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Why Horn Formulas Matter in Computer Science: Initial Structures and Generic Examples

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We introduce the notion of *generic examples* as a unifying principle for various phenomena in computer science such as *initial structures* in the area of abstract data types, and *Armstrong relations* in the area of data bases. Generic examples are also useful in defining the semantics of logic programming, in the formal theory of program testing and in complexity theory. We characterize initial structures in terms of their genericity properties and give a syntactic characterization of first-order theories *admitting initial structures*. The latter can be used to explain why Horn formulas have gained such a predominant role in various areas of computer science. © 1987 Academic Press, Inc.

INTRODUCTION

Verification by example has always been an alternative to formal deduction. Historically, in mathematics, it usually also preceded the development of formal deduction methods. The Babylonians "knew" that $(x+y)^2 = x^2 + 2xy + y^2$ but they did not have a notational system which allowed them to carry out a formal, i.e., algebraic, proof. Instead they wrote $(3+5)^2 = 3^2 + 2 \times 3 \times 5 + 5^2$, from which they immediately concluded all the other instances of the general formula. The choice of the particular instance $x = 3$, $y = 5$ is important here. It is clear why $x = 1$, $y = 2$ would confuse the matter, and we informally describe an appropriate choice of an instance as finding a "generic" example. The art of finding "generic" examples has been pushed to the extreme in Euclidean plane geometry, where we convince ourselves of many theorems by just drawing *one* picture of a non-degenerate case. The class of problems where this is possible, incidentally, has also a decision procedure which is much faster than the general decision procedure for Euclidean geometry, due to Wu [Wu83]. The generalization of this approach to other areas of reasoning is usually highly non-trivial. In algebraic geometry, for example, a satisfactory definition of "generic points" was only found in this century.

In computer science one is often concerned with the specification and analysis of algorithms and programs. Methods for formal specification and verification of programs have been developed intensively without leaving too much impact on the practical programmers. They are all very much in the spirit of formal deduction.

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The use of "generic" examples can be observed occasionally with various degrees of explicitness. Strassen [Str74] and his school have used the generic points of algebraic geometry with considerable success to obtain results in algebraic complexity theory. Recent work in the mathematical foundation of program testing, as presented in the survey edited by Chandrasekaran and Radicchi [CR81], focuses on various notions of "generic" input (see also [DW83]). In data base theory Armstrong has introduced a kind of "generic" relation for functional dependencies and Fagin has investigated the possibilities of generalizing this for implicational dependencies [Fa82]. In [DS78] some notion of "genericity" is explored for formal evaluation and automated theorem proving. Last, but not least, there is Zloof's approach to data base query languages where queries are specified by giving "generic" examples, an approach he most recently generalized to operate on more complex systems in office automation [Zl82]. It is not surprising that specification and verification by example is more appealing to the computer engineer than formal deduction; a look at Euclidean geometry can be revealing again. People involved in surveying and drawing plans, in general, have very little use for formal deduction Euclidean style, but are very much aware of the role of the "generic" non-degenerate configuration.

The purpose of this paper is to introduce some variation of notions of "genericity" which arise in abstract specification of data structures, in relational data bases and in logic programming. What these three areas of computer science have in common is the use of first-order logic as its basic specification language. In each of these areas Horn formulas play an important role. In algebraic specification of abstract data structures one first used pure equational logic with the semantics of initial structures as a specification language (hence algebraic) and later felt the need to extend this to conditional equations which are universal Horn formulas without relation symbols. In relational data bases various ad hoc specification languages where introduced, such as the arrow notation between finite sets of attribute names, to express functional and multivalued dependencies, but it was soon realized by Fagin, Beeri, and others, that implicational dependencies, which are Horn formulas without function symbols, could capture all the previously considered cases. In logic programming Horn formulas are used both as a specification and a programming language because, as Kowalski put it, they allow a *procedural* interpretation.

Various attempts exist in the literature to explain why Horn formulas are the right class of formulas to be used in the respective contexts. Mahr and Makowsky [MM83] prove that under certain assumptions for the semantics of algebraic specifications (i.e., the existence of initial-term models) conditional equations form the largest specification language satisfying these assumptions. This result appears here, in a slightly modified version, as Theorem 3.9. Makowsky and Vardi [MV84] characterize various classes of data base dependencies in terms of preservation properties under operations on relations which come from data manipulation. In logic programming it was shown by Andreka and Nemeti [AN75], and independently, by Tarnlund [Tarn77] that Horn logic is enough to program every recursive function, a result, stated in slightly different form in a different context, proven already by Aanderaa and independently by Börger; for an excellent survey see [Bo84]. Note, however, that the result is already implicit in Turing's original papers, though Turing himself seemed to reject proof sequences as a computational model. The same result is also buried in [SH61], were a proof of Church's theorem is given which uses only universal Horn formulas, though the concept of Horn formulas does not appear in the book.

Our main result in this paper is a characterization of Horn formulas in terms of the existence of $\exists^+$-generic structures. It simultaneously extends and unifies the results of [MM83], [Ma84], and [VM84] and remedies the objections raised to [MM83] by Tarlecki. It states that a first-order theory T admits initial ($=\exists^+$-generic) models iff there is a set of definable partial functions such that adding those functions to the vocabulary of T gives us a theory T_1 which is equivalent to a universal Horn theory. Additionally, if T is finite, this set of definable partial functions can be chosen to be finite, too. The significance of this result for database theory and logic programming will be discussed in Section 7.

In detail the paper is organized as follows: In Section 1 we characterize propositional Horn formulas via the existence of generic assignments, a simple result which seems new and is needed for our further investigations.

In Section 2 we introduce A-genericity and $\exists^+$-genericity and relate these definitions to initiality. We prove a basic definability theorem for initial models; we characterize initial-term models as A-generic models and initial models as $\exists^+$-generic pseudo-term models.

In Section 3 we characterize first-order theories which admit initial-term models as the universal Horn theories. This theorem was already proved in [MM83], but here we present a different proof based on the results of Section 1 and therefore avoid the difficulties arising from the application of a theorem of Mal'cev, as in [MM83].

In Section 4 we establish the intersection property of first-order theories admitting $\exists^+$-generic structures and review some classical model theoretic results on first-order theories with the intersection property. From this we get that theories admitting $\exists^+$-generic models can always be axiomatized by universal-existential sentences.

In Section 5 we apply a theorem of Rabin [Ra60], which characterizes first-order theories with the intersection property, to obtain our main result. We show that a first-order theory admits initial models iff it is a partially functional $\forall\exists$-Horn theory.

In Section 6 we discuss briefly an application of our main theorem to the simultaneous solvability of systems of equations and inequations. In Section 7, finally, we discuss the relevance of these characterizations for the working computer scientist and directions of further research.

1. Characterizing Propositional Horn Formulas

In this section we introduce *generic assignments* for propositional logic and characterize propositional Horn formulas as the largest class of propositional formulas (up to logical equivalence) which admits generic assignments (Theorem 1.9). The main purpose of this section is to introduce the reader to the concepts of *generic assignment*, *admitting generic assignments*, and the type of reasoning which underlies the rest of the paper. We assume that the material presented in this section was well known among many logicians. At least Weisspfenning convinced me, after having read the manuscript, that he taught similar material in his logic courses in Heidelberg, at least since 1975.

1.1. Definitions. Let $p_1, p_2, \ldots, p_n, \ldots$ be propositional variables. A *literal* is a propositional variable p_i or a negated propositional variable $\neg p_i$. A *clause* is a (finite) disjunction of literals. A *Horn clause* is a clause with at most one non-negated literal. We also write Horn clauses in the form $p_1 \wedge p_2 \wedge \cdots \wedge p_n \rightarrow q$, where q is either a propositional variable or the symbol **false**.

Since every formula of propositional logic can be written in conjunctive normal form, every such formula is equivalent to a set of clauses. A *propositional Horn formula* is a (finite) conjunction of Horn clauses.

1.2. Definition. Let V be the set of propositional variables. A function $z: V \rightarrow \{0, 1\}$ is called an *evaluation function* or *assignment*. If ϕ is a formula of propositional logic we define $\phi\langle z \rangle$ inductively as

(i) $p_i\langle z \rangle = z(p_i)$, $\mathbf{false}\langle z \rangle = 0$.

(ii) If ϕ is $\phi_1 \wedge \phi_2$ then $\phi\langle z \rangle = \min(\phi_1\langle z \rangle, \phi_2\langle z \rangle)$.

(iii) If ϕ is $\phi_1 \vee \phi_2$ then $\phi\langle z \rangle = \max(\phi_1\langle z \rangle, \phi_2\langle z \rangle)$.

(iv) If ϕ is $\neg\psi$ then $\phi\langle z \rangle = 1 - \psi\langle z \rangle$.

(v) If Σ is a set of propositional formulas we put $\Sigma\langle z \rangle = \min_{\phi \in \Sigma}\{\phi\langle z \rangle\}$.

1.3. Definitions. A formula ϕ of propositional logic is

(i) *satisfiable* if there is an assignment z such that $\phi\langle z \rangle = 1$.

(ii) *valid* if for every assignment z, $\phi\langle z \rangle = 1$.

A set Σ of formulas of propositional logic is

(iii) *satisfiable* if there is an assignment z such that $\phi\langle z \rangle = 1$ for every $\phi \in \Sigma$, or equivalently, if there is an assignment z such that $\Sigma\langle z \rangle = 1$.

(iv) A formula ϕ is a *consequence* of a set of formulas Σ if for every assignment z such that $\Sigma\langle z \rangle = 1$ we also have $\phi\langle z \rangle = 1$. We write $\Sigma \models \phi$ if ϕ is a consequence of Σ.

