Approximation properties of modified Gamma operators ✩

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Abstract

In this paper we present a survey of rates of pointwise approximation of modified Gamma operators $G_n$ for locally bounded functions and absolutely continuous functions by using some inequalities and results of probability theory with the method of Bojanic–Cheng. In the paper a kind of locally bounded functions is introduced with different growth conditions in the fields of both ends of interval $(0, +\infty)$, and it is found out that the operators have different properties compared to the Gamma operators discussed in [X.M. Zeng, Approximation properties of Gamma operators, J. Math. Anal. Appl. 311 (2005) 389–401]. And we obtain two main theorems. Theorem 1 gives an estimate for locally bounded functions which subsumes the approximation of functions of bounded variation as a special case. Theorem 2 gives an estimate for absolutely continuous functions which is best possible in the asymptotical sense.

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1. Introduction

Let $f$ be a real function defined on $(0, +\infty)$ and satisfying the following two growth conditions:

\begin{equation}
(i) \quad \left| f(t) \right| \leq M e^{\beta/t} \quad (\beta \geq 0, \ M > 0, \ t \to 0),
\end{equation}

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(ii) \( |f(t)| \leq Mt^p \quad (p \in N, \ M > 0, \ t \to +\infty)\), \hspace{1cm} (2)

where the same \( M \) may vary at its each occurrence in the context, and \( N \) is the set of natural number.

The modified Gamma operators \( G_n \) applied to \( f \) is

\[
G_n(f; x) = \int_0^{+\infty} \varphi_n(x; u)f\left(\frac{n}{u}\right) du \quad (n \geq p), \hspace{1cm} (3)
\]

where \( \varphi_n(x; u) = e^{-ux}u^n\frac{e^{n+1}}{n!} \).

In this paper we present a survey of the properties of pointwise approximation of modified Gamma operators \( G_n \) to the class of locally bounded functions \( \Phi_B \) and the absolutely continuous functions \( \Phi_{DB} \), respectively. The two classes of functions \( \Phi_B \) and \( \Phi_{DB} \) are defined as follows:

\[
\Phi_B = \{ f \mid f \text{ is bounded on every finite subinterval of } (0, +\infty) \},
\]

\[
\Phi_{DB} = \{ f \mid f(x) = \int_1^x h(t) dt + f(1); \ 0 < x < +\infty, \ h(t) \text{ is bounded on every finite subinterval of } (0, +\infty) \}.
\]

Furthermore, for a function \( f \in \Phi_B \), we introduce the following metric:

\[
\Omega_x(f; \lambda) = \sup_{t \in [x-\lambda, x+\lambda]} |f(t) - f(x)|, \quad x \in (0, +\infty), \ 0 \leq \lambda < x, \hspace{1cm} (4)
\]

where \( x \in (0, +\infty) \) is fixed, \( \lambda \geq 0 \). The properties of \( \Omega_x(f; \lambda) \) can be founded in [2].

The modified Gamma operators \( G_n \) have been studied in [1] by use of the equivalence of the K-functional and moduli of smoothness. In article [2], the classical Gamma operators is studied. Here the modified Gamma operators can apply to a function which is unbounded in both ends of its domain of definition \((0, +\infty)\).

2. Approximation for locally bounded functions

In this section we study the rate of convergence of modified Gamma operator \( G_n \) for \( f \in \Phi_B \), our main result is as follows.

**Theorem 1.** Let \( f \in \Phi_B \) and let \( f \) satisfy the conditions (1) and (2). If \( f(x+) \) and \( f(x-) \) exist at a fixed point \( x \in (0, +\infty) \), then for \( n \geq \max\{2p, \frac{2p+1}{x}\} \) we have

\[
|G_n(f; x) - \frac{f(x+) + f(x-)}{2} + \frac{2(f(x+) - f(x-))}{3\sqrt{2\pi n}}| \leq \frac{10}{n} \sum_{k=1}^{n} \Omega_x\left(g_x; \frac{x}{\sqrt{k+1}}\right) + O\left(\frac{1}{n}\right), \hspace{1cm} (5)
\]

where

\[
g_x(t) = \begin{cases} 
  f(t) - f(x+), & x < t < +\infty; \\
  0, & t = x; \\
  f(t) - f(x-), & 0 < t < x.
\end{cases} \hspace{1cm} (6)
\]
We point out that Theorem 1 subsumes the case of approximation of functions of bounded variation, from Theorem 1 we get immediately that

**Corollary 1.** Let \( f \) be a function of bounded variation on every subinterval of \((0, +\infty)\) and let \( f(t) \) satisfy the conditions (1) and (2). Then for a fixed \( x \in (0, +\infty) \) for \( n \geq \max\{2p, \frac{2\beta+1}{x}\} \) we have

\[
\left| G_n(f; x) - \frac{f(x+) + f(x-)}{2} + \frac{2(f(x+) - f(x-))}{3\sqrt{2\pi n}} \right|
\leq \frac{10}{n} \sum_{k=1}^{n} \frac{x^+}{\sqrt{x^+ + 1}} (g_x) + O\left(\frac{1}{n}\right),
\]

where \( \int_a^b f \) indicates the total variation of \( f \) on \([a, b]\).

**Corollary 2.** Under the conditions of Theorem 1, if \( \Omega_x(g_x; \lambda) = o(\lambda) \), then we have

\[
G_n(f; x) = \frac{f(x+) + f(x-)}{2} - \frac{2}{3\sqrt{2\pi n}} (f(x+) - f(x-)) + o\left(\frac{1}{\sqrt{n}}\right).
\]

To prove Theorem 1 we need some preliminary results.

