty \int_0^w \left(\frac{p^a - \frac{h}{w}}{\lambda^2} e^{-2w/\lambda}\right) dw = \int_{h - p^a}^\infty (wp^a - h) \frac{4}{\lambda^2} e^{-2w/\lambda} dw
\]

\[
= \left[ \left( \frac{2h}{\lambda} - \frac{2p^aw}{\lambda} - p^a \right) e^{-2w/\lambda} \right]_{h - p^a}^\infty
\]

\[
= \left( - \frac{2h}{\lambda} + \frac{2h}{\lambda} + p^a \right) e^{-2h \lambda^{-a}/\lambda}
\]

\[
= p^a e^{-2h \lambda^{-a}}.
\]

The last equation above follows from the fact that \(\rho \lambda = \sigma\) and \(r \sigma = \rho\). The result now follows.

**Generating Data Accesses**

The output of the discrete approximation is a sequence of data accesses. This sequence represents the access activity of the programs that are using the database. This sequence of access times will not be a Poisson process,
but will satisfy two of the three axioms for a Poisson process. During any
interval of time the average number of accesses will be a constant times the
length of the interval. This constant is called the access intensity, abbrevi-
ated $\eta$. This property will be verified in Theorem 4. In addition, the
probability that two or more accesses occur during an interval of time
should have lower order than the length of the interval. This is shown in
Theorem 5.

The generation of data accesses for a data granule with identifier $i$ is
done as follows. Generate a Poisson process starting at 0, with intensity 1.
This would be fine if the uniform measure were being used, but the
measures $\overline{Pr}_t(i)$ are required. The Poisson process can be adjusted by
applying the inverse function $F^{-1}$ to the ordinary Poisson points, where $F$
is $\eta \int_{\text{ins}(i)} \overline{Pr}_t(i) \, dt$. Remarkably, all of these steps can be done in closed
form.

The first step is to compute $\overline{Pr}_t$ by applying Axiom 5 to the set $A = \{i\}$. Since $\overline{W}_t$ has a constant value on $A$, it follows that

$$\overline{Pr}_t(A) = \int_A \frac{\ln(q)}{\ln(p)} \left( \frac{q}{p} \right)^{W_t(i)} \, d\Pr_t,$$

$$= \frac{\ln(q)}{\ln(p)} \left( \frac{q}{p} \right)^{W_t(i)} \Pr_t(A).$$

Since $\mu(D_t) = \sigma$, it follows that $\Pr_t(A) = 1/\sigma$. When $0 \leq t - s \leq u$, we find that

$$\overline{Pr}_t(A) = \frac{1}{\sigma} \ln(q) \left( \frac{q}{p} \right)^{(1/\ln(p))/\ln((s+u-t)/u)}$$

$$= \frac{1}{\sigma} \ln(q) \exp \left( (\ln(q) + \ln(p)) \left( \frac{1}{\ln(p)} \ln \left( \frac{s + u - t}{u} \right) \right) \right)$$

$$= \frac{1}{\sigma} \ln(q) \exp \left( \frac{\ln(q)}{\ln(p)} - 1 \right) \ln \left( \frac{s + u - t}{u} \right)$$

$$= \frac{1}{\sigma} R \left( \frac{s + u - t}{u} \right)^{R-1},$$

where $R = \ln(q)/\ln(p)$. Similarly, when $u \leq t - s \leq u + v$, we have

$$\overline{Pr}_t(A) = \frac{1}{\sigma} R \left( \frac{t - s - u}{v} \right)^{R-1}. $$
The expression in parentheses above will occur frequently. Write $H$ for this expression. More precisely,

$$H(s, t, u, v) = \begin{cases} 
\frac{s + u - t}{u}, & \text{when } s \leq t \leq s + u; \\
\frac{t - s - u}{v}, & \text{when } s + u \leq t \leq s + u + v; \\
0, & \text{otherwise.}
\end{cases}$$

By the calculation above, $H$ is related to $W_i$ by

$$\frac{\ln(q)}{\ln(p)} \left( \frac{q}{p} \right)^{W_i} = R(H(s, t, u, v))^{R-1}, \quad \text{when } s \leq t \leq s + u + v. \quad (7)$$