(v) Let Σ_1, Σ_2 be two sets of propositional formulas. We write $\Sigma_1 \models \Sigma_2$ if $\Sigma_1 \models \phi$ for every $\phi \in \Sigma_2$. We say that Σ_1 and Σ_2 are *equivalent*, if both $\Sigma_1 \models \Sigma_2$ and $\Sigma_2 \models \Sigma_1$.

The next definition introduces the central notion of this paper.

1.4. DEFINITIONS. Let Σ be a set of propositional formulas.

(i) Let z be an assignment. We say that *z is generic for Σ* if $\Sigma\langle z\rangle = 1$ and $z(p_i) = 1$ iff $\Sigma \models p_i$. In other words, z makes Σ true by giving only those variables the value 1 for which also all other assignments z' with $\Sigma\langle z'\rangle = 1$ give the value 1.

(ii) Let z_i, $i \in I$, be a set of assignments. We define $\bigcap_{i \in I} z_i$ to be the assignment z with $z(p_j) = \min_{i \in I} z_i(p_j)$.

(iii) Let $z_{\min,\Sigma}$ the assignment $\bigcap_{\Sigma\langle z\rangle = 1} z$.

1.5. EXAMPLES. (i) The formula $p_1 \wedge p_2$ has a generic assignment with $z(p_1) = z(p_2) = 1$. The generic assignment is the assignment $z_{\min, p_1 \wedge p_2}$.

(ii) The formula $p_1 \vee p_2$ has no generic assignment since neither p_1 nor p_2 are consequences of $p_1 \vee p_2$.

The following is immediate from the definitions:

1.6. THEOREM. (i) *For every set of propositional formulas Σ there is* at most one *generic assignment.*

(ii) *If a set Σ of propositional formulas has a generic assignment then it is equal to $z_{\min,\Sigma}$.*

(iii) *Let Σ be a set of Horn clauses. Then Σ is satisfiable iff there is a generic assignment for Σ (which, by* (ii) *is equivalent to $\Sigma\langle z_{\min,\Sigma}\rangle = 1$).*

Sketch of Proof. (i) and (ii) follow from the definition. (iii) follows easily from the fact that *unit resolution* is complete for testing satisfiability of Horn clauses, as shown by Henschen and Wos in [HW74]. ▮

1.7. DEFINITION. Let Σ be a set of propositional formulas. We say that Σ *admits generic assignments* if for every set Δ of non-negated literals either $\Sigma \cup \Delta$ is not satisfiable or $\Sigma \cup \Delta$ has a generic assignment. Note that a non-negated literal which appears in Δ may appear negated in Σ.

The next examples illustrate the difference between *having* a generic assignment and *admitting* a generic assignment. The difference resembles certain robustness assumptions in topology or statistics: A set of propositional formulas Σ may have a generic assignment quite accidentally. If Σ admits a generic assignment we require that Σ has generic assignments in every extension of Σ by sets of atomic formulas Δ. The set Δ plays here the role of a neighborhood.

1.8. EXAMPLES. (i) By the above theorem every set Σ of Horn formulas admits generic assignments.

(ii) Let $\phi = p_1 \vee p_2 \vee \neg p_3$. Clearly $z \equiv 0$ is a generic assignment for ϕ and ϕ is not a Horn formula. To see that ϕ does not admit generic assignments we look at $\Delta = \{p_3\}$. The only candidate for a generic assignment for $\phi \wedge p_3$ is z' defined by $z'(p_1) = z'(p_2) = 0$ and $z'(p_3) = 1$. But we easily verify that $\phi \wedge p_3\langle z' \rangle = 0$.

The following theorem characterizes propositional Horn formulas in terms of generic assignments:

1.9. THEOREM. *Let Σ be a set of propositional formulas. Then Σ admits generic assignments iff Σ is equivalent to the set Σ_H of Horn formulas θ such that $\Sigma \models \theta$.*

Proof. Clearly, if Σ_H is equivalent to Σ, for every assignment z, $\Sigma\langle z \rangle = \Sigma_H\langle z \rangle$. So Σ admits generic assignments iff Σ_H does. But the latter is a set of Horn formulas and we can apply Theorem 1.6.

Conversely, let Σ admit generic assignments and let $\phi \in \Sigma$ be such that $\phi \notin \Sigma_H$. So ϕ is not a Horn clause. We can assume $\phi = p_1 \vee p_2 \vee \cdots p_n \vee C$, where C is a clause containing only negated literals.

Without loss of generality we can assume that (1): for no $i = 1,..., n$ we have that $\Sigma \models p_i$. For otherwise, if $\Sigma \models p_1$, say, then $\Sigma \cup \phi \models p_2 \vee p_3 \vee \cdots \vee p_n \vee C$.

We now put $\Delta = \{p_i: \neg p_i \in C\}$.

Claim. $\Sigma \cup \Delta$ has no generic assignment.

Assume, for contradiction, that z is a generic assignment for $\Sigma \cup \Delta$. So $\Delta\langle z \rangle = 1$ and $\phi\langle z \rangle = 1$. By (1) and Definition 1.4, we get that $p_i\langle z \rangle = 0$ for every $i = 1,..., n$. From this we conclude that $\phi\langle z \rangle = 0$, a contradiction. ∎

1.10. INTERPRETATION. We can think of non-negated literals as *facts* and of Horn clauses as *rules*. A generic assignment then corresponds to a world where only those assertions are true which are either facts or follow from the facts by application of the rules. This is nothing else than the *closed world assumption* as introduced in [Re78]. The requirement that Σ *admits generic assignments* can be viewed as a robustness requirement: The existence of generic assignments is not affected by a change of the facts, though the nature of the generic assignment is.

The purpose of the rest of this paper is to explore how Theorem 1.9 has to be generalized to first-order logic and what one can learn from this generalization. Two important points became apparent already in the discussion of propositional logic: The distinction between *having* and *admitting* generic assignments and the *equivalence* of generic assignments for Σ and the minimal assignment $z_{\min,\Sigma}$. These points will reappear in Sections 2 and 4, respectively.

2. Initial Models and Genericity

In this section we introduce various notions which are analogs to the generic and minimal assignments introduced in the previous section. Many of the complications encountered in this and the following sections stem from the fact that we allow both function and relation symbols to occur in the vocabulary of first-order formulas. As is well known, functions can be looked at as special cases of relations (and vice versa). We have to be careful in the choice of definitions to make them invariant under the translation between functions and relations. In the absence of relations the evolving picture is much simpler and is described completely in Section 3. Sections 4 and 5 treat the case where relations are also allowed.

Let us first recall some standard definitions. From this section on we deal with first-order languages *with equality*. *Vocabularies* (=*similarity types*) are allowed to be *many-sorted* and may include *function symbols*, *relation symbols*, and *constant symbols*. Vocabularies are denoted by τ, σ. A τ*-structure* $\mathbf{A}$ is a collection of *universes* (=sets) $A_1,..., A_n$, for each sort in τ one universe, together with interpretations for all the function, relation, and constant symbols in τ. τ*-terms*, *atomic formulas*, and τ*-formulas* are defined as usual. If T is a set of τ-formulas, ϕ is a τ-formula, and $\mathbf{A}$ is a τ-structure, we write $A \models T$ if the universal closure of all the formulas $\phi \in T$ is true in $\mathbf{A}$. We write $T \models \phi$ if, in every τ-structure $\mathbf{A}$ such that $\mathbf{A} \models T$, we also have $\mathbf{A} \models \phi$. We call sets of τ-formulas *theories* and formulas *without free variables* also τ*-sentences*. We call τ-structures also *models* and denote by $\mathrm{Mod}(T)$ the class of τ-structures $\mathbf{A}$ such that $\mathbf{A} \models T$. We write often $\bar{t}$ for a sequence of terms $t_1, t_2,..., t_n$ and $\exists\bar{x}$ $(\forall\bar{x})$ instead of $\exists x_1, \exists x_2,..., \exists x_n$ $(\forall x_1, \forall x_2,..., \forall x_n)$.

A τ-formula is *universal* (*existential*) if its prenex normal form is a formula with universal (existential) quantifiers only. We then speak of $\forall$-formulas and $\exists$-formulas, respectively. A τ-formula is an $\forall\exists$-formula if its prenex normal form is a formula whose quantifier string is a string of universal quantifiers followed by a string of existential quantifiers.

A first-order clause is a quantifier-free formula which is a disjunction of atomic or negated atomic formulas (literals) possibly containing the constant **false**. A (*first-order*) *Horn clause* is a clause with at most one non-negated literal. The Horn clause is *strict* if it contains at least one negated literal. A *Horn formula* is a formula in prenex normal form whose quantifier-free part is a conjunction of first-order Horn clauses. A *strict Horn formula* is a Horn formula whose quantifier-free part consists of a conjunction of strict Horn clauses.

Let τ, τ' be a vocabularies such that $\tau \subset \tau'$ and $\mathbf{A}$ be a τ-structure. A τ'-structure $\mathbf{B}$ is an *expansion* of $\mathbf{A}$ ($\mathbf{A}$ is the *restriction* of $\mathbf{B}$) if for every sort in τ, $\mathbf{A}$ and $\mathbf{B}$ have the same universes and for every relation, function, and constant symbol in τ their respective interpretations in $\mathbf{A}$ and $\mathbf{B}$ are the same.

2.1. Definitions. (i) Let $\mathbf{K}$ be a class of τ-structures closed under isomorphisms and $\mathbf{A} \in \mathbf{K}$. We say that $\mathbf{A}$ *is initial in* $\mathbf{K}$ (*is an initial model for* $\mathbf{K}$) if for every structure $\mathbf{B} \in \mathbf{K}$ there is a *unique* homomorphism h_B: $\mathbf{A} \to \mathbf{B}$.