**Lemma 1.** For \( x \in (0, +\infty) \), \( k \) is a positive integer and \( 0 \leq k \leq p \), we have

\[
G_n(t^k; x) = \frac{n^k}{n(n-1)(n-2)\cdots(n-k+1)} x^k.
\]

**Proof.** Direct computation gives

\[
G_n(t^k; x) = \frac{x^n + 1}{n!} \int_0^{+\infty} e^{-u} u^n \frac{n^k}{y^k} du = \frac{x^n n^k}{n!} \int_0^{+\infty} e^{-u} u^{-k} du = \frac{n^k \Gamma(n-k+1)}{n!} x^k = \frac{n^k}{n(n-1)(n-2)\cdots(n-k+1)} x^k.
\]

Lemma 1 is proved. \( \square \)

**Lemma 2.** For \( x \in (0, +\infty) \), we have

\[
G_n((t-x)^4; x) = \frac{3n + 18}{(n-1)(n-2)(n-3)} x^4,
\]

\[
G_n((t-x)^6; x) = \frac{15n^2 + 430n + 600}{(n-1)(n-2)\cdots(n-5)} x^6,
\]

\[
G_n(e^{2\beta/t}; x) \leq e^{4\beta/t} \quad (nx - 1 \geq 2\beta).
\]
Proof. We prove the last inequality only since the other equalities are easy to obtain through direct computation by Lemma 1.

\[ G_n(e^{2\beta/t}; x) = \frac{x^{n+1}}{n!} \int_0^{+\infty} u^n e^{-u} e^{2\beta u/n} du = \left(1 + \frac{2\beta}{nx - 2\beta}\right)^{n+1} \]

\[ = \left(1 + \frac{2\beta}{nx - 2\beta}\right)^{\frac{nx - 2\beta}{2\beta} \frac{(n+1)2\beta}{nx - 2\beta}} \leq e^{4\beta/x}. \]

Lemma 2 is proved. \( \square \)

3. Proof of Theorem 1

Let \( f \) satisfy the conditions of Theorem 1, then \( f \) can be expressed as

\[ f(t) = \frac{f(x+) + f(x-)}{2} + g_x(t) + \frac{f(x+) - f(x-)}{2} \text{sign}(t - x) \]

\[ + \delta_x(t) \left[ f(x) - \frac{f(x+) + f(x-)}{2} \right], \quad (13) \]

where \( g_x(t) \) is defined in (6), \( \text{sign}(t) \) is sign function and

\[ \delta_x(t) = \begin{cases} 1, & t = x; \\ 0, & t \neq x. \end{cases} \quad (14) \]

Obviously,

\[ G_n(\delta_x; x) = 0. \quad (15) \]

In order to use Lemma 3 in [2] to estimate \( G_n(\text{sign}(t - x); x) \), we need to introduce a series of random variables. Let \( \{\xi_k\}_{k=1}^{+\infty} \) be a sequence of independent random variables with the same Gamma distribution and their probability density functions are

\[ P_{\xi_i}(t) = \begin{cases} xe^{-xt}, & t \geq 0; \\ 0, & t < 0, \end{cases} \]

where \( x \in (0, +\infty) \) is a parameter, and \( i = 1, 2, \ldots, n \). Then by direct computations we get

\[ E(\xi_1) = \frac{1}{x}, \quad \sigma^2 := E(\xi_1 - E(\xi_1))^2 = \frac{1}{x^2} \quad (\sigma > 0), \]

\[ E(\xi_1 - E(\xi_1))^3 = \frac{2}{x^3}, \quad E(\xi_1 - E(\xi_1))^4 = \frac{9}{x^4}. \]

Let \( \eta_{n+1} = \sum_{k=1}^{n+1} \xi_k \). \( F_{n+1}(t) \) indicates the distribution of the random variable \( \sum_{k=1}^{n+1} (\xi_k - E(\xi_i))/\sigma \sqrt{n + 1} \), the distribution of \( \eta_{n+1} \) is

\[ P(\eta_{n+1} \leq y) = \frac{x^{n+1}}{\Gamma(n+1)} \int_0^y u^n e^{-ux} du. \]

Direct calculation shows that \( E\eta_{n+1} = \frac{n+1}{x} \). Therefore
\[ G_n \left( \text{sign}(t - x); x \right) \]
\[ = \frac{x^{n+1}}{\Gamma(n+1)} \int_0^\infty t^n e^{-tx} \, dt - \frac{x^{n+1}}{\Gamma(n+1)} \int_{\frac{n}{x}}^\infty t^n e^{-tx} \, dt \]
\[ = 2 \left( P\left( \eta_{n+1} + \frac{n+1-x}{\sqrt{n+1}x} \leq \frac{-1}{\sqrt{n+1}} \right) - 1 \right) - 2F_{n+1} \left( \frac{-1}{\sqrt{n+1}} \right) - 1 \]
\[ = 2 \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{1}{\sqrt{n+1}}} e^{-\frac{t^2}{2}} \, dt + \frac{2/\sqrt{n+1}}{6\sqrt{n+1}} \frac{1}{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2(n+1)}} + O\left( \frac{1}{n} \right) \right) - 1 \]
\[ = -\frac{4}{3\sqrt{2\pi n}} + O\left( \frac{1}{n} \right), \tag{16} \]

where the forth equality is obtained by use of Lemma 3 in [2], and the last equality is obtained by direct computation.