Now that we know $\overline{Pr}_i(A)$, the next step is to anti-differentiate it. When $s \leq \tau \leq s + u$,

$$F(\tau) = \eta \int_s^\tau \overline{Pr}_i(A) \; dt$$

$$= \eta \int_s^\tau \frac{1}{\sigma} R \left( \frac{s + u - t}{u} \right)^{R-1} \; dt$$

$$= \eta \left[ -u \left( \frac{s + u - t}{u} \right)^R \right]_s^\tau$$

$$= \frac{\eta}{\sigma} \left( u - \left( \frac{s + u - \tau}{u} \right)^R \right).$$

Similarly, when $s + u \leq \tau \leq s + u + v$,

$$F(\tau) = F(s + u) + \eta \int_{s + u}^\tau \overline{Pr}_i(A) \; dt$$

$$= F(s + u) + \eta \int_{s + u}^\tau \frac{1}{\sigma} R \left( \frac{t - s - u}{v} \right)^{R-1} \; dt$$

$$= \frac{\eta}{\sigma} u + \frac{\eta}{\sigma} \left[ v \left( \frac{t - s - u}{v} \right)^R \right]_{s + u}^\tau$$

$$= \frac{\eta}{\sigma} \left( u + v \left( \frac{\tau - s - u}{v} \right)^R \right).$$

In terms of $H$, the function $F$ is defined by

$$F'(t) = \frac{\eta}{\sigma} R \left( H(s, t, u, v) \right)^{R-1}. \quad (8)$$
The last step is to invert \( \omega = F(\tau) \). This is straightforward, if somewhat tedious. Write \( G \) for the inverse \( F^{-1} \) of \( F \).

\[
G(\omega) = s + u - u \left( \frac{\eta u - \sigma \omega}{\eta u} \right)^Q, \quad \text{when } 0 \leq \omega \leq \frac{\eta u}{\sigma}
\]

\[
= s + u + u \left( \frac{\sigma \omega - \eta u}{\eta v} \right)^Q, \quad \text{when } \frac{\eta u}{\sigma} \leq \omega \leq \frac{\eta (u + v)}{\sigma},
\]

where \( Q = 1/R = \frac{\ln(p)}{\ln(q)} \).

The following summarizes the algorithm for generating the sequence of accesses to data granules. This algorithm is easy to program. This program is discussed in more detail in [1]. The time and space complexities of the algorithm are \( O(1) \) per access and \( O(1) \) per granule, respectively.

**Simulation Algorithm.**

1. **Parameters.** The process is determined by four parameters, three of which are independent. These parameters are the database size or \( u \), the lifetime or \( \lambda \), the access rate or \( \eta \), and the fractal dimension or \( Q \). The lifetime and access rate are not independent, since the lifetime just determines the unit of time which is then used by the access rate. One could, without loss of generality, choose the unit of time so that \( \lambda = 1 \).

2. **Insertion times.** The sequence of insertion times \( \text{ins}(i) \) of the data granules in \( \Omega \) is a Poisson process with intensity \( \rho = \sigma / \lambda \).

3. **First and second periods.** The first and second periods, \( u(i) \) and \( v(i) \), of the data granule with identifier \( i \) are obtained by choosing them independently as exponential random variables having mean \( \lambda/2 \).

4. **Deletion times.** The deletion time \( \text{del}(i) \) of the data granule with identifier \( i \) is given by \( \text{del}(i) = \text{ins}(i) + u(i) + v(i) \).

5. **Access times.** The access times for the data granule with identifier \( i \) are obtained by applying the function \( G \) to a Poisson process on the interval \((0, \eta(u(i) + v(i))/\sigma)\). This process has intensity 1. The function \( G \) is defined by:

\[
G(\omega) = \begin{cases} 
  s + u - u \left( \frac{\eta u - \sigma \omega}{\eta u} \right)^Q, & \text{when } 0 \leq \omega \leq \frac{\eta u}{\sigma}; \\
  s + u + u \left( \frac{\sigma \omega - \eta u}{\eta v} \right)^Q, & \text{when } \frac{\eta u}{\sigma} \leq \omega \leq \frac{\eta (u + v)}{\sigma}; \\
  0, & \text{otherwise}.
\end{cases}
\]
6. **Merging access times.** The accesses generated as above are merged in order by access time to form a single sequence of access times labeled by granule identifiers.

