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Primitive Ovoids in $O_8^+(q)$

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Kleidman (1988) has classified the 2-transitive ovoids in finite polar spaces. We show that any ovoid $\mathcal{O}$ in $O_8^+(q)$ for which $Aut(\mathcal{O})$ permutes the points of $\mathcal{O}$ primitively, is either the Cooperstein ovoid in $O_8^+(5)$, or is 2-transitive (and so appears in Kleidman's list). © 2000 Academic Press

1. INTRODUCTION

The classification theorem of finite simple groups is followed by several classifications of combinatorial structures whose automorphism groups act 2-transitively on their points. Kantor [8], Kleidman [12], and Taylor [20] have classified 2-transitive finite linear spaces, ovoids in finite polar spaces, and two-graphs respectively. The natural extension in this direction is to classify structures whose automorphism groups act primitively on their points. Although we do not have an explicit list of finite primitive groups, we use the O'Nan-Scott theorem about primitive permutation groups along with other number theoretical constraints found in ovoids to reduce the problem to the case of almost simple groups. Our concentration on $O_8^+(q)$ is justified by the facts that $O_8^+(q)$ provides one of the richest geometries in polar spaces and many questions about ovoids in $O_8^+(q)$ remain unanswered. We state our main result in Section 2. Our results show that there is only one ovoid which is primitive but not 2-transitive, namely the Cooperstein ovoid in $O_8^+(5)$.

The organization of this paper is as follows. In Section 2, we introduce our notation and state our main result as Theorem 2.1. In Section 3, we include several lemmas and in Section 4, we prove Theorem 2.1.

2. NOTATION AND THE LIST OF PRIMITIVE OVOIDS

More details of the following definitions can be found in [1, 2, 7]. An *orthogonal space of type* $O_8^+(q)$ is a pair (V, Q) such that V is a vector

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space of dimension 8 over $GF(q)$ and $Q: V \to GF(q)$ is a quadratic form, i.e., $Q(\lambda x) = \lambda^2 Q(x)$ and $Q(x+y) = Q(x) + Q(y) + (x, y)$, for all $\lambda \in GF(q)$; $x, y \in V$, where $(\, , \,)$ is a bilinear form and Q is nondegenerate of hyperbolic type. The *points* of $O_8^+(q)$ are one-dimensional subspaces and $\langle v \rangle$ is called a *singular* point if $Q(v) = 0$. An *ovoid* $\mathcal{O}$ in $O_8^+(q)$ space is a set of singular points such that every maximal totally singular subspace contains just one point in $\mathcal{O}$, or equivalently, $\mathcal{O} = \{\langle v_i \rangle : 1 \leqslant i \leqslant q^3 + 1\}$ consists of singular points such that $(v_i, v_j) = 0$ iff $i = j$.

The *orthogonal group* $O_8^+(q)$ is the subgroup of $GL(V)$ which fixes the quadratic form. The *generalized orthogonal group* $GO_8^+(q)$ is the subgroup of $GL(V)$ which fixes the *quadric* (the set of singular points), and hence $|GO_8^+(q) : O_8^+(q)| = q - 1$. Let $G = GO_8^+(q)$. The *automorphism group* of an ovoid $\mathcal{O}$, denoted $Aut(\mathcal{O})$, is defined as $Aut(\mathcal{O}) = G_{\mathcal{O}}/G_{(\mathcal{O})}$, where $G_{\mathcal{O}}$ is the stabilizer of $\mathcal{O}$ in G and $G_{(\mathcal{O})}$ is the pointwise stabilizer of $\mathcal{O}$ in G. A *primitive* (*or* 2-*transitive*) ovoid is an ovoid $\mathcal{O}$ whose automorphism group acts primitively (or 2-transitively) as a permutation group on $\mathcal{O}$. Two ovoids are *isomorphic* if there is a group element in the generalized orthogonal group that takes one ovoid to the other. We use $Lie(p)$ as the set of simple groups of Lie type (classical or exceptional) over fields of characteristic p and define $Lie(p')$ as $\bigcup_{r \neq p} Lie(r)$. Our main result is stated in the following theorem.