Thus combining (13), (15), (16) we have the following result:

\[ \left| G_n(f; x) - \frac{f(x+)+f(x-)}{2} - \frac{2(f(x+)-f(x-))}{3\sqrt{2\pi n}} \right| \leq \left| G_n(g_x; x) \right| + O\left( \frac{1}{n} \right). \tag{17} \]

Let

\[ K_{n+1}(x; t) = P(\eta_{n+1} \leq t) = \frac{x^{n+1}}{\Gamma(n+1)} \int_0^t u^n e^{-ux} \, du, \]

and the Lebesgue–Stieltjes integral presentation of \( G_n \) can be written as follows:

\[ G_n(g_x; x) = \int_0^{+\infty} g_x \left( \frac{n}{t} \right) d_t K_{n+1}(x; t). \]

We decompose \( |G_n(g_x; x)| \) into four parts as follows:

\[ G_n(g_x; x) = \Delta_{1,n}(g_x) + \Delta_{2,n}(g_x) + \Delta_{3,n}(g_x) + \Delta_{4,n}(g_x), \tag{18} \]

where

\[ \Delta_{1,n}(g_x) = \int_0^{\frac{2(n+1)}{3\sqrt{n+1}}} g_x \left( \frac{n}{t} \right) d_t K_{n+1}(x; t), \]

\[ \Delta_{2,n}(g_x) = \int_{\frac{2(n+1)}{3\sqrt{n+1}}}^{\frac{n+1+\sqrt{n+1}}{3x}} g_x \left( \frac{n}{t} \right) d_t K_{n+1}(x; t), \]

\[ \Delta_{3,n}(g_x) = \int_{\frac{n+1+\sqrt{n+1}}{3x}}^{\frac{n+1+\sqrt{n+1}}{3x}} g_x \left( \frac{n}{t} \right) d_t K_{n+1}(x; t), \]

\[ \Delta_{4,n}(g_x) = \int_{\frac{n+1+\sqrt{n+1}}{3x}}^{+\infty} g_x \left( \frac{n}{t} \right) d_t K_{n+1}(x; t). \]
For \( f \in \Phi_B \) satisfying the conditions (1) and (2), equivalently we have
\[
\begin{align*}
\left| f(t) \right| & \leq M e^{\beta/t} \quad (0 < t \leq 1); \\
\left| f(t) \right| & < Mt^p \quad (t \geq 1),
\end{align*}
\]
where \( \beta \geq 0 \), \( M > 0 \) and \( p \in N \). Then the estimate of \( |\Delta_{1,n}(g_x)| \) is given in two cases:

(1) when \( \frac{2(n+1)}{3x} \leq n \) for \( n \geq 2p \), then
\[
|\Delta_{1,n}(g_x)| \leq M \int_0^{\frac{2(n+1)}{3x}} \frac{n^p}{t^p} d_i K_{n+1}(x; t)
\]
\[
\leq M \left( \frac{n-2}{2(n+1)x} \right)^2 \int_0^{\frac{2(n+1)}{3x}} \left( \frac{n}{t} - x \right)^2 \frac{n^p}{t^p} d_i K_{n+1}(x; t)
\]
\[
\leq M \left( \frac{n-2}{2(n+1)x} \right)^2 \left( \int_0^{\frac{2(n+1)}{3x}} \left( \frac{n}{t} - x \right)^2 dt \right)^{1/2} \left( \int_0^{\frac{2(n+1)}{3x}} \frac{n^p}{t^p} dt \right)^{1/2}
\]
\[
= M \left( \frac{n-2}{2(n+1)x} \right)^2 \sqrt{\frac{3n+18}{(n-1)(n-2)(n-3)x^2}}
\]
\[
\times \left( \frac{n^p}{(n(n-1)(n-2)(n-3))^2} \right)^{1/2}
\]
\[
\leq M \frac{n^p}{n x^p}, \quad (19)
\]
where the forth inequality is obtained by Schwarz’s inequality, and the fifth equality is obtained by Lemma 2;

(2) when \( \frac{2(n+1)}{3x} > n \) for \( n \geq \max\{2p, \frac{2\beta + 1}{x}\} \), then
\[
\Delta_{1,n}(g_x) = \int_0^n g_x \left( \frac{n}{t} \right) d_i K_{n+1}(x; t) + \int_n^\infty g_x \left( \frac{n}{t} \right) d_i K_{n+1}(x; t). \quad (20)
\]

The method to the estimation of absolute value of the first term of (20) is the same as in the proof of (19), here we write out the result as follows:
\[
\left| \int_0^n g_x \left( \frac{n}{t} \right) d_i K_{n+1}(x; t) \right| \leq \frac{M}{n(1-x)^2} x^{p+2}. \quad (21)
\]
Using the same method as in the proof of (19) and in view of (12), we have the estimate of the second term of (20) as follows:
\[
\left| \int_{n}^{\frac{n+1}{x}} g_x \left( \frac{n}{t} \right) \, dt \right| K_{n+1}(x; t) \leq M \int_{n}^{\frac{n}{x}} e^{\frac{\beta t}{n}} \int_{n}^{\frac{n+1}{x}} \left| g_x - g_x(x) \right| \, dt K_{n+1}(x; t) \leq \frac{M}{n} e^{\frac{2\beta}{n}}. \tag{22}
\]

From (20)–(22) we have
\[
\left| \Delta_{1,n}(g_x) \right| \leq \frac{M}{n} e^{\frac{2\beta}{n}} + \frac{M}{n(1-x)^2} x^{p+2}. \tag{23}
\]