**Proof of the Axioms**

Having found a method for generating the data accesses, it remains to verify that this method results in an "almost" Poisson sequence of times and that the data granules are accessed according to Axiom 5. The other axioms having been verified earlier, this will complete the proof that the discrete approximation satisfies the axioms. These results all follow from the following basic formulas.

**Theorem 3.** Let \((s_1, s_2)\) and \((t_1, t_2)\) be subintervals of \(R\), and let \((u_1, u_2)\) and \((v_1, v_2)\) be subintervals of the positive real numbers. Let

\[
A = \{ i \in \Omega : s(i) \in (s_1, s_2), u(i) \in (u_1, u_2) \text{ and } v(i) \in (v_1, v_2) \}.
\]

Then the measure of \(A\) is

\[
\mu(A) = \frac{4\rho}{\lambda^2} \int_{u_1}^{u_2} \int_{v_1}^{v_2} e^{-2(u+v)/\lambda} \, ds \, dv, \]

and the mean number of data accesses to data granules in \(A\) during the time interval \((t_1, t_2)\) is

\[
\frac{4\rho \eta}{\sigma \lambda^2} \int_{t_1}^{t_2} \int_{u_1}^{u_2} \int_{v_1}^{v_2} R \left( H(s, t, u, v) \right) R^{-1} e^{-2(u+v)/\lambda} \, ds \, dv \, du \, dt.
\]

**Proof.** Let \(B\) be the set of data granules \(i\) for which \(s(i), u(i), \) and \(v(i)\) take values in the infinitesimal intervals \((s, s + ds),\) \((u, u + du),\) and \((v, v + dv)\). Since these three random variables are independent, the insertion times \(s(i)\) form a Poisson process having intensity

\[
2 \rho e^{-2\lambda/\lambda} \frac{2}{\lambda} e^{-2\sigma/\lambda} \, dv.
\]

Multiplying this by \(ds\) gives the expected number of data granules in \(B\). Integrating this over \(u, v, \) and \(s\) gives the expected number of data granules in \(A\).

For each data granule \(i\) in \(B\), the data accesses of \(i\) falling in an interval \((a, b)\) are obtained by applying \(G\) to the points of a Poisson process in the interval \((F(a), F(b))\), having intensity 1. The mean number of such ac-
cesses will be $F(b) - F(a)$. In particular, $F'(t) \, dt$ accesses will be in the infinitesimal interval $(t, t + dt)$. By formula (8), we have that

$$F'(t) \, dt = \frac{\eta}{\sigma} R(H(s, t, u, v))^{R-1} \, dt.$$  

Multiplying this by the expected number of data granules in $B$ and integrating over $t, s, u,$ and $v$, yields the expected number of accesses to granules in $A$. The theorem now follows.

Theorem 3 is the basic tool for all the remaining results. The first application is to compute the expected number of accesses during an interval of time. This result shows that the process satisfies the first axiom of the Poisson process. The second axiom concerns the likelihood of more than one access occurring during a short interval of time. This second axiom is shown in Theorem 5 below. As noted earlier, the third axiom of the Poisson process does not hold.

**Theorem 4.** The expected number of data accesses during the time interval $(t_1, t_2)$ is $\eta(t_2 - t_1)$.

**Proof.** By Theorem 3, the expected number of data accesses during $(t_1, t_2)$ is

$$\frac{4\rho \eta}{\sigma \lambda^2} \int_{t_1}^{t_2} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty R(H(s, t, u, v))^{R-1} e^{-2u+v}/\lambda \, ds \, dv \, du \, dt. \quad (9)$$