2.1. THEOREM. *Let $\mathcal{O}$ be a primitive ovoid in $O_8^+(q)$ space. Then $\mathcal{O}$ is either the Cooperstein ovoid in $O_8^+(5)$, or a 2-transitive ovoid (and so appears in Kleidman's list [12]). All the primitive ovoids in $O_8^+(q)$ are shown in Table I. Each entry corresponds to a unique isomorphism class.*

TABLE I

Primitive Ovoids in $O_8^+(q)$

$Aut(\mathcal{O})$	Restrictions	Reference
S_9	$q = 2$	[7, Sect. 3]
$Sp_6(2)$	$q = 3$, induced from $O_7(3)$	[7, Sect. 3]
S_{10}	$q = 5$, Cooperstein Ovoid	[5, Sect. 2]
Aut($PSU_3(q)$)	$5 \leqslant q \equiv 0, 2 \ (mod\ 3)$ induced from $O_7(q)$ if $q \equiv 0 \ mod\ 3$	[7, Sect. 4]
Aut($PSL_2(q^3)$)	q even, $q \geqslant 4$	[7, Sect. 7]
Aut($^2G_2(q)$)	q an odd power of 3, $q \geqslant 27$, induced from $O_7(q)$	[7, Sect. 6]

3. PRELIMINARIES

Our approach is to use the O'Nan-Scott theorem to reduce the case of a primitive group to a case of an almost simple group. In the following lemma, we prove a useful number theoretical result that will help us rule out some of the cases in the O'Nan-Scott theorem. An alternative proof of this lemma with elementary number theory can be found in [6].

3.1. LEMMA. *Let p be a prime and k, n, m be positive integers. If $p^k + 1 = n^m$ then one of the following is true.*

1. $m = 1$ *or*
2. $m = p = 2$, $n = k = 3$.

Proof. Assume $p^k + 1 = n^m$ and $m > 1$. Since $p^k = (n-1)\sum_{i=0}^{m-1} n^i$, we have $n = p^r + 1$ and $m = sp$ for some integers $r, s \geqslant 1$. Now we can rewrite the above equation as $p^k + 1 = (p^r + 1)^m$. The result follows from [14, Theorem 7]. ∎

3.2. LEMMA. *Let T be a nonabelian finite simple group. Suppose T embeds in $PGL_n(F)$, $n \leqslant 8$ and $char(F) = p$. Then T must be one of the following groups. (Note: In these groups, only one entry is included from each of the isomorphism classes given in [13, p. 43].)*

1. *Sporadic groups*
   M_{11}, M_{12}, M_{22}, J_1, J_2.
2. *Alternating groups*
   A_5, A_6, A_7, A_8, A_9, A_{10}.
3. $T \in Lie(p')$
   $^2B_2(8)$, $PSL_2(7)$, $PSL_2(8)$, $PSL_2(11)$, $PSL_2(13)$, $PSL_2(17)$, $PSL_3(4)$, $PSp_4(3)$, $PSp_6(2)$, $PSU_3(3)$, $PSU_4(3)$, $P\Omega_8^+(2)$.
4. $T \in Lie(p)$
   $^3D_4(r)$, $^2B_2(r)$ $(r = 2^{2m+1})$, $^2G_2(r)$ $(r = 3^{2m+1})$, $G_2(r)$, $PSL_n(r) (n \leqslant 8)$, $PSU_n(r) (3 \leqslant n \leqslant 8)$, $\Omega_7(r)$ (r odd), $PSp_{2m}(r)$ $(m = 2, 3, 4)$, $P\Omega_8^+(r)$, $P\Omega_8^-(r)$.