Since \( |g_x(x)| = 0 \) and \( \int_{0}^{+\infty} d_t K_{n+1}(x; t) = 1 \), we obtain
\[
\left| \Delta_{3,n}(g_x) \right| \leq \left| \int_{\frac{n+1}{x} + \sqrt{\frac{n+1}{x}}}^{\frac{n+1}{x} - \sqrt{\frac{n+1}{x}}} \left( g_x \left( \frac{n}{t} \right) - g_x(x) \right) \, dt K_{n+1}(x; t) \right|
\]
\[
\leq \left| \int_{\frac{n+1}{x} + \sqrt{\frac{n+1}{x}}}^{\frac{n+1}{x} - \sqrt{\frac{n+1}{x}}} \Omega_x \left( g_x - g_x(x) \right) \, dt K_{n+1}(x; t) \right|
\]
\[
\leq \left| \int_{\frac{n+1}{x} + \sqrt{\frac{n+1}{x}}}^{\frac{n+1}{x} - \sqrt{\frac{n+1}{x}}} \Omega_x \left( g_x - g_x(x) \right) \, dt K_{n+1}(x; t) \right|
\]
\[
\leq \Omega_x \left( g_x - g_x(x) \right) \leq \frac{1}{n} \sum_{k=1}^{n} \Omega_x \left( g_x - g_x(x) \right). \tag{24}
\]

Suppose \( 0 < t \leq \frac{n+1}{x} \), and note \( E(\eta_{n+1} - \frac{n+1}{x}) = \frac{n+1}{x^2} \), by use of Chebyshev’s inequality we have
\[
K_{n+1}(x; t) = P(\eta_{n+1} - \frac{n+1}{x} \leq t) = P \left( \left| \eta_{n+1} - \frac{n+1}{x} \right| \geq \frac{n+1}{x} - t \right)
\]
\[
\leq \frac{E(\eta_{n+1} - \frac{n+1}{x})^2}{\left( \frac{n+1}{x} - t \right)^2} = \frac{n+1}{(n+1-tx)^2}. \tag{25}
\]

Next we estimate \( |\Delta_{2,n}(g_x)| \). Note \( g_x(x) = 0 \), using integration by parts we have
\[
\left| \Delta_{2,n}(g_x) \right| = \left| \int_{\frac{n+1}{x} + \sqrt{\frac{n+1}{x}}}^{\frac{n+1}{x} - \sqrt{\frac{n+1}{x}}} \left( g_x \left( \frac{n}{t} \right) - g_x(x) \right) \, dt K_{n+1}(x; t) \right|
\]
\[
\leq \int_{\frac{n+1}{x} + \sqrt{\frac{n+1}{x}}}^{\frac{n+1}{x} - \sqrt{\frac{n+1}{x}}} \Omega_x \left( g_x - g_x(x) \right) \, dt K_{n+1}(x; t) \tag{26}
\]
\[
\leq \int_{\frac{n+1}{x} + \sqrt{\frac{n+1}{x}}}^{\frac{n+1}{x} - \sqrt{\frac{n+1}{x}}} \Omega_x \left( g_x - g_x(x) \right) \, dt K_{n+1}(x; t) \tag{27}
\]
\[= \Omega_x \left( g_x; \frac{n}{t} - x \right) K_{n+1}(x; t) \left|_{\frac{n+1}{x} - \frac{\sqrt{n+1}}{x}}^{\frac{\sqrt{n+1}}{x}} \right. + \int_{\frac{2}{3} \frac{n+1}{x}}^{\frac{n+1}{x} - \frac{\sqrt{n+1}}{x}} K_{n+1}(x; t) \, dt \left( -\Omega_x \left( g_x; \frac{n}{t} - x \right) \right). \] (28)

Note that \( K_{n+1}(x; t) \leq 1 \), it follows that

\[\Omega_x \left( g_x; \frac{n}{t} - x \right) K_{n+1}(x; t) \left|_{\frac{n+1}{x} - \frac{\sqrt{n+1}}{x}}^{\frac{\sqrt{n+1}}{x}} \right. + \int_{\frac{2}{3} \frac{n+1}{x}}^{\frac{n+1}{x} - \frac{\sqrt{n+1}}{x}} K_{n+1}(x; t) \, dt \left( -\Omega_x \left( g_x; \frac{n}{t} - x \right) \right) \leq \Omega_x \left( g_x; \frac{x}{\sqrt{n+1}} \right). \] (29)

Using inequality (25) we obtain

\[\int_{\frac{2}{3} \frac{n+1}{x}}^{\frac{n+1}{x} - \frac{\sqrt{n+1}}{x}} K_{n+1}(x; t) \, dt \left( -\Omega_x \left( g_x; \frac{n}{t} - x \right) \right) \leq \int_{\frac{2}{3} \frac{n+1}{x}}^{\frac{n+1}{x} - \frac{\sqrt{n+1}}{x}} \frac{n+1}{(n+1-tx)^2} \, dt \left( -\Omega_x \left( g_x; \frac{n}{t} - x \right) \right). \] (30)

and using integration by parts to (30), it turns into

\[\frac{n+1}{(n+1-tx)^2} \left( -\Omega_x \left( g_x; \frac{n}{t} - x \right) \right) \left|_{\frac{2}{3} \frac{n+1}{x}}^{\frac{n+1}{x} - \frac{\sqrt{n+1}}{x}} \right. + 2(n+1)x \int_{\frac{2}{3} \frac{n+1}{x}}^{\frac{n+1}{x} - \frac{\sqrt{n+1}}{x}} \Omega_x \left( g_x; \frac{n}{t} - x \right) \, dt. \] (31)