We first evaluate the “innermost” integrand, namely

$$\int_{-\infty}^\infty R(H(s, t, u, v))^{R-1} \, ds$$

$$= \int_{t-u-v}^{t-u} R\left(\frac{t - s - u}{v}\right)^{R-1} \, ds + \int_{t-u}^t R\left(\frac{s + u - t}{u}\right)^{R-1} \, ds$$

$$= \left[-v\left(\frac{t - s - u}{v}\right)\right]_{t-u-v}^{t-u} + \left[u\left(\frac{s + u - t}{u}\right)\right]_{t-u}^t$$

$$= u + v.$$
Substituting this into formula (9) and replacing \( u \) by \( w = u + v \) yields

\[
\frac{4 \rho \eta}{\sigma \lambda^2} \int_{t_1}^{t_2} \int_{0}^{\infty} \int_{0}^{\infty} (u + v) e^{-2(u+v)/\lambda} \, dv \, du \, dt
\]

\[
= \frac{4 \rho \eta}{\sigma \lambda^2} \int_{t_1}^{t_2} \int_{0}^{\infty} \int_{0}^{w} we^{-2w/\lambda} \, dv \, dw \, dt
\]

\[
= \frac{4 \rho \eta}{\sigma \lambda^2} \int_{t_1}^{t_2} \int_{0}^{\infty} w^2 e^{-2w/\lambda} \, dw \, dt
\]

\[
= \frac{4 \rho \eta}{\sigma \lambda^2} \int_{t_1}^{t_2} \frac{\lambda^3}{4} \, dt
\]

\[
= \frac{\eta \rho \lambda}{\sigma} (t_2 - t_1) = \eta(t_2 - t_1).
\]

This completes the proof.

The next result shows that the stochastic process satisfies the second axiom of the Poisson process. Let \( N(t_1, t_2) \) denote the number of accesses that occur during the interval \([t_1, t_2)\).

**Theorem 5.** \( \Pr(N(t_1, t_2) > 1) = o(t_2 - t_1) \) as \( t_2 - t_1 \to 0 \).

**Proof.** Let \( g \) be a granule having insertion time \( s \), first period \( u \) and second period \( v \). Write \( A(g) \) for the event that there are two or more accesses to \( g \) during the interval \([t_1, t_2)\). Let \( \omega_1 = F(t_1) \), \( \omega_2 = F(t_2) \), and \( x = \eta(\omega_2 - \omega_1) \). By the definition of \( F \), the accesses to \( g \) are Poisson with respect to \( \omega \), where \( \omega = F(t) \). Therefore,

\[
\Pr(A(g)) = \left( \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right) \exp(-x)
\]

\[
= x^2 \left( \frac{1}{2!} + \frac{x}{3!} + \cdots \right) \exp(-x)
\]

\[
\leq x^2 \exp(x) \exp(-x) = x^2.
\]

By the Mean Value Theorem, \( \omega_2 - \omega_1 = F'(t_0)(t_2 - t_1) \) for some \( t_0 \) in \((t_1, t_2)\). By formula (8), we then have that

\[
x = \eta(\omega_2 - \omega_1) - \frac{\eta}{\sigma} R(H(s, t_0, u, v))^{R-1}(t_2 - t_1).
\]
To simplify the rest of the argument, we replace \([t_1, t_2)\) by the infinitesimal interval \([t, t + dt)\). Then \(t = t_0 = t_1, \ t_2 = t + dt\) and formula (11) becomes

\[
x = \frac{\eta R \, dt}{\sigma H^{1-R}}.
\]

Note that \(0 < R \leq 1\) so that \(1 < 1/(1 - R) \leq \infty\). Choose \(a\) so that

\[
1 < a < \min\left(2, \frac{1}{1 - R}\right).
\]

By (10), (13), and the fact that \(x\) is infinitesimal, we have

\[
\Pr(A(g)) \leq x^2 \leq x^a.
\]

The (expected) probability of the event \((N(t, t + dt) > 1)\) is obtained from formula (14) by integrating over \(s, u, \) and \(v\) as in Theorem 3. Thus,

\[
\Pr(N(t, t + dt) > 1) \leq \frac{4\rho}{\lambda^2} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty x^a \exp(-2(u + v)/\lambda) \, ds \, du \, dv
\]

\[
= \frac{4\rho}{\lambda^2} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \left(\frac{\eta R \, dt}{\sigma H^{1-R}}\right)^a \exp(-2(u + v)/\lambda) \, ds \, du \, dv
\]

\[
= \frac{4\rho \eta^a R^a (dt)^a}{\lambda^2 \sigma^a} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty H^{aR-a} \exp(-2(u + v)/\lambda) \, ds \, du \, dv.
\]