Proof. Let T be a nonabelian simple group. A lower bound for n when T embeds in $PGL_n(F)$ is known. These results are directly obtained from those lower bounds given in [13, p. 186] for alternating groups, [13, p. 187] for sporadic groups, [18] for $T \in Lie(p')$, and [13, p. 200] for $T \in Lie(p)$. ∎

LEMMA. *Let G_0 be a nonabelian simple group that can be embedded in $PGL_n(F)$, $n \leqslant 8$, $char(F) = p$, and G be a group such that $G_0 \trianglelefteq G \leqslant Aut(G_0)$.*

Suppose G_0 is a sporadic, or an alternating, or $G_0 \in Lie(p')$. Suppose G has a maximal subgroup of index q^3+1 not containing G_0, where q is a power of p. Then one of the following possibilities occurs.

1. $G_0 = A_8$ *and* $q = 3$.
2. $G_0 = A_9$ *and* $q = 2, 5$.
3. $G_0 = A_{10}$ *and* $q = 5$.
4. $G_0 = PSL_2(7)$ *and* $q = 3$.
5. $G_0 = PSL_2(8)$ *and* $q = 2, 3$.
6. $G_0 = PSp_6(2)$ *and* $q = 3$.
7. $G_0 = PSU_3(3)$ *and* $q = 3$.
8. $G_0 = PSL_4(2) \cong A_8$ *and* $q = 3$.
9. $G_0 = PSU_4(3)$ *and* $q = 5$.

Proof. Let G_0 be a nonabelian simple group as mentioned in Lemma 3.3. Lemma 3.2 gives us the possible candidates for G_0. For these candidates, the indices of the maximal subgroups of G when $G_0 \trianglelefteq G \leqslant Aut(G_0)$ are given in [4]. If we check these indices the only possibilities are as mentioned in Lemma 3.3. ∎

3.4 LEMMA. *Let G_0 be a nonabelian simple group that can be embedded in $PGL_n(F)$, $n \leqslant 8$, $char(F) = p$, and G be a group such that $G_0 \trianglelefteq G \leqslant Aut(G_0)$. Suppose $G_0 \in Lie(p)$. and G has a maximal parabolic subgroup H of index q^3+1 not containing G_0, where q is a power of p. Then one of the following possibilities occurs.*

1. $G_0 = {}^2G_2(r)$, $r = 3^{2m+1} = q$, *one conjugacy class of H.*
2. $G_0 = PSL_2(r)$, $r = q^3$, *one conjugacy class of H.*
3. $G_0 = PSU_3(r)$, $r = q$, *one conjugacy class of H.*

Proof. Let G_0 be a nonabelian simple group as mentioned in Lemma 3.4. Lemma 3.2 gives us the possible candidates for G_0. For these candidates, the indices and conjugacy classes of the maximal parabolic subgroups of G when $G_0 \trianglelefteq G \leqslant Aut(G_0)$ are given in [11] for ${}^3D_4(r)$, [19] for ${}^2B_2(r)$, [10] for ${}^2G_2(r)$, [10] or [16] for $G_2(r)$, r odd, [3] for $G_2(r)$, r even, [13, Propositions 4.1.4, 4.1.17, 4.1.22] for $PSL_n(r)$, [13, Propositions 4.1.4, 4.1.18] for $PSU_n(r)$, [13, Propositions 4.1.3, 4.1.19] for $PSP_n(r)$, [13, Propositions 4.1.6, 4.1.20] for $\Omega_7(r)$, [13, Propositions 4.1.6, 4.1.7, 4.1.20] for $P\Omega_8^-(r)$, and [13, Propositions 4.1.6, 4.1.7, 4.1.20] or [9] for $\Omega_8^+(r)$. If we check these subgroups' indices the only possibilities are as mentioned in Lemma 3.4. ∎

3.5 LEMMA. *Let (V, Q) be an orthogonal space with an n-dimensional vector space V over $GF(q)$ and a nondegenerate quadratic form Q. Assume $\mathcal{O}$ is an ovoid in the space such that $\mathcal{O}$ is not induced from any subgeometry. Let G be the group of all $g \in GO_8^+(q)$ which fix the quadric. Then $G_{(\mathcal{O})}$ (the pointwise stabilizer of $\mathcal{O}$ in G) is $Z(GL(V))$.*