By easy calculation to the first term of (31) becomes

\[-\Omega_x \left( g_x; \frac{x}{\sqrt{n+1}} \right) + \frac{4}{n+1} \Omega_x \left( g_x; \frac{n-2}{2(n+1)} x \right) \]

\[\leq -\Omega_x \left( g_x; \frac{x}{\sqrt{n+1}} \right) + \frac{4}{n+1} \Omega_x \left( g_x; \frac{1}{2} x \right), \] (32)

and the second term of (31) becomes

\[\frac{1}{n+1} \int_{\frac{1}{3} \frac{n+1}{x}}^{\frac{n+1}{x} - \frac{\sqrt{n+1}}{x}} \Omega_x \left( g_x; \frac{\frac{1}{\sqrt{n+1}} - \frac{1}{n+1}}{1 - \frac{1}{\sqrt{n}}} x \right) \, du. \] (33)

Combining (28)–(33) we have
Finally, in order to calculate the top boundary of \(|\Delta_{2,n}(g_x)|\), we decompose \(\Delta_{2,n}(g_x)\) into two parts as follows:

\[
\Delta_{2,n}(g_x) = \int_{\frac{n+1}{x} + \sqrt{\frac{n+1}{x}}}^{\frac{2(n+1)}{x}} g_x \left( \frac{n}{t} \right) d_t K_{n+1}(x; t) + \int_{\frac{2(n+1)}{x}}^{+\infty} g_x \left( \frac{n}{t} \right) d_t K_{n+1}(x; t) := I_1 + I_2.
\]

Using the similar method as in the estimation of \(|\Delta_{2,n}(g_x)|\), we have

\[
|I_1| \leq \frac{2}{n+1} \Omega_x \left( g_x; \frac{n+2}{2(n+1)} x \right) + \frac{1}{n+1} \int_{1}^{\frac{n+1}{x} + \sqrt{\frac{n+1}{x}}} \Omega_x \left( g_x; \frac{n+1}{1 + \frac{1}{\sqrt{u}}} x \right) du
\]

\[
\leq \frac{2}{n+1} \Omega_x \left( g_x; \frac{1}{\sqrt{2}} x \right) + \frac{1}{n+1} \int_{1}^{2} \Omega_x \left( g_x; \frac{n+1}{1 + \frac{1}{\sqrt{u}}} x \right) du
\]

\[
+ \frac{1}{n+1} \int_{2}^{n+1} \Omega_x \left( g_x; \frac{1}{\sqrt{u}} x \right) du
\]

\[
\leq \frac{4}{n+1} \sum_{k=1}^{n} \Omega_x \left( g_x; \frac{1}{\sqrt{k+1}} x \right)
\]

\[
\leq \frac{4}{n} \sum_{k=1}^{n} \Omega_x \left( g_x; \frac{1}{\sqrt{k+1}} x \right).
\]

(34)

As for the estimate of \(|I_2|\), we only write out the result since the proof of the result is almost the same as that of \(|\Delta_{1,n}(g_x)|\). Namely

1) when \(\frac{2(n+1)}{x} \geq n\) for \(n \geq \frac{2\beta+1}{x}\), then

\[
|I_2| = \left| \int_{\frac{2(n+1)}{x}}^{+\infty} g_x \left( \frac{n}{t} \right) d_t K_{n+1}(x; t) \right| \leq \frac{M}{n} e^{\beta/x}.
\]

(35)
(2) when \( \frac{2(n+1)}{x} < n \) for \( n \geq \max\{2p, \frac{2\beta+1}{x}\} \), then

\[
|I_2| = \left| \int_{\frac{2(n+1)}{x}}^{+\infty} g_x \left( \frac{n}{t} \right) d_t K_{n+1}(x; t) \right| \leq \frac{M}{n} x^p + \frac{M}{n(x-1)^2} e^{2\beta/x}. \tag{38}
\]

Thus, by (17)–(19), (23), (24), (34), (36)–(38), Theorem 1 is proved.

4. Approximation for some absolutely continuous functions

In this section we study the rate of convergence of modified Gamma operators \( G_n \) for function \( f \in \Phi_{DB} \). Similarly, we write out the main result:

**Theorem 2.** Let \( f \in \Phi_{DB} \) and let \( f \) satisfy the conditions (1) and (2). If \( h(x+) \) and \( h(x-) \) exist at a fixed point \( x \in (0, +\infty) \), then for \( n \geq \max\{2p, \frac{2\beta+1}{x}\} \) we have

\[
|G_n(f; x) - f(x)| \leq \left| \frac{\tau}{\sqrt{2\pi n}} \right| + \frac{10x}{n} \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \Omega_x \left( \Phi_x; \frac{x}{k+1} \right) + \frac{\tau x + M(x^p + e^{2\beta/x})}{n^{\frac{3}{2}}}, \tag{39}
\]

where \( \tau = h(x+) - h(x-) \), \([a]\) indicates the integer part of \( a \), and

\[
\Phi_x(t) = \begin{cases} 
  h(t) - h(x+), & x < t < +\infty; \\
  0, & t = x; \\
  h(t) - h(x-), & 0 \leq t < x.
\end{cases} \tag{40}
\]

Similarly, some elementary lemmas are useful for the prove of Theorem 2.

**Lemma 3.** Let \( x \in (0, +\infty) \), we have

\[
G_n(|t-x|; x) = \frac{2x n! e^{-n}}{n^n}. \tag{41}
\]

Since the proof is similar to Lemma 4 in [2], here we omit it.

**Corollary 3.** [2, Corollary 3] Let \( x \in (0, +\infty) \), we have

\[
\left| G(|t-x|; x) - \sqrt{\frac{2}{n\pi}} x \right| \leq \frac{x}{15n^{\frac{3}{2}}}. \tag{42}
\]

Estimation (42) is the best possible, that is to say, it cannot be asymptotically improved.