By (13), we know that \(aR - a > -1\), and as a result the innermost integral of formula (15) converges to

\[
\int_{-\infty}^{\infty} H^{aR-a} ds = \int_{t-u-v}^{t-u-v} \left(\frac{s - u - v}{u}\right)^{aR-a} ds + \int_{t-u}^{t-u} \left(\frac{s + u - t}{u}\right)^{aR-a} ds
\]

\[
= \frac{u + v}{aR - a + 1}.
\]

Substituting this into formula (15) yields

\[
\Pr(N(t, t + dt) > 1) \leq \frac{4\rho \eta^a R^a (dt)^a}{\lambda^2 \sigma^a} \int_0^\infty \int_0^\infty \frac{u + v}{aR - a + 1} \exp(-2(u + v)/\lambda) \, du \, dv
\]

\[
= \frac{4\rho \eta^a R^a (dt)^a}{\lambda^2 \sigma^a} \cdot \frac{\lambda^3}{4(aR - a + 1)}
\]

\[
= \frac{\eta^a R^a}{\sigma^{a-1} (aR - a + 1)} (dt)^a = o(dt).
\]
The last equality above is due to the fact that $a > 1$. The theorem now follows. Note that the convergence of $\Pr(N(t_1, t_2) > 1)/(t_2 - t_1)$ to zero proceeds more slowly when $Q$ is larger. This is as one would expect since the bursts will be more intense when $Q$ is larger.

The last step is to show that Axiom 5 is satisfied.

**Theorem 6.** The probability that a data granule from $A \subseteq \Omega$ is accessed at time $t$, given that some data granule is accessed at time $t$ is

$$\frac{1}{\sigma} \int_{D \cap A} \frac{\ln(q)}{\ln(p)} \left( \frac{q}{p} \right)^{W_i} d\mu.$$

**Proof.** By Theorem 5, the probability that a set of granules is accessed during an infinitesimal time interval $(t, t + dt)$ is the same as the expected number of accesses during the same interval. By Theorem 3, the probability that a granule from $A$ is accessed during this interval is

$$\frac{4\rho \eta}{\sigma \lambda^2} \int \int \int_{D \cap A} R(H(s, t, u, v))^{R-1} e^{-2(u+v)/\lambda} ds dv du dt. \quad (16)$$

By the first part of Theorem 3, the differential of $\mu$ is

$$d\mu = \frac{4\rho}{\lambda^2} e^{-2(u+v)/\lambda} ds dv du.$$

Therefore, formula (16) can be rewritten in the form:

$$\int_{D \cap A} \frac{\eta}{\sigma} R(H(s, t, u, v))^{R-1} d\mu dt = \frac{1}{\sigma} \int_{D \cap A} \frac{\ln(q)}{\ln(p)} \left( \frac{q}{p} \right)^{W_i} d\mu \eta dt. \quad (17)$$

The last equation above follows from formula (7). By Theorems 4 and 5, the probability that anything in $D_t$ is accessed during $(t, t + dt)$ is $\eta dt$. Dividing formula (17) by this gives the desired result, and the theorem follows.

5. **Goodness of Fit**

In this section data from various sources in the literature is compared with simulations of the stochastic process. A detailed and extensive series of empirical tests is now in progress, and a report will soon be forthcoming.
Empirical studies of data access patterns predate the development of computers. Beginning with Zipf [15], a large literature has appeared in this area. Zipf proposed that the data granules be ordered by rank. The most frequently accessed data granule is given rank 1, the next most frequently is given rank 2, and so on. The rank-frequency pairs are then graphed on a log-log graph. Zipf observed that the resulting "curve" is approximately a straight line whose slope is approximately \(-1\). This distribution is called "Zipf's law."

The most striking feature of Zipf's law is its apparent ubiquity. Zipf and later researchers have studied such phenomena as the distribution of the populations of cities, the assignment of species to genera [7, 8], the usage of words in essays [12], and citations to scientific papers, to mention just a few. Most of these deal with data access patterns of various kinds, but do not consider the dynamic aspects of data access.

There have been many attempts to explain Zipf's law and to give it a theoretical foundation. Fedorowicz gives an excellent discussion of these in [3]. The implications for file storage requirements and performance are discussed in [2, 4].