(ii) If **K** is of the form Mod(T), where T is some first-order theory, we also say that **A** *is initial for T.*

(iii) A τ-structure **A** is a *term model* (*reachable model*) if for every $a \in \mathbf{A}$ there is a τ-term t such that its interpretation $\mathbf{A}(t)$ in **A** is the element a.

(iv) **A** is a *prime model for* **K**, if for every $\mathbf{B} \in \mathbf{K}$ there is an embedding $h: \mathbf{A} \to \mathbf{B}$.

(v) **A** is an *initial* (*prime*) *term model for* **K** if it is both an initial (prime) model for **K** and a term model.

2.2. *Remarks.* (i) If **A** is initial for some **K** then **A** is *rigid*, i.e., there are no non-trivial automorphisms of **A**.

(ii) If **K** has, up to isomorphisms, exactly one term model **A** then **A** is both initial and prime.

(iii) Every prime term model **A** in **K** is also an initial term model, but not conversely.

Prime models were introduced in model theory by Robinsons [Rob63] generalizing the *algebraic* concept of a *prime field.* Initial objects originate in *category theory* generalizing the concept of *free groups.* The main difference between the two consists in the class of morphisms; in model theory one deals mainly with embeddings whereas in category theory one prefers homomorphisms. In computer science initial structures are introduced by the ADJ-group in the context of algebraic specifications of abstract data types in [ADJ75].

2.3. Examples. (i) Let τ_{suc} consist of one unary function symbol f and one constant symbol c. The τ_{suc}-structure $\langle N, \mathrm{successor}, 0\rangle$ is an initial term model among all the τ_{suc}-structures. It is not a prime model.

(ii) Let τ_{sucrel} consist of one binary relation symbol R and one constant symbol c. Let T_{sucrel} be the τ_{sucrel}-sentence asserting that R is the graph of a total function. $\langle N, \mathrm{successorrelation}, 0\rangle$ is an initial model among all the τ_{sucrel}-structures satisfying T_{sucrel}, but it is neither a term model nor a prime model. Note that T_{sucrel} has been obtained from the previous example by translating the function symbol into a relation symbol.

(iii) Let τ_{Peano} consist of one unary function symbol f, two binary function symbols A, M, one binary relation symbol $R_<$ and one constant symbol c. Let T_{Peano} be the usual Peano axioms for τ_{Peano}. The τ_{Peano}-structure $\langle N, \mathrm{successor}, +, *, <, 0\rangle$ is an initial term model for T_{Peano} which is also a prime model.

(iv) Let $\tau_{P(\mathrm{inf})}$ consist of one unary function symbol f, two binary function symbols A, M, one binary relation symbol $R_<$ and two constant symbols c, d. Let $T_{P(\mathrm{inf})}$ be the usual Peano axioms for τ_{Peano} augmented by the set of atomic sentences $f^n(c) < d$. The models of $T_{P(\mathrm{inf})}$ are non-standard models of T_{Peano}. Their structure is rather complex, cf. Smorynski's survey [Sm84]. $T_{P(\mathrm{inf})}$ has no initial model, no prime model and no term model. Note that we obtained $T_{P(\mathrm{inf})}$ from T_{Peano} by adding only atomic sentences. Note that in the non-standard models of T_{Peano} the first-order induction scheme is still true, though second-order induction is false. The standard model of T_{Peano} is the only model of T_{Peano} in which "true" induction holds.

(v) Let τ_{dense} consist of one binary relation symbol $R_<$ and two constant symbols c, d. Let T_{dense} be the set of τ_{dense}-sentences which assert that $R_<$ is a linear dense ordering with c the first and d the last element. T_{dense} has a prime model, the rational interval $[0, 1]$ with the natural ordering, but no term model nor an initial model, since the rational interval is not rigid.

Next we introduce the concept of *generic* structures for first-order theories and relate it to initial structures.

2.4. DEFINITIONS. Let $\mathbf{K}$ be a class of τ-structures closed under isomorphisms and $\mathbf{A} \in \mathbf{K}$. Let Σ be a set of first-order sentences (i.e., formulas without free variables).

(i) We say that $\mathbf{A}$ *is generic in* $\mathbf{K}$ *for* Σ if for every $\phi \in \Sigma$ we have that $\mathbf{A} \models \phi$ iff for every $\mathbf{B} \in \mathbf{K}$ we have that $\mathbf{B} \models \phi$.

(ii) If Σ is the set of atomic τ-formulas we say *A-generic* instead of generic for Σ.

(iii) Let $\exists^+$ be the set of τ-formulas of the form $\exists \bar{x} \wedge_{i=1}^{n} \phi_i$ with each ϕ_i an atomic formula and $\exists$ be the set of τ-formulas of the form $\exists \bar{x} \psi(\bar{x})$ with ψ quantifier-free.

(iv) If Σ is the set of $\exists^+$-sentences we say $\exists^+$*-generic* instead of generic for Σ.

2.5. *Remarks.* (i) If $\Sigma_0 \subset \Sigma$ and $\mathbf{A}$ is Σ-generic then $\mathbf{A}$ is also Σ_0-generic.

(ii) If $\mathbf{A}$ is prime for $\mathbf{K}$ then $\mathbf{A}$ is A-generic.

(iii) If $\mathbf{A}$ is a A-generic term model then $\mathbf{A}$ is an initial-term model.

2.6. EXAMPLES. (i) The τ_{suc}-structure $\langle N, \mathrm{successor}, 0 \rangle$ is $\exists^+$-generic in the class of all τ_{suc}-structures. $\langle N, \mathrm{successor}, 0 \rangle$ is actually generic for all τ_{suc}-sentences.

(ii) Let τ be any vocabulary containing only function symbols and at least one constant symbol for each sort. Let $\mathbf{F}_\tau$ be the free-term structure for τ, i.e., the structure consisting of all τ-terms with the natural interpretation of all the symbols. $\mathbf{F}_\tau$ is generic in the class of all τ-structures for all τ-sentences.

(iii) We call a τ-theory *complete* if T has a model and for every τ-sentence ϕ either $T \models \phi$ or $T \models \neg\phi$. If T is a complete theory then every model of T is generic for the set of τ-sentences.

(iv) Let τ_{dense} and T_{dense} be as in Example 2.3(v) and let T_{lin} be the τ_{dense}-sentence asserting that $R_<$ is a linear ordering with c as its first and d as its last element. The τ_{dense}-structure $\langle [0,1], <, 0, 1 \rangle$ is generic in $\text{Mod}(T_{\text{dense}})$ for Σ, the set of all τ_{dense}-sentences, since T_{dense} is a complete theory. $\langle [0,1], <, 0, 1 \rangle$ is A-generic in $\text{Mod}(T_{\text{lin}})$ but not $\exists^+$-generic. The structure $\langle \{0,1\}, <, 0, 1 \rangle$ is $\exists^+$-generic in $\text{Mod}(T_{\text{lin}})$.

(v) Let τ_{Peano} and T_{Peano} be as in Example 2.3(iii). The structure $\langle N, \text{successor}, +, *, <, 0 \rangle$ is $\exists^+$-generic in $\text{Mod}(T_{\text{Peano}})$, but not generic for all τ_{Peano}-sentences, by Godel's incompleteness theorem.

2.7. THEOREM ($\exists^+$-genericity). *Let* **K** *be a class of* τ*-structures closed under isomorphisms and* $\mathbf{A}_I$ *be initial for* **K**. *Then* $\mathbf{A}_I$ *is* $\exists^+$*-generic.*

Proof. Let $\exists \bar{x} \phi(\bar{x})$ be a τ-sentence with ϕ a conjunction of atomic formulas. We have to show that $\mathbf{A}_I \models \exists \bar{x} \phi(\bar{x})$ iff for every $\mathbf{B} \in \mathbf{K}$ we have that $\mathbf{B} \models \exists \bar{x} \phi(\bar{x})$. Assume, for contradiction, there is $\mathbf{B} \in \mathbf{K}$ with $\mathbf{B} \models \forall \bar{x} \neg \phi(\bar{x})$. Since $\mathbf{A}_I$ is initial there is a homomorphism $h: \mathbf{A}_I \to \mathbf{B}$. Let $\bar{a} \in \mathbf{A}_I$ be such that $\mathbf{A}_I \models \phi(\bar{a})$ and $\bar{b} \in \mathbf{B}$ be the image of $\bar{a}$ under h. Since h is a homomorphism $\mathbf{B} \models \phi(\bar{b})$, a contradiction. The other direction is trivial. ∎

2.8. COROLLARY. *Let* **K** *be a class of* τ*-structures closed under isomorphisms and* $\mathbf{A} \in \mathbf{K}$. *Then* **A** *is an initial term model for* **K** *iff* **A** *is an A-generic term model.*

Proof. Use Remarks 2.5(i), (iii), and Theorem 2.7. ∎

2.9. DEFINITIONS. Let T be a set of τ-sentences, let **A** be a τ-structure and $a \in A$ be an element of the universe of **A**.

(i) a is *definable over* **A** if there is a τ-formula $\phi_a(x)$ with x the only free variable of ϕ such that $\mathbf{A} \models \phi_a(a)$ and if $\mathbf{A} \models \phi_a(b)$ for any $b \in A$ then $\mathbf{A} \models a = b$. We call ϕ_a the defining formula of a.

(ii) a is $\exists^+$*-definable* ($\exists$*-definable, atomically definable*) over **A** if a is definable over **A** and the defining formula is an $\exists^+$-formula ($\exists$-formula, conjunction of atomic formulas).