**Proof.** By Lemma 3 and using Stirling’s formula:

\[
n! = \sqrt{2\pi n} \left( \frac{n}{e} \right)^n e^n, \quad (12n + 1)^{-1} < c_n < (12n)^{-1},
\]
we obtain
\[ \sqrt{\frac{2}{n\pi}} x - G_n(|t - x|; x) = \sqrt{\frac{2}{n\pi}} x (1 - e^{c_n}) \]
and by direct computations we have
\[ \sqrt{\frac{2}{\pi}} \frac{x}{15n^3} \leq \sqrt{\frac{2}{n\pi}} x (1 - e^{c_n}) \leq \frac{x}{15n^3}, \quad (43) \]
and Corollary 3 is proved. \( \Box \)

By direct computation we obtain
\[ G_n(f; x) - f(x) = \frac{h(x+) - h(x-)}{2} G_n(|t - x|; x) + L_{n,x}(\phi_x) - R_{n,x}(\phi_x) - T_{n,x}(\phi_x), \quad (44) \]
where
\[ L_{n,x}(\phi_x) = \int_0^{\frac{n}{x}} \left( \int_0^{\frac{n}{x}} \phi_x(u) \, du \right) d_t K_{n+1}(x; t), \quad (45) \]
\[ R_{n,x}(\phi_x) = \int_{\frac{2n}{x}}^{\frac{2n}{x}} \left( \int_{\frac{n}{x}}^{\frac{2n}{x}} \phi_x(u) \, du \right) d_t K_{n+1}(x; t), \quad (46) \]
\[ T_{n,x}(\phi_x) = \int_{\frac{2n}{x}}^{+\infty} \left( \int_{\frac{n}{x}}^{\frac{2n}{x}} \phi_x(u) \, du \right) d_t K_{n+1}(x; t). \quad (47) \]
The integral (45) can be decomposed into two parts as follows:
\[ L_{n,x}(\phi_x) = \int_0^{\frac{n}{x}} \left( \int_0^{\frac{n}{x}} \phi_x(u) \, du \right) d_t K_{n+1}(x; t) + \int_{\frac{n}{x}}^{+\infty} \left( \int_0^{\frac{n}{x}} \phi_x(u) \, du \right) d_t K_{n+1}(x; t), \quad (48) \]
Using integration by parts to the second term of (48), it equals to
\[
\int_x^x \phi_x(u) \, du K_{n+1}(x; t) \left|_{-\frac{n}{2t}}^{\frac{n}{2t}} \right. + n \int_{\frac{n}{2x}}^{\frac{n}{x}} K_{n+1}(x; t) \frac{n}{t^2} \phi_x \left( \frac{n}{t} \right) \, dt \\
= -\int_x^x \phi_x(u) \, du K_{n+1}(x; \frac{n}{2x}) + \int_x^x K_{n+1}(x; \frac{n}{y}) \phi_x(y) \, dy. \quad (49) \]
The second term of (49) is obtained by putting \( y = \frac{n}{t} \), and it can be decomposed into two parts as follows:

\[
\int_{x + \frac{x}{\sqrt{n+1}}}^{x + \frac{x}{\sqrt{n+1}}} K_{n+1}\left( x; \frac{n}{y} \right) \phi_x(y) \, dy + \int_{x + \frac{x}{\sqrt{n+1}}}^{2x} K_{n+1}\left( x; \frac{n}{y} \right) \phi_x(y) \, dy := A_1(x) + A_2(x). \tag{50}
\]

Then

\[
\left| A_1(x) \right| \leq \int_{x + \frac{x}{\sqrt{n+1}}}^{x + \frac{x}{\sqrt{n+1}}} \Omega_x(\phi_x; y - x) \, dy \leq \frac{x}{\sqrt{n+1}} \Omega_x(\phi_x; \frac{x}{\sqrt{n+1}})
\leq \frac{x}{\sqrt{n}} \sum_{k=1}^{[\sqrt{n}]} \Omega_x(\phi_x; \frac{x}{k+1}). \tag{51}
\]

Decompose \( A_2(x) \) into two parts

\[
A_2(x) = \int_{x + \frac{x}{\sqrt{n+1}}}^{\frac{3}{2}x} K_{n+1}\left( x; \frac{n}{y} \right) \phi_x(y) \, dy + \int_{\frac{3}{2}x}^{2x} K_{n+1}\left( x; \frac{n}{y} \right) \phi_x(y) \, dy. \tag{52}
\]

We have

\[
\left| \int_{\frac{3}{2}x}^{2x} K_{n+1}\left( x; \frac{n}{y} \right) \phi_x(y) \, dy \right| \leq \int_{\frac{3}{2}x}^{2x} K_{n+1}\left( x; \frac{n}{\frac{3}{2}x} \right) \left| \phi_x(y) \right| \, dy \leq \frac{5x}{n} \sup_{t \in \left[ \frac{3}{2}x, 2x \right]} \left| \phi_x(t) \right|. \tag{53}
\]

And to the first term of (52), we can use the inequality (25) and replace \( y \) by \( x + \frac{x}{t} \), then the term

\[
\int_{x + \frac{x}{\sqrt{n+1}}}^{\frac{3}{2}x} K_{n+1}\left( x; \frac{n}{y} \right) \phi_x(y) \, dy \leq \int_{x + \frac{x}{\sqrt{n+1}}}^{\frac{3}{2}x} \frac{n+1}{(n + 1 - \frac{nx}{y})^2} \Omega_x(\phi_x; y - x) \, dy
\leq x \int_{2}^{\frac{n+1}{2}} \frac{n+1}{(n + 1 - \frac{nt}{t+1})^2} \Omega_x\left( \phi_x; \frac{x}{t} \right) \, dt
\leq \frac{n+1}{n} \sum_{k=1}^{[\sqrt{n}]} \Omega_x\left( \phi_x; \frac{x}{k+1} \right). \tag{54}
\]