For simplicity, all graphs in the rest of this paper are assumed to be log-log graphs, with horizontal axis representing the rank and vertical axis representing the frequency, as specified by Zipf. Both axes range from 1 to a power of ten, and all axis lines represent a factor of ten. The following graph was made using data from Sichel [12]. It plots the frequencies of the words used in an essay by Macaulay:

![Graph 1](image1)

By contrast, the corresponding Zipf distribution should have the graph:

![Graph 2](image2)
The real data differs in two ways from the pure Zipf distribution. First, the curve of real data is not a straight line, but rather is usually convex to some extent. The amount of curvature varies from experiment to experiment. Second, the upper left part of the graph exhibits a great deal of variance from one experiment to another.

The next graph below shows a run of the simulation algorithm. The parameters were chosen to produce 8041 accesses, with about 2000 distinct granules being accessed, as in the word occurrence graph above:

Notice that the curve is convex. The convex shape of the curve produced by the simulation algorithm is most marked when the fractal dimension is low, as seen in the following:

In this case, the upper left part of the curve will exhibit very little variance. When the fractal dimension is high, such as in the next graph, the curve will be almost straight (i.e., Zipfian), and the upper left part of the curve will exhibit larger variance:
The last graph is from data in Fedorowicz [4]:

The data here were averaged by unequal-sized groups, and then a Zipf distribution was fitted to the resulting curve. The averaging has eliminated any irregularities that may have existed in the upper left part of the curve, but the overall convex shape of the curve is still apparent.

6. PROBLEMS AND FUTURE WORK

A number of interesting questions arise from the stochastic process developed here. It was pointed out above that \( W_f \) is only one possible choice for the random variables \( V_r \). Another possibility is to choose the sum \( w = u + v \) first, and then "allocate" \( u \) and \( v \) in a second step. For example, one could take \( u = v = w/2 \). In this process, the activity level rises and falls symmetrically about the midpoint of the lifetime of a data granule. Other allocations are also possible. It would be interesting to know what form the function \( V_r \) can take in general.

Large computer systems usually keep data on very large, but relatively slow, secondary storage media. Data that is being used at any one time is kept in a faster, but smaller storage area called the "buffer" or the "cache." When a data access is needed, the system first checks to see if it is in the buffer or cache. If it is, then an expensive access to the secondary storage
area is avoided. Performance is improved if such “cache hits” occur with a high probability. Typically a buffer or cache holds a fixed number $N$ of data granules. As a result, the cache hit probability is an autocorrelation: it is the probability that a given access is the same as a previous one. For example, in the “least recently used” algorithm, the probability of a cache hit is the probability that a given access is the same as one of the $N$ most recent distinct data accesses.

Running the simulation algorithm can produce unexpected results. This is partly due to the non-Poissonian character of the model. Although the model cannot be decomposed into Poissonian and non-Poissonian parts, it is possible to estimate what fraction of the process exhibits Poissonian-like behavior and what fraction occurs in the occasional bursts. These two fractions are not disjoint and there is a part of the process that belongs to neither of them. Preliminary computations give the following formulas:

1. **Burst part.** The fraction of the process that consists of intense bursts is

   $$\sigma^{-1/Q} = \frac{\sigma^{-Q}}{Q},$$

   where $Q = \ln(p)/\ln(q)$ is the fractal dimension.

2. **Poissonian part.** The fraction of the process that is similar to a Poisson process is

   $$\frac{\Gamma(3-R)}{2^{1-R}(\eta \lambda)^R},$$

   where $R = \ln(q)/\ln(p) = 1/Q$.

Fractal phenomena have been discovered already in study of cache hits and misses in [14]. These fractal phenomena are geometric, whereas the term “fractal” in Section 3 above is probabilistic rather than geometric. It would be interesting to see if the stochastic models introduced above actually exhibit true geometric fractal behavior. This question is now being studied.

7. **Conclusion**

A new class of stochastic processes has been developed that reflects the observed behavior of computer systems and communication networks better than existing models. In this class of processes, skewness of access and
the evolution of both the database itself and the skewness are incorporated into the process in a systematic way. A general class of continuous processes as well as a specific discrete approximation are developed. The discrete approximation is both analytically tractable and easy to simulate.

REFERENCES