(iii) a is *definable over* T if there is a τ-formula $\phi_a(x)$ with x the only free variable of ϕ_a such that $\mathbf{A} \models \phi_a(a)$ and $T \models \forall x \forall y \, (\phi_a(x) \wedge' \phi_a(y) \to x = y)$.

(iv) a is $\exists^+$*-definable* ($\exists$*-definable, atomically definable*) over T if a is definable over T and the defining formula is an $\exists^+$-formula ($\exists$-formula, conjunction of atomic formulas).

(v) We say that $\mathbf{A} \models T$ is a *pseudo-term model* ($\exists$-term model) *of* T if every element $a \in A$ is $\exists^+$-definable ($\exists$-definable) over T.

Note that if an element a is definable over T, then it has a definition $\phi(x)$, which is only a definition (i.e., has a unique element satisfying ϕ) among models of T. By the compactness of first-order logic there is a finite subset T_a of T, such that ϕ is already a definition over T_a. If **A** is a pseudo-term model of T, every element a of **A** is definable over some finite T_a, but if **A** is infinite, it may well happen that for no finite $T^* \subset T$ is it the case that **A** is a pseudo-term model of T^*.

2.10. EXAMPLES. (i) In a term model every element is atomically definable.

(ii) Let **A** be a τ-structure which is a term model. Let τ_{rel} be obtained from τ by replacing every n-ary function symbol by an $n+1$-ary relation symbol and $\mathbf{A}_{\mathrm{rel}}$ be the τ_{rel}-structure obtained from **A** by the natural interpretation of the relation symbols. Then $\mathbf{A}_{\mathrm{rel}}$ is a pseudo-term model.

(iii) Let $\mathbf{C}_{\mathrm{alg}}$ be the field of algebraic numbers. The definable elements of $\mathbf{C}_{\mathrm{alg}}$ are exactly the rational numbers. An algebraic irrational number a is only *weakly definable* in the sense that there is an atomic formula $\phi(x)$ such that $\mathbf{C}_{\mathrm{alg}} \models \phi(a)$ and only finitely many other elements $b \in C_{\mathrm{alg}}$ also satisfy $\mathbf{C}_{\mathrm{alg}} \models \phi(b)$.

(iv) Let $\mathbf{R}_{\mathrm{alg}}$ be the field of real algebraic numbers. It is easy to see that $\mathbf{R}_{\mathrm{alg}}$ is not a term model. The atomically definable elements of $\mathbf{R}_{\mathrm{alg}}$ are again the rational numbers. Using the fact that the positive numbers x are exactly the numbers satisfying $\exists y(x^2 = y)$ it is easily verified that $\mathbf{R}_{\mathrm{alg}}$ is a pseudo-term model.

(v) Let **A** be a τ-structure which is a pseudo-term model. One would naturally ask whether we can find new functions f_i $(i \in I)$ on **A** such that $\langle \mathbf{A}, f_i \rangle_{i \in I}$ is a term model. Example (iv) shows that this is not the case. However, if we allow the f_i to be *partial functions* then the answer is yes.

Note that we could transform the above example trivially into a term model by adding (infinitely many) constants. The advantage of adding partial functions lies in the possibility of adding only one such function of arity $n+3$ for all polynomials of degree n whose value gives the smallest root of that polynomial in a given open interval, provided it exists. In this case the domain of this function is definable by a quantifierfree formula, as one can see from a constructive version of a special case of Sturm's theorem (cf. [KK66, Chap. 4]).

The following theorem shows that in an initial model of a theory T is always a pseudo-term model. In the proof we make use of the method of diagrams, so we need two more definitions.

2.11. DEFINITIONS. Let **A** be a τ-structure and let τ_A be the vocabulary obtained from τ by adding a new constant symbol $\mathbf{c}_a$ for every $a \in A$.

(i) The *atomic diagram* $\mathbf{D}_A$ is the set of τ_A-atomic sentences true in **A** when every constant symbol $\mathbf{c}_a$ is interpreted by a.

(ii) The *negative diagram* $\mathbf{D}_A^-$ is the set of negated τ_A-atomic sentences true in **A** when every constant symbol $\mathbf{c}_a$ is interpreted by a.

2.12. THEOREM ($\exists^+$-definability). *Let T be a first-order theory and let $\mathbf{A}_I$ be an initial model of T. Then every $a \in A$ is definable over T by a $\exists^+$-formula ϕ_a. In other words $\mathbf{A}_I$ is a pseudo-term model.*

Proof. Let $\mathbf{A}_I$ be the initial model of T and $a_0 \in A$. For every $a \in A$ let $\mathbf{a}$ be a constant symbol whose interpretation in $\mathbf{A}_I$ is a. Let $\mathbf{D}_A$ be the atomic diagram of $\mathbf{A}_I$ and $\mathbf{D}_{A'}$ the result of replacing every constant symbol $\mathbf{a}$ of $\mathbf{D}_A$ by $\mathbf{a}'$.

Claim. $T \cup \mathbf{D}_A \cup \mathbf{D}_{A'} \cup \{\mathbf{a}_0 \neq \mathbf{a}_0'\}$ is inconsistent.

For otherwise, let $\mathbf{B}$ be a model of it. So $\mathbf{B} \models T$. We now define two homomorphisms f and f' from $\mathbf{A}_I$ to $\mathbf{B}$, herewith contradicting the initiality of $\mathbf{A}_I$. Put $f(a) = \mathbf{a}^B$ and $f'(a) = \mathbf{a}'^B$. Since $\mathbf{B} \models \mathbf{D}_A \cup \mathbf{D}_{A'}$ this clearly defines homomorphisms. Since $\mathbf{B} \models \mathbf{a}_0 \neq \mathbf{a}_0'$ we clearly have $f \neq f'$. Therefore we conclude that

$$T \cup \mathbf{D}_A \cup \mathbf{D}_{A'} \models \mathbf{a}_0 = \mathbf{a}_0'.$$

Now we use compactness to find a finite set of atomic formulas $A_0, A_1, ..., A_k$ and constant symbols $\mathbf{a}_0, \mathbf{a}_1, ..., \mathbf{a}_n, \mathbf{a}_0', \mathbf{a}_1', ..., \mathbf{a}_n'$ such that

$$T \models \bigwedge_{i=0}^{k} A_i(\mathbf{a}_0, \bar{\mathbf{a}}) \wedge \bigwedge_{i=0}^{k} A_i(\mathbf{a}_0', \bar{\mathbf{a}}') \rightarrow \mathbf{a}_0 = \mathbf{a}_0',$$

where $\bar{\mathbf{a}} = (\mathbf{a}_1, ..., \mathbf{a}_n)$, and $\bar{\mathbf{a}}' = (\mathbf{a}_1', ..., \mathbf{a}_n')$. Now put

$$\phi_{a_0} = \exists x_1, x_2, ..., x_n, x_1', x_2', ..., x_n' \bigwedge_{i=0}^{k} (A_i(\bar{x}) \wedge' A_i(\bar{x}')).$$

Clearly ϕ_{a_0} is the required formula. ∎

We now are in a position to characterize initial models as pseudo-term models which are $\exists^+$-generic.

2.13. THEOREM. *Let T be a first-order theory and let* $\mathbf{A}$ *be a model of T. Then* $\mathbf{A}$ *is initial (for T) iff* $\mathbf{A}$ *is a $\exists^+$-generic pseudo-term model.*

Proof. Assume $\mathbf{A}$ is initial. So, by Theorem 2.12, $\mathbf{A}$ is a pseudo-term model and, by Theorem 2.7, $\mathbf{A}$ is $\exists^+$-generic. So assume that $\mathbf{A}$ is a $\exists^+$-generic pseudo-term model, and let $\mathbf{B}$ be an arbitrary model of T. We define a unique homomorphism $\mathbf{A} \rightarrow \mathbf{B}$ in the following way: For every $a \in A$ and let $\phi_a(x)$ be the $\exists^+$-formula which defines a over T. So $\mathbf{A} \models \exists! x \phi_a(x)$. Since $\mathbf{A}$ is $\exists^+$-generic also $\mathbf{B} \models \exists! x \phi_a(x)$. Let $b \in B$ such that $\mathbf{B} \models \phi_a(b)$. So we put $h(a) = b$. ∎

3. Characterizing First-Order Theories Which Admit Initial Term Models

In this section we characterize first-order theories which admit initial-term models. Such a characterization was first given in [MM83], based on a theorem due to Mal'cev [Mal56]. In [Mal56] there is a minor mistake as pointed out by

[Mo59], which propagated into [MM83] in as far as one had to assume that every first-order theory admitting initial-term models also has a trivial model. In this section we reprove the main result of [MM83], based on the characterization of propositional Horn formulas as presented in Section 1. A proof of the same theorem using a modified version of Mal'cev's theorem was also given by Tarlecki [Tar84]. We first give the first-order version of a set of sentences admitting initial models.

3.1. DEFINITIONS. Let **K** be a class of τ-structures closed under isomorphisms.

(i) We say that **K** *admits initial-term models* if for every σ and for every set Δ of atomic (variable free) σ-sentences either Δ has no model in **K** or $\mathbf{K} \cap \mathrm{Mod}(\Delta)$ has an initial-term model (i.e., there is an initial-term model in **K** which satisfies Δ).

(ii) We say that **K** *strongly admits term models* if for every σ and for every set Δ of *atomic or negated atomic* (variable-free) σ-sentences either Δ has no model in **K** or there is an initial-term model in **K** which satisfies Δ.

Note. In (i) and (ii) the sentences in Δ may contain relation, function, or constant symbols not occuring in τ.