Noting (25), it follows that

\[
K_{n+1}\left( x; \frac{n}{2x} \right) \leq \frac{n+1}{(n + 1 - \frac{n}{2})^2} \leq \frac{5}{n+1}. \tag{55}
\]
By (55) and easy evaluations of the first term of (49) gives
\[
\left| - \int_{\frac{x}{2}}^{2x} \phi_x(u) \, du \, K_{n+1} \left( x; \frac{n}{2x} \right) \right| \leq \frac{5x}{n} \sup_{t \in [x, 2x]} |\phi_x(t)|. \tag{56}
\]

From (48)–(56) it follows that
\[
|L_{n,x}(\phi_x)| \leq \left| \int_{0}^{\frac{n}{t_p}} \left( \int_{\frac{x}{2}}^{x} \phi_x(u) \, du \right) \, d_t \, K_{n+1} \left( x; \frac{n}{t} \right) \right| + \frac{9x}{n} \sum_{k=1}^{[\sqrt{n}]} \Omega_x \left( \phi_x; \frac{x}{k+1} \right) \\
+ \frac{10x}{n} \sup_{t \in [x, 2x]} |\phi_x(t)|. \tag{57}
\]

For the first term of (57) that is denoted by \( \Lambda \), using the same method as in the estimation of \( |\Delta_{1,n}(g_x)| \), we have
(1) when \( \frac{n}{2x} < n \Rightarrow x > \frac{1}{2} \) for \( n \geq \frac{2\beta+1}{x} \), then
\[
\Lambda \leq M \int_{0}^{n^{p/2}} d_t \, K_{n+1} \left( x; \frac{n}{t} \right) \leq \frac{M}{x^3} \int_{0}^{n^{p/2}} \left( \frac{n}{t} - x \right) \, d_t \, K_{n+1} \left( x; \frac{n}{t} \right) \\
\leq \frac{M}{x^3} \int_{0}^{\frac{n^{p/2}}{t_p}} \left( \frac{n}{t} - x \right)^6 \, d_t \, K_{n+1} \left( x; \frac{n}{t} \right) \\
= \frac{M}{x^3} \left( \frac{15n^2 + 430n + 600}{(n-1)(n-2) \cdots (n-5)} x^6 \right)^{\frac{1}{2}} \left( \frac{n^{2p}}{t^{2p}} \right) \, d_t \, K_{n+1} \left( x; \frac{n^{2p}}{t^{2p}} \right) \\
\leq \frac{M}{n^2 x^p}, \tag{58}
\]
where the last equality is obtained by Lemma 2;
(2) when \( \frac{n}{2x} > n \Rightarrow x < \frac{1}{2} \) for \( n \geq \max\{2p, \frac{2\beta+1}{x}\} \), then
\[
\Lambda \leq \frac{M}{n^2 (1-x)^3} x^{p+3} + \frac{M}{n^2} e^{2\beta/x}. \tag{59}
\]

Donate
\[
M_{1x} = \left( \frac{M}{n^{3/2}} x^p \right) \chi_{(1/2, +\infty)} + \left( \frac{M}{n^{3/2} (1-x)^3} x^{p+3} + \frac{M}{n^{3/2}} e^{2\beta/x} \right) \chi_{(0, 1/2)},
\]
where \( \chi_I \) indicates the characteristic function of interval \( I \).

By (57)–(59) we have
\[
|L_{n,x}(\phi_x)| \leq \frac{9x}{n+1} \sum_{k=1}^{[\sqrt{n}]} \Omega_x \left( \phi_x; \frac{x}{k+1} \right) + \frac{10x}{n} \sup_{t \in [x, 2x]} |\phi_x(t)| + M_{1x}. \tag{60}
\]
It is easy to find out that
\[ M_{1x} \leq \frac{M}{n^2} x^p + \frac{M}{n^2} e^{2\beta/x}. \]

In the same way as in the proof of (58)–(60) we can obtain

(1) when \( \frac{2n}{x} \geq n \Rightarrow x \leq 2 \) for \( n \geq \frac{2\beta+1}{x} \), then
\[ |T_{n,x}(\phi_x)| \leq \frac{M}{n^2} e^{\frac{2\beta}{x}} \quad (61) \]

(2) when \( \frac{2n}{x} < n \Rightarrow x > 2 \) for \( n \geq \max\{2p, \frac{2\beta+1}{x}\} \), then
\[ |T_{n,x}(\phi_x)| \leq \frac{M}{n^2} x^p + \frac{M}{n^2 (x-1)} e^{\frac{2\beta}{x}}. \quad (62) \]

Donate
\[ M_{2x} = \left( \frac{M}{n^2} e^{\frac{2\beta}{x}} \right) \chi(0,2) + \left( \frac{M}{n^2} x^p + \frac{M}{n^2 (x-1)} e^{\frac{2\beta}{x}} \right) \chi(2,\infty), \]

therefore
\[ |T_{n,x}(\phi_x)| \leq M_{2x}, \quad (63) \]

also it is easy to find out
\[ M_{2x} \leq \frac{M}{n^2} x^p + \frac{M}{n^2} e^{\frac{2\beta}{x}}. \]

For the estimation of \( |R_{n,x}(\phi_x)| \), we define a new kernel as \( K'_{n,x}(x; t) = 1 - K_{n+1}(x; t) \), if \( t \geq \frac{n+1}{x} \), then
\[ K'_{n,x}(x; t) = P(\eta_{n+1} \geq t) \leq P \left( \left| \eta_{n+1} - \frac{n+1}{x} \right| \geq t - \frac{n+1}{x} \right) \]
\[ \leq \frac{E(\eta_{n+1} - \frac{n+1}{x})^2}{(n+1-tx)^2} = \frac{n+1}{(n+1-tx)^2}. \quad (64) \]

Integrating by parts to \( R_{n,x}(\phi_x) \), we have
\[ |R_{n,x}(\phi_x)| = \left| \int_{\frac{n}{t}}^{\frac{2n}{x}} \left( \int_{\frac{n}{t}}^{x} \phi_x(u) \, du \right) d_t K'_{n,x}(x; t) \right| \quad (65) \]
\[ = \left| \int_{\frac{n}{t}}^{x} \phi_x(u) \, d_u K'_{n,x}(x; t) \bigg|_{\frac{n}{t}}^{\frac{2n}{x}} + n \int_{\frac{n}{t}}^{\frac{2n}{x}} \frac{K'_{n,x}(x; t)}{t^2} \phi_x \left( \frac{n}{t} \right) \, dt \right| \quad (66) \]
\[ = \left| \int_{\frac{n}{x}}^{x} \phi_x(u) \, d_u K'_{n,x} \left( x; \frac{2n}{x} \right) \right| + \left| \int_{\frac{n}{y}}^{x} K'_{n,x} \left( x; \frac{n}{y} \right) \phi_x(y) \, dy \right|. \quad (67) \]
By (64) and the first term of (67)

\[
\left| \int_{\frac{x}{2}}^{x} \phi_x(u) \, du \, K'_{n,x}(x; \frac{2n}{x}) \right| \leq \frac{3x}{2(n+1)} \Omega_x \left( \phi_x; \frac{x}{2} \right) \leq \frac{3x}{2n} \sum_{k=1}^{\left\lceil \sqrt{n} \right\rceil} \Omega_x \left( \phi_x; \frac{x}{k+1} \right). \tag{68}
\]

On the other hand the second term of (67)

\[
\left| \int_{\frac{x}{2}}^{x} K'_{n,x}(x; \frac{n}{y}) \phi_x(y) \, dy \right| \leq \int_{\frac{x}{2}}^{x} K'_{n,x}(x; \frac{n}{y}) \Omega_x(\phi_x; x-y) \, dy \tag{69}
\]

\[
= \int_{x-\frac{x}{\sqrt{n+1}}}^{x} K'_{n,x}(x; \frac{n}{y}) \Omega_x(\phi_x; x-y) \, dy
\]

\[
+ \int_{\frac{x}{2}}^{x-\frac{x}{\sqrt{n+1}}} K'_{n,x}(x; \frac{n}{y}) \Omega_x(\phi_x; x-y) \, dy. \tag{70}
\]

Using (64) to (70), we obtain

\[
\int_{x-\frac{x}{\sqrt{n+1}}}^{x} K'_{n,x}(x; \frac{n}{y}) \Omega_x(\phi_x; x-y) \, dy \leq \frac{x}{\sqrt{n+1}} \Omega_x \left( \phi_x; \frac{x}{\sqrt{n+1}} \right)
\]

\[
\leq \frac{x}{n} \sum_{k=1}^{\left\lceil \sqrt{n} \right\rceil} \Omega_x \left( \phi_x; \frac{x}{k+1} \right). \tag{71}
\]

\[
\int_{\frac{x}{2}}^{x-\frac{x}{\sqrt{n+1}}} K'_{n,x}(x; \frac{n}{y}) \Omega_x(\phi_x; x-y) \, dy \leq \frac{n+1}{(n+1-\frac{nt}{t+1})^2} \Omega_x \left( \phi_x; \frac{x}{t} \right) \frac{1}{t^2} \, dt
\]

\[
\leq \frac{(n+1)x}{4(n-1)^2} \int_{\frac{x}{2}}^{\sqrt{n+1}} \Omega_x \left( \phi_x; \frac{x}{t} \right) \, dt
\]

\[
\leq \frac{x}{2n} \sum_{k=1}^{\left\lceil \sqrt{n} \right\rceil} \Omega_x \left( \phi_x; \frac{x}{k+1} \right). \tag{72}
\]

From (67), (68), (70)–(72) we have

\[
|R_{n,x}(\phi_x)| \leq \frac{3x}{n} \sum_{k=1}^{\left\lceil \sqrt{n} \right\rceil} \Omega_x \left( \phi_x; \frac{x}{k+1} \right). \tag{73}
\]

Therefore combining (60), (63), and (73) gives the result of (39), Theorem 2 is proved.

In the final paragraph we will show that the estimate of Theorem 2 is asymptotically optimal. In fact, let \( f(t) = |t - x|, h(t) = \text{sign}(t - x), \tau = h(x+) - h(x-) = 2, \phi_x = 0. \)
By (39) and (43) we have
\[
\sqrt{\frac{2}{\pi}} \frac{x^{\frac{1}{3}}}{15n^{\frac{1}{2}}} \leq \left| G_n\left(|t - x|, x\right) - \sqrt{\frac{2}{\pi}} \frac{2x + M(x^p + e^{2\beta})}{n^{\frac{3}{2}}} \right| \leq \frac{2x + M(x^p + e^{2\beta})}{n^{\frac{3}{2}}}. \tag{74}
\]
Therefore (39) cannot be further asymptotically improved.

References


Further reading