(iii) We say that **K** is *closed under substructures* if whenever $\mathbf{A} \in \mathbf{K}$ and $\mathbf{B} \subset \mathbf{A}$ is a substructure of **A** then $\mathbf{B} \in \mathbf{K}$.

(iv) Let T be a first-order theory. We ay that T is *preserved under substructures* if whenever $\mathbf{A} \models T$ and $\mathbf{B} \subset \mathbf{A}$ is a substructure of **A** then $\mathbf{B} \models T$.

We first show that classes of structures admitting initial-term models are closed under substructures. This theorem was inspired by [Mal56] and first stated in [MM83].

3.2. THEOREM (Mahr and Makowsky). *Let* **K** *be a class of τ-structures closed under isomorphisms.*

(i) *If* **K** *strongly admits term models, then* **K** *is closed under substructures.*

(ii) *If* **K** *admits initial term models, then* **K** *is closed under substructures.*

Proof. (i) Let $\mathbf{A} \in \mathbf{K}$ and $\mathbf{B} \subset \mathbf{A}$. Put $\Delta = \mathbf{D}_B$ to be the atomic diagram of **B** and put Δ^- to be the negated atomic diagram of **B**. Clearly **A** can be expanded to a model of $\Delta \cup \Delta^-$ and $\mathbf{A} \in \mathbf{K}$ so $\Delta \cup \Delta^-$ has a term model $\mathbf{B}_0$ in **K**.

Claim. $\mathbf{B} = \mathbf{B}_0$.

We have that $\mathbf{B} \subset \mathbf{B}_0$ since $\mathbf{B}_0$ is a model of $\Delta \cup \Delta^-$ and $\mathbf{B} = \mathbf{B}_0$ since $\mathbf{B}_0$ is a term model.

To prove (ii) we replace $\Delta \cup \Delta^-$ by Δ. The proof is essentially the same, except to show that $\mathbf{B}_0 \subset \mathbf{B}$ in the claim we have to use that $\mathbf{B}_0$ is an initial term model, and therefore, by Theorem 2.8, an A-generic model of Δ and therefore $\mathbf{B}_0 \models \Delta^-$. ∎

Classes of structures closed under isomorphisms and substructures were characterized by Tarski [Ta52].

3.3. THEOREM (Tarski). *Let T be a first-order theory which is preserved under substructures. Then T is equivalent to a universal theory $T_\forall$. Additionally, if T is finite so is $T_\forall$.*

3.4. COROLLARY. *Let T be a first-order theory.*

(i) *If T strongly admits term models, then T is equivalent to a universal theory $T_\forall$.*

(ii) *If T admits initial term models, then T is equivalent to a universal theory $T_\forall$.*

Additionally, in both cases, if T is finite so is $T_\forall$.

Proof. Immediate from Theorems 3.2 and 3.3. ∎

The next theorem characterizes first-order theories which admit initial-term models. A similar theorem was proved in [MM83], but there the proof was based on a theorem due to Mal'cev [Ma56], which led to confusion, due to an oversight in [Mal56], pointed out in [Mo59]. The proof here is based on the same idea as the proof of Theorem 1.9 and is self-contained.

3.5. THEOREM. *Let T be a first-order theory which admits initial-term models. Then T is equivalent to a universal Horn theory T_H. Additionally, if T is finite so is T_H.*

Proof. Since T admits a term model, T is equivalent to a universal theory $T_\forall$, by Corollary 3.4. Now let T_H be the set of all universal Horn sentences ϕ such that $T \models \phi$.

Claim. T_H is equivalent to $T_\forall$.

Assume, for contradiction, that there is $\forall \bar{x} \psi \in T_\forall$ such that $T_H \cup \{\exists \bar{x} \neg \psi\}$ has a model. W.l.o.g. ψ is a clause of the form

$$\psi(\bar{x}) = A_1(\bar{x}) \vee \cdots \vee A_k(\bar{x}) \vee \neg A_{k+1}(\bar{x}) \vee \cdots \vee \neg A_n(\bar{x})$$

with $k \geqslant 2$. Let ψ_i be obtained from ψ by omitting $A_i(\bar{x})$. W.l.o.g. we can assume that ψ is *minimal*, i.e., for no ψ_i do we have that $T \models \psi_i$. Let $\bar{x} = (x_1, ..., x_m)$ and let $\bar{c} = (c_1, c_2, ..., c_m)$ be new constant symbols. Let Δ be $\{A_{k+1}(\bar{c}), ..., A_n(\bar{c})\}$.

$$\text{Clearly, } T_H \cup \{\exists \bar{x} \neg \psi\} \cup \Delta \text{ has a model.} \qquad (1)$$

Subclaim 1. $T \cup \Delta$ has a model. Assume, for contradiction, that this is not true. So we have $T \models \forall \bar{x} \neg A_{k+1}(\bar{x}) \vee \cdots \vee \neg A_n(\bar{x})$. But $\forall \bar{x} \neg A_{k+1}(\bar{x}) \vee \cdots \vee \neg A_n(\bar{x})$ is a universal Horn formula, so $\forall \bar{x} \neg A_{k+1}(\bar{x}) \vee \cdots \vee \neg A_n(\bar{x})$ is in T_H, which contradicts (1).

Subclaim 2. $T \cup \Delta$ has no initial-term model. Assume, for contradiction, that **A** is an initial-term model for $T \cup \Delta$. By Theorem 2.8, **A** is $\mathcal{A}$-generic. Since we have chosen ψ to be minimal, for no $i = 0, \ldots, k$ do we have that $T \models A_i(\bar{c})$. So, by the $\mathcal{A}$-genericity of **A** we have that

$$\mathbf{A} \models \neg A_1(\bar{c}) \wedge \cdots \wedge \neg A_k(\bar{c}). \tag{2}$$

But we also have that $\mathbf{A} \models \psi$ and $\mathbf{A} \models \Delta$, so we get

$$\mathbf{A} \models A_1(\bar{c}) \vee \cdots \vee A_k(\bar{c}). \tag{3}$$

But clearly, (2) and (3) are a contradiction. We conclude therefore that $k \leqslant 1$ and that T is equivalent to T_H. If, additionally, T is finite, a standard compactness argument gives that T_H can be chosen to be finite, too. ∎

3.6. Definitions. Let **K** be a class of τ-structures closed under isomorphisms.

(i) We say that **K** is *closed under products* if whenever $\mathbf{A}_i \in \mathbf{K}$ $(i \in I)$ is a family of τ-structures then $\prod_{i \in I} \mathbf{A}_i \in \mathbf{K}$.

(ii) Let T be a first-order theory. We say that T is *preserved under products* if whenever $\mathbf{A}_i \models T$ $(i \in I)$ is a family of τ-structures then $\prod_{i \in I} \mathbf{A}_i \models T$.

The following characterization of universal Horn formulas is due to McKinsey [McK43].

3.7. Theorem (McKinsey). *A first-order theory T is equivalent to a universal Horn theory iff T is closed under products and substructures.*

We can now give an easy proof of the fact that every universal Horn theory admits initial term models.

3.8. Theorem. *Let T be a universal Horn theory (over a vocabulary τ containing at least one constant symbol). Then T admits initial-term models.*

Proof. We first show that T has an initial-term model. By Corollary 2.8 it suffices to show that T has an $\mathcal{A}$-generic-term model. Let A_i $(i \in I)$ be a list of all the atomic sentences which are not consequences of T and let $\mathbf{A}_i$ be a model of $T \cup \{\neg A_i\}$. Put $\mathbf{B} = \prod_{i \in I} \mathbf{A}_i$. Since T is a universal Horn theory, by Theorem 3.7, $\mathbf{B} \models T$. By the definition of the product $\mathbf{B} \models \neg A_i$ for every $i \in I$. Now let $\mathbf{B}_0$ be the term submodel of **B**. Since T is a universal theory, by Theorem 3.3, $\mathbf{B}_0 \models T$ and, since $\neg A_i$ is quantifier-free, $\mathbf{B}_0 \models \neg A_i$ for every $i \in I$. Therefore $\mathbf{B}_0$ is an $\mathcal{A}$-generic-term model of T.

To see that T admits initial-term models it suffices to observe that for every set of atomic sentences Δ and every universal Horn theory T the theory $T \cup \Delta$ is again a universal Horn theory. ∎

We can now collect the results of this section into one theorem.

3.9. THEOREM. *For a first-order theory T the following are equivalent*:

(i) *T admits initial-term models*;

(ii) *T is preserved under products and substructures*;

(iii) *T is equivalent to a universal Horn theory.*

Proof. (i) → (ii) Use Theorems 3.5, 3.7, and 3.8.

(ii) → (iii) This is Theorem 3.7.

(iii) → (i) This is Theorem 3.8. ∎

4. FIRST-ORDER THEORIES WITH THE INTERSECTION PROPERTY

In this section we show that first-order theories admitting initial models have the intersection property. This will allow us to use Theorem 4.7 of Rabin [Ra60, Ra62] here, together with the results of Sections 2 and 3, to characterize, in Section 5, first-order theories which admit initial models.

CONVENTION. *Let* $\bar{x} = x_0, \ldots, x_n$ *and* $\bar{y} = y_0, \ldots, y_n$. *We write in this and the following sections* $\exists!\, \bar{x}\phi(\bar{x})$ *for the conjunction of the two formulas* $\exists \bar{x}\, \phi(\bar{x})$ *and* $\forall \bar{x}\, \forall \bar{y}(\phi(\bar{x}) \wedge \phi(\bar{y}) \rightarrow \bigwedge_{i=0}^{i=n} x_i = y_i)$.

4.1. DEFINITIONS. (i) Let **K** be a class of structures closed under isomorphisms. **K** is said to have the *Intersection Property* if for every $\mathbf{A}, \mathbf{A}_1, \mathbf{A}_2 \in \mathbf{K}$ with $\mathbf{A}_i \subset \mathbf{A}$ $(i = 1, 2)$ either the intersection $\mathbf{A}_1 \cap \mathbf{A}_2 \in \mathbf{K}$ or $\mathbf{A}_1 \cap \mathbf{A}_2 =$ *empty*.

(ii) A first order theory T is said to have the *Intersection Property* if Mod(T) has the Intersection Property.

(iii) A first order theory T is said to have the *Infinite Intersection Property* if with every family $\mathbf{A}_i$ $(i \in I)$ such that $\mathbf{A}_i \subset \mathbf{B}$, $\mathbf{B} \models T$ and $\mathbf{A}_i \models T$ either $\bigcap_{i \in I} \mathbf{A}_i \models T$ or $\bigcap_{i \in I} \mathbf{A}_i =$ *empty*.

In the case of $\mathbf{K} = \text{Mod}(T)$ for some first order theory T if **K** has the Intersection Property then it has also the Infinite Intersection Property, cf. [Ro63].

4.2. THEOREM (Robinson). *A first-order theory T has the intersection property iff T has the infinite intersection property.*

4.3. EXAMPLES. (i) The class of all groups, abelian groups, rings, fields, algebraically closed fields all have the intersection property.

(ii) The theory T_{dense} from Example 2.6(iv) does not have the intersection property. We can easily find two dense subsets of the interval [0, 1] whose intersection is discrete.

(iii) If T is preserved under substructures then T has the intersection property. Therefore, by Theorem 3.3, every universal theory has the intersection property.

4.4. THEOREM. *Let T be a first-order theory which admits initial models. Then T has the intersection property.*

Proof. Let $\mathbf{B} = \mathbf{A}_1 \cap \mathbf{A}_2$ and let $\mathbf{D}_B$ be the atomic diagram of $\mathbf{B}$.

Claim 1. For every $\mathbf{C} \models T \cup \mathbf{D}_B$ we have

(i) $\mathbf{B} \subset \mathbf{C}$;

(ii) if $\bar{b} \in \mathbf{B}$ and $\mathbf{C} \models \exists!\, x\phi(x, \bar{b})$, where ϕ is a conjunction of atomic formulas, then $\mathbf{B} \models \exists!\, x\phi(x, \bar{b})$.

(i) follows since we require $\mathbf{C} \models \mathbf{D}_B$. To see (ii) we can asume that $\mathbf{C}$ is initial for $T \cup \mathbf{D}_B$. Clearly $\mathbf{A}, \mathbf{A}_1, \mathbf{A}_2$ have expansions $\bar{\mathbf{A}}, \bar{\mathbf{A}}_1, \bar{\mathbf{A}}_2$ satisfying $T \cup \mathbf{D}_B$ and there are unique homomorphisms h, h_1, h_2, respectively, mapping $\mathbf{C}$ into the corresponding structure. By the uniqueness of these homomorphisms and since $\mathbf{A}_i \subset \mathbf{A}$ $(i = 1, 2)$ we have that $Rg(h_1) = Rg(h_2) = Rg(h)$. Now let $\exists!\, x\phi(x, \bar{b})$ as required and $c \in \mathbf{C}$ be such that $\mathbf{C} \models \phi(c, \bar{b})$. So $h(c) \in \mathbf{A}_1 \cap \mathbf{A}_2 = \mathbf{B}$. This proves Claim 1.

Claim 2. Assume that $\mathbf{C}$ is initial for $T \cup \mathbf{D}_B$. Then $\mathbf{B} \cong \mathbf{C}$.

To see this we use the $\exists^+$-definability theorem 2.12 and Claim 1. ∎

First-order theories which have the intersection property were studied in the early days of model theory by Robinson and Rabin [Rab62, Rob51]. Then those theories were called *convex theories*, and they were studied mainly with an eye on classical algebra and possible generalizations of algebraic concepts to models of arbitrary first-order theories. We need two theorems of this work for our characterization of theories admitting initial models.

4.5. THEOREM (Robinson, Chang, and Łos and Suszko, cf. [Rob63]). *Let T be a first-order theory with the intersection property. Then T is equivalent to a set of $\forall\exists$-sentences.*

The above theorem is a condensed form of two theorems: Robinson's theorem asserts that a theory with the intersection property is a theory which is preserved under unions of chains; and the theorem of Chang and, independently, of Łos and Suszko asserts that a theory which is preserved under a union of chains is equivalent to a set of $\forall\exists$-sentences.

4.6. COROLLARY. *Let T be a theory which admits initial models. Then T is equivalent to a set of $\forall\exists$-sentences.*

Proof. Theorem 4.4 and Theorem 4.5. ∎

The next theorem, due to Rabin [Rab60], characterizes theories with the intersection property.

Notation. Let $\phi(\bar{x}, \bar{y})$ be a first-order formula with free variables $\bar{x} = x_1, x_2, \ldots, x_r$ and $\bar{y} = y_1, y_2, \ldots, y_n$ and let k be a natural number. We denote by $N(k, \bar{y}, \phi(\bar{x}, \bar{y}))$ the first-order formula which says that there are exactly k different n-tuples $\bar{y}$ satisfying the formula $\phi(\bar{x}, \bar{y})$.

4.7. THEOREM (Rabin [Rab60]). *A necessary and sufficient condition for a first-order theory T to have the intersection property is that for every $\forall\exists$-sentence $\forall\bar{x}\,\exists\bar{y}\,\psi(\bar{x}, \bar{y})$ which is a consequence of T, there exist two sequences of quantifier-free formulas*

$$\sigma_1(\bar{x}, \bar{u}), \sigma_2(\bar{x}, \bar{u}), \ldots, \sigma_m(\bar{x}, \bar{u})$$

and

$$\theta_1(\bar{x}, \bar{y}, \bar{z}), \theta_2(\bar{x}, \bar{y}, \bar{z}), \ldots, \theta_m(\bar{x}, \bar{y}, \bar{z})$$

and a sequence of natural numbers $k_1, k_2, \ldots, k_m$ such that

$$T \models \forall\bar{x}(\forall\bar{u}\,\sigma_1(\bar{x}, \bar{u}) \vee \forall\bar{u}\,\sigma_2(\bar{x}, \bar{u}) \vee \cdots \vee \forall\bar{u}\,\sigma_m(\bar{x}, \bar{u})), \tag{a}$$

and for $1 \leqslant i \leqslant m$,

$$T \models \forall\bar{x}(\forall\bar{u}\,\sigma_i(\bar{x}, \bar{u}) \rightarrow N(k_i, \bar{y}, \exists\bar{z}(\psi(\bar{x}, \bar{y}) \wedge' \theta_i(\bar{x}, \bar{y}, \bar{z})))). \tag{b}$$

In the next section we want to give a similar characterization for first-order theories admitting initial models. Our proof will depend on Rabin's theorem together with the $\exists^+$-definability theorem 2.12, Theorem 3.5, and Theorem 4.6.

Our next goal is to show the existence of initial models for certain theories which have the intersection property and which are preserved under products. For this we need some more definitions.

4.8. DEFINITIONS. Let T be a first-order theory with the intersection property. A model $\mathbf{A}_0$ of T is a *core model* if there is no proper submodel $\mathbf{B} \subset \mathbf{A}_0$ such that $\mathbf{B} \models T$. If $\mathbf{A}$ is a model of T and $\mathbf{A}_0 \subset \mathbf{A}$, $\mathbf{A}_{0'} \models T$ is a core model we say that $\mathbf{A}_0$ is a *T-core of* $\mathbf{A}$.

4.9. LEMMA. *Let T be a first-order theory with the intersection property and at least one constant symbol. Then every model $\mathbf{A}$ of T has a T-core $\mathbf{A}_0$.*

Proof. Let $\mathbf{A}_0$ be the intersection of all submodels of $\mathbf{A}$ which satisfy T. The constant symbol is needed to ensure that this intersection is not empty. ∎

4.10. PROPOSITION. *Let T be a first-order theory with the intersection property. Then every core model of T is an $\exists$-term model.*

Proof. The proof is a slight modification of the proof of Theorem 2.12. Let $\mathbf{A}$ be a core model of T and $a_0 \in A$. For every $a \in A$ let $\mathbf{a}$ be a constant symbol whose interpretation in $\mathbf{A}_I$ is a. Let $\mathbf{D}_A$ be the atomic diagram of $\mathbf{A}_I$ and $\mathbf{D}_{A'}$ the result of replacing every constant symbol $\mathbf{a}$ of $\mathbf{D}_A$ by $\mathbf{a}'$. Let Diff_A, $\mathrm{Diff}_{A'}$ be the set of negated atomic formulas $\{\mathbf{a} \neq \mathbf{b}: a, b \in A\}$, $\{\mathbf{a}' \neq \mathbf{b}': a, b \in A\}$, respectively.

Claim. $T \cup \mathbf{D}_A \cup \mathbf{D}_{A'} \cup \mathrm{Diff}_A \cup \mathrm{Diff}_{A'} \cup \{\mathbf{a}_0 \neq \mathbf{a}'_0\}$ is inconsistent.

For, otherwise, let $\mathbf{B}$ be a model for it. The sets of sentences Diff_A, $\mathrm{Diff}_{A'}$ guarantee that there are two isomorphic copies of $\mathbf{A}$ contained as submodels in $\mathbf{B}$. The formula $\mathbf{a}_0 \neq \mathbf{a}'_0$ guarantees that they are not identical and the intersection property guarantees that the intersection is isomorphic to a proper submodel of $\mathbf{A}$ satisfying T. But this contradicts the fact that $\mathbf{A}$ is a core model. Therefore we conclude that

$$T \cup \mathbf{D}_A \cup \mathbf{D}_{A'} \models \mathbf{a}_0 = \mathbf{a}'_0.$$

Now we use compactness to find a finite set of atomic formulas and inequalities $A_0, A_1, \ldots, A_k$ and constant symbols $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n$, $\mathbf{a}'_0, \mathbf{a}'_1, \ldots, \mathbf{a}'_n$ such that

$$T \models \bigwedge_{i=0}^{k} A_i(\mathbf{a}_0, \bar{\mathbf{a}}) \wedge' \bigwedge_{i=0}^{k} A_i(\mathbf{a}'_0, \bar{\mathbf{a}}') \rightarrow \mathbf{a}_0 = \mathbf{a}'_0,$$

where $\bar{\mathbf{a}} = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$, and $\bar{\mathbf{a}}' = (\mathbf{a}'_1, \ldots, \mathbf{a}'_n)$. Now put

$$\phi_{a_0} = \exists x_1, x_2, \ldots, x_n, x'_1, x'_2, \ldots, x'_n \bigwedge_{i=0}^{k} (A_i(\bar{x}) \wedge A_i(\bar{x}')).$$

Clearly ϕ_{a_0} is the required formula. ∎

In the above proof we had to use negated atomic formulas; this is why we could not get the core model to be an $\exists^+$-term model, i.e., a pseudo-term model. As we shall see below, what we really need are core models which are pseudo-term models.

4.11. PROBLEM. Let T be a first-order theory with the intersection property which is also preserved under products. Are the core models then pseudo-term models?

4.12. DEFINITION. A first-order theory T is *pseudo-algebraic* if T is preserved under products, has the intersection property and if every core model of T is a pseudo-term model.

4.13. THEOREM. *A pseudo-algebraic first-order theory T has an initial model $\mathbf{A}_I$.*

Proof. By Theorem 2.13 it suffices to show that T has an $\exists^+$-generic pseudo-term model $\mathbf{A}_I$. So let α_i $(i \in I)$ be all the $\exists^+$-sentences which are not consequences of T and let $\mathbf{B}_i \models T \cup \{\neg\alpha_i\}$. Put $\mathbf{B} = \prod_{i \in I} \mathbf{B}_i$ and let $\mathbf{A}_I$ be the T-core of $\mathbf{B}$.

Claim 1. $\mathbf{A}_I \models T$.

Clear, since T has the intersection property and is preserved under products.

Claim 2. $\mathbf{A}_I$ is $\exists^+$-generic.

We have to show that $\mathbf{A}_I \models \neg\alpha_i$ for every $i \in I$. We first observe that $\mathbf{B} \models \neg\alpha_i$. Assume, for contradiction, that $\mathbf{B} \models \alpha_i$. It is easily checked that then for every $j \in I$, $\mathbf{B}_j \models \alpha_i$, since α_i is an $\exists^+$-formula. So, in particular, $\mathbf{B}_i \models \alpha_i$, a contradiction. To conclude that $\mathbf{A}_I \models \neg\alpha_i$ it suffices to observe that α_i is a universal formula and to apply Theorem 3.3.

To conclude the proof use our assumption that every core model of T is a pseudo-term model. ∎

A converse of Theorem 4.13 will proved in the next section.

5. Characterizing First-Order Theories Which Admit Initial Models

The purpose of this section is to characterize first-order theories which admit initial models. We first want to show that such a theory is equivalent to a $\forall\exists$-Horn theory.

5.1. Theorem. *Let T be a first-order theory which admits initial models. Then*

(i) *T is equivalent to a $\forall\exists$-Horn theory $T_{\forall\exists H}$.*

(ii) *If T is finite, so is $T_{\forall\exists H}$.*

To prove Theorem 5.1 we first construct an auxiliary theory T^* in which every $\exists^+$-definable element is represented as a term.

5.2. Definitions. Let T be a first-order theory over a vocabulary τ.

(i) We say that T *has enough terms* if for every $\exists^+$-formula $\exists\bar{z}\,\alpha(y, \bar{z})$ such that $T \models \exists!\, y(\exists\bar{z}\,\alpha(y, \bar{z}))$ there is a τ-term t such that $T \models \exists\bar{z}\,\alpha(t, \bar{z})$.

(ii) We define T^* as follows: Let $\exists\bar{z}_i\,\alpha_i(y, \bar{z}_i)$ $(i \in I)$ be an enumeration of all the $\exists^+$-formulas over τ such that $T \models \exists!\, y(\exists\bar{z}\,\alpha(y, \bar{z}))$. Let $\mathbf{c}_i$ be a set of constant symbols not in τ. Put $\Sigma = \{\exists\bar{z}_i\,\alpha_i(\mathbf{c}_i, \bar{z}_i) : i \in I\}$. Now we put $T^* = T \cup \Sigma$.

5.3. Lemma. *T^* has enough terms.*

Proof. We have to show that for every $\exists^+$-formula $\exists\bar{z}\,\alpha(y, \bar{z}, \bar{\mathbf{c}})$ such that

$T \models \exists! y(\exists \bar{z}\, \alpha(y, \bar{z}, \bar{\mathbf{c}}))$ there is a τ^*-term t such that $T \models \exists \bar{z}\, \alpha(t, \bar{z}, \bar{\mathbf{c}})$. Assume for simplicity that $\bar{\mathbf{c}} = (\mathbf{c}_1, \mathbf{c}_2)$. Put

$$\beta(y) = \exists u_1\, \exists u_2(\exists \bar{z}\, \alpha(y, \bar{z}, u_1, u_2) \wedge \exists \bar{z}_1\, \alpha_1(u_1, \bar{z}_1) \wedge \exists \bar{z}_1\, \alpha_2(u_2, \bar{z}_1)).$$

But $\beta(y) = \alpha_i$ for some $i \in I$. ∎

5.4. Lemma. *Let T be a first-order theory which admits initial models and T^* the theory obtained from T. Then*

(i) *Every model $\mathbf{A}^*$ of T^* is also a model of T;*
(ii) *Every model $\mathbf{A}$ of T has a unique expansion to a τ^*-model $\mathbf{A}^*$ of T^*;*
(iii) *T^* admits initial term models.*

Proof. (i) and (ii) are trivial. To prove (iii) we first observe that T^* admits initial models since T admits initial models and every initial model is $\exists^+$-generic by Theorem 2.7. Now we use Theorem 2.7 together with Lemma 5.3 to conclude that the initial models of T^* are term models. ∎

Proof of Theorem 5.1. (i) Let T be a first-order theory which admits initial models. So, by Lemma 5.4, T^* admits initial-term models. Now we apply Theorem 3.5 and obtain a τ^*-theory T_H which is universal Horn and equivalent to T^*. Next we eliminate the newly introduced constant symbols from T_H and replace them by their defining formulas. One easily checks that in this way we obtain an $\forall\exists$-Horn theory $T_{\forall\exists}$ over the vocabulary τ. It is also easily verified that T and $T_{\forall\exists}$ are equivalent.

(ii) is a standard compactness argument. ∎

Next we want to state a generalization, or rather an analog of Rabin's Theorem 4.7 for theories which admit initial models.

5.5. Theorem. *Let T be a first-order theory which admits initial models. Then for every $\forall\exists$-sentence $\forall \bar{x}\, \exists \bar{y}\, \psi(\bar{x}, \bar{y})$ which is a consequence of T, there exist two sequences of formulas,*

$$\sigma_1(\bar{x}, \bar{u}), \sigma_2(\bar{x}, \bar{u}), \ldots, \sigma_m(\bar{x}, \bar{u})$$

and

$$\chi_1(\bar{x}, \bar{y}, \bar{z}), \chi_2(\bar{x}, \bar{y}, \bar{z}), \ldots, \chi_m(\bar{x}, \bar{y}, \bar{z})$$

where σ_i are quantifier free formulas and χ_i are $\exists^+$-formulas, such that

$$T \models \forall \bar{x}(\forall \bar{u}\, \sigma_1(\bar{x}, \bar{u}) \vee \forall \bar{u}\, \sigma_2(\bar{x}, \bar{u}) \vee \cdots \vee \forall \bar{u}\, \sigma_m(\bar{x}, \bar{u})), \tag{a}$$

and for $1 \leqslant i \leqslant m$,

$$T \models \forall \bar{x}(\forall \bar{u}\, \sigma_i(\bar{x}, \bar{u}) \rightarrow \exists! \bar{y}(\exists \bar{z}(\psi(\bar{x}, \bar{y}) \wedge \chi_i(\bar{x}, \bar{y}, \bar{z})))). \tag{b}$$

Proof. We first apply Theorem 4.4 to establish the intersection property and then Theorem 4.5. This gives us that T is equivalent to an $\forall\exists$-theory $T_{\forall\exists}$.

Now let $\forall\bar{x}\,\exists\bar{y}\,\psi(\bar{x}, \bar{y}) \in T_{\forall\exists}$. Using Theorem 4.7 we can find quantifier-free formulas

$$\sigma_1(\bar{x}, \bar{u}), \sigma_2(\bar{x}, \bar{u}),..., \sigma_m(\bar{x}, \bar{u})$$

and

$$\theta_1(\bar{x}, \bar{y}, \bar{z}), \theta_2(\bar{x}, \bar{y}, \bar{z}),..., \theta_m(\bar{x}, \bar{y}, \bar{z}),$$

and a sequence of natural numbers $k_1, k_2,..., k_m$ such that

$$T_{\forall\exists} \models \forall\bar{x}(\forall\bar{u}\,\sigma_1(\bar{x}, \bar{u}) \vee \forall\bar{u}\,\sigma_2(\bar{x}, \bar{u}) \vee \cdots \vee \forall\bar{u}\,\sigma_m(\bar{x}, \bar{u})), \tag{a}$$

and for $1 \leqslant i \leqslant m$,

$$T_{\forall\exists} \models \forall\bar{x}(\forall\bar{u}\,\sigma_i(\bar{x}, \bar{u}) \rightarrow N(k_i, \bar{y}, \exists\bar{z}(\psi(\bar{x}, \bar{y}) \wedge \theta_i(\bar{x}, \bar{y}, \bar{z}))). \tag{b}$$

Our next goal is to show that we can w.l.o.g. assume that $k_1 = k_2 = \cdots = k_m = 1$.

Let $k_1 > 1$. Let $\mathbf{A}_I$ be the initial model of T and let $\bar{a}_1, \bar{a}_2,..., \bar{a}_{k_1}$ be distinct tuples of $\mathbf{A}_I$ such that

$$\mathbf{A}_I \models \forall\bar{x}(\forall\bar{u}\,\sigma_i(\bar{x}, \bar{u}) \rightarrow \exists\bar{z}(\psi(\bar{x}, \bar{a}_i \wedge \theta_i(\bar{x}, \bar{a}_j, \bar{z}))).$$

for every $j = 1,..., k_1$. By Theorem 2.12 there is an $\exists^+$-formula $\alpha_j(\bar{v})$ defining $\bar{a}_j$, i.e., such that $\mathbf{A}_I \models \alpha_j(\bar{a}_j)$ and if $\bar{b} \in A_I$ and $\mathbf{A}_I \models \alpha_j(\bar{b}_j)$ then $\bar{a} = \bar{b}$.

Now we define $\theta_{ij} = \theta_i(\bar{x}, \bar{y}, \bar{z}) \wedge \alpha_j(\bar{y})$. Clearly we have

$$\mathbf{A}_I \models \forall\bar{x}(\forall\bar{u}\,\sigma_i(\bar{x}, \bar{u}) \rightarrow \exists!\,\bar{y}(\exists\bar{z}(\psi(\bar{x}, \bar{y})) \wedge \theta_{ij}(\bar{x}, \bar{y}, \bar{z}))).$$

Using argument similar to that in 2.12 we conclude that

$$T_{\forall\exists} \models \forall\bar{x}(\forall\bar{u}\,\sigma_i(\bar{x}, \bar{u}) \rightarrow \exists!\,\bar{y}(\exists\bar{z}(\psi(\bar{x}, \bar{y})) \wedge \theta_{ij}(\bar{x}, \bar{y}, \bar{z}))). \quad \blacksquare$$

5.6. Definition. We call a first-order theory which satisfies the conclusion of Theorem 5.5 *partially functional*. This is justified since Theorem 5.5 says that every $\forall\exists$-formula which is a consequence of T can be Skolemized with finitely many partial functions.

5.7. Corollary. *Let T be a first-order theory which admits initial models. Then every $\exists$-term model $\mathbf{A}$ is a pseudo-term model.*

Proof. Let $a \in A$ and let $\alpha(x)$ be its $\exists$-formula defining it. Now we use Theorem 5.5 with $x = y$ for ψ. Clearly $\forall x\,\exists y\,(x = y)$ is a consequence of T. Now we apply (a) and (b) of Theorem 5.5. $\blacksquare$

We need another well-known result from model theory, see, e.g., [CK73].

5.8. THEOREM. *Let T be an $\forall\exists$-Horn theory. Then T is preserved under products.*

Putting everything together we obtain

5.9. THEOREM (main theorem). *Let T be a first-order theory. The following are equivalent*:

(i) *T admits initial models;*

(ii) *T is equivalent to a partially functional $\forall\exists$-Horn theory.*

(iii) *T is pseudo-algebraic.*

Proof. (i) → (ii) Use Theorems 5.1 and 5.5.

(ii) → (iii) Clearly T is preserved under products, by Theorem 5.8. To see, that T has the intersection property we use Theorem 4.7. To verify that every $\exists$-term model is a pseudo-term model we use Corollary 5.7.

(iii) → (i) This is Theorem 4.13. ▮

6. THE INDEPENDENCE PROPERTY FOR INEQUALITIES

In this section we give an application of our characterization of admitting initial models to the simultaneous solvability of inequations. A similar problem is addressed in [Col84] in the context of unification algorithms which also preserve inequations.

Let Σ be a partially functional $\forall\exists$-Horn theory and let

$$I = \{t_{i1} \neq t_{i2} : i \in \omega\}$$

be an infinite set of inequations. Assume further that I and Σ have only finitely many free variables $x_1, ..., x_n$. We are interested in the satisfiability of $\Sigma \cup I$. In the case that Σ consists only of a set of equations E, this is the problem of finding a simultaneous solution $(c_1, ..., c_n)$ of E and I. Our result shows that this is possible iff it can be done for each inequation separately. More precisely, we have

6.1. THEOREM. *Let Σ and I be as above. Then $\Sigma \cup I$ is satisfiable iff for every $i \in \omega$ $\Sigma \cup \{t_{i1} \neq t_{i2}\}$ is satisfiable.*

Proof. The *only if* part is trivial. For the *if* part let $c_1, ..., c_n$ be new constant symbols not occuring in $\Sigma \cup I$ and let Σ^*, I^* be obtained from Σ and I respectively by substituting the free variables x_j by c_j. Clearly, Σ^* is still a partially functional $\forall\exists$-Horn theory. Therefore, Σ^* admits an initial model $\mathbf{A}$, by Theorem 5.9. By Theorem 2.13, $\mathbf{A}$ is an $\exists^+$ pseudo-term model.

Now assume that for every $i \in \omega$, $\Sigma \cup \{t_{i1} \neq t_{i2}\}$ is satisfiable. So it is not the case that $\Sigma^* \models t_{i1} = t_{i2}$. Therefore $\mathbf{A} \models t_{i1} \neq t_{i2}$ for every $i \in \omega$ and $\mathbf{A} \models I^*$. ▮

The same proof shows that we can add to I a set of negated atomic formulas or even to a set of negated $\exists^+$-formulas.

7. Conclusions

We have given a characterization of universal Horn theories in terms of the existence of initial, or equivalently, A-generic term models (Theorem 3.9) and a characterization of partially functional $\forall\exists$-Horn theories in terms of the existence of initial, or equivalently, $\exists^+$-generic pseudo-term models (Theorem 5.9).

The significance of this characterization can be explained as follows:

We singled out a property of the class of universal Horn formulas (admitting initial term models) which characterize this class of formulas (Sect. 3).

We analyzed the notion of an initial (term) model and show that it is equivalent to the notion of a $\exists^+$-generic (A-generic) model. This property (admitting initial-term models, admitting A-generic models) garantees that

(i) the "closed-world assumption" for databases and "negation as failure" for logic programming are reasonable concepts;

(ii) logic programs allow a procedural interpretation, because there is a *unique* "generic" mathematical structure in which to interprete logic programs;

(iii) we have the independence property for the simultaneous solution of inequations (Theorem 6.1);

(iv) that the three previous advantages are preserved under the addition of new facts to a databasis or a logic program.

We then discussed (Sects. 4–5) what happens if we drop the requirement that the initial model has to be a term model and show that in this latter case the class of formulas characterized by this liberalized property is almost the class of universal Horn formulas in the following sense: The only way it can fail to be universal Horn is by a wrong choice of the primitive symbols involved, namely that the sorts overlap in the wrong way and that certain functions were represented by their graphs (as relations). Our main theorem (5.9) really says that if a first-order theory admits initial models, then the sorts and relation and function symbols can be chosen in a natural way such that we obtain an equivalent theory which is universal Horn. Sets of first-order formulas of that latter type are called *partially functional $\forall\exists$-theories.*

We have, therefore, shown an intimate relationship between two semantic concepts (initially and $\exists^+$-genericity) and a syntactic restriction (partially functional $\forall\exists$-theories). We believe that it is this intimate relationship between these concepts which answers the question posed in the title of this paper.

The paper also sheds more light onto the question why in [ADJ75] initial structures were proposed as the framework for abstract data types. We have given in Theorem 2.13 a characterization of initial structures as $\exists^+$-generic pseudo-term models. For somebody not familiar with category theory this may be more appealing since it relates directly to our concept of verification by example. However, this characterization has also its technical merits for it provides the missing link between the category-theoretic concept and the model-theoretic tools needed to prove 5.9.

Last but not least, we have added yet another explanation as to why Horn formulas play such an important role in the various branches of computer science. We have shown that universal Horn theories (partially functional $\forall\exists$-theories) are exactly the framework in which the notion of a generic example can be applied. This should prevent other researchers from trying to generalize logic programming or the semantics for abstract data types to larger classes of first-order formulas, as done in [CM80]. If it has to be generalized then the direction chosen by Burstall and Goguen in [GB84] seems to be *much more appropriate.*

The reader should not misunderstand our point: We do not claim that one has to restrict oneself to Horn formulas when dealing with logic programming or databases. We only propose an *explanatory paradigm*: If there are reasons that the existence of initial models, $\exists^+$-generic models, are crucial for the activity one has in mind then the restriction to Horn formulas is necessary. More elaborate model-theoretic variations of our characterization theorems may be found in [Vo85], which was written after our [Ma85].

Further research should pursue this and similar paradigms to identify the syntactic restrictions which come from semantic requirements. This amounts to exploring how far *abstract model theory* (as described in [BF85]) can be made more fruitful for computer science and to what extent the direction taken in [GB84] can be further pushed to encompass also the definition and specification of programming languages and programming environments in general.

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