Proof. Since $\mathcal{O}$ is not induced from any subgeometry, we can find a basis $\{v_1, v_2, ..., v_n\}$ for V such that each $\langle v_i \rangle \in \mathcal{O}$. We also have $(v_i, v_j) = 0$ if and only if $i = j$. Let $g \in G_{(\mathcal{O})}$. Thus, $g(v_i) = \lambda_i v_i$ for nonzero λ_i. Since $g \in G$, there exists nonzero λ such that $(g(x), g(y)) = \lambda(x, y)$ for all $x, y \in V$. Thus, $(g(v_i), g(v_j)) = \lambda(v_i, v_j) = \lambda_i \lambda_j (v_i, v_j)$. If $i \neq j$ then $\lambda = \lambda_i \lambda_j$ since $(v_i, v_j) \neq 0$. This implies that $\lambda_1 = \lambda_2 = \cdots = \lambda_n$. Thus $g \in Z(GL(V))$. It is clear that $Z(GL(V)) \subseteq G_{(\mathcal{O})}$. Therefore, $G_{(\mathcal{O})} = Z(GL(V))$. ∎

4. PROOF OF THEOREM 2.1

Let $\mathcal{O}$ be an ovoid in $O_8^+(q)$ space, where $q = p^k$, p a prime and k a positive integer. Let G be a primitive automorphism group of $\mathcal{O}$ with socle N. Thus G is a permutation group of degree $n = |\mathcal{O}| = q^3 + 1$. It is known that $O_8^+(2)$ and $O_8^+(3)$ spaces contain unique ovoids up to isomorphism [7]. Any 2-transitive ovoid in $O_8^+(q)$ appears in Kleidman's list [12]. Therefore, we can assume hereafter that G is not 2-transitive and $q > 3$.

Let N be the socle of G. According to the O'Nan-Scott Theorem [15], $N \cong T^m$ with $m \geqslant 1$, where T is a simple group with the following possibilities for G.

1. Affine groups, $T = Z_s$ for some prime s and $n = s^m$ (By Lemma 3.1, no possibility occurs when $q > 3$),

2. Almost simple groups, here $m = 1$, T is a nonabelian simple group and $T \leqslant G \leqslant Aut(T)$,

3. In this case $N \cong T^m$ with $m \geqslant 2$ and T a nonabelian simple group.

The following three possibilities occur.

(a) Simple diagonal action, $n = |T|^{m-1}$ (By Lemma 3.1, the only possibility is $m = 2$),

(b) Product action, $n = t^l$ for some integers t, l with $l > 1$ (This does not occur by Lemma 3.1),

(c) Twisted wreath action, $n = |T|^k$ (This does not occur by Lemma 3.1).

Thus, we have reduced our situation to the two cases [A] G is an almost simple group, or [B] $N \cong T^2$, T is a nonabelian simple group with $|T| = q^3+1$. By Lemma 3.5, if $\mathcal{O}$ is eight dimensional then $Aut(\mathcal{O}) \leqslant PGL_8(q)$ and if $\mathcal{O}$ is seven dimensional $Aut(\mathcal{O}) \leqslant PGL_7(q)$. Thus, Lemma 3.2 gives us the possible candidates for the nonabelian simple groups appearing in statements A & B above. If we check the orders of the groups [4] given in Lemma 3.2, none of the groups has the order equal to q^3+1 for any q.

Thus, we have reduced our work to the situation where $G_0 \trianglelefteq G \leqslant Aut(G_0)$ and G_0 is one of the simple groups given in Lemma 3.2. Since any point stabilizer G_x is maximal and $[G : G_x] = q^3+1$, we can check the indices of maximal subgroups of G. Assume $H = G_x$. Since G_0 is transitive on $\mathcal{O}$, H cannot contain G_0. Since $G_0 H = G$, $[G : H] = [G_0 : G_0 \cap H]$.

When G_0 is sporadic, by Lemma 3.3, no possibility occurs with a maximal subgroup of index q^3+1. When G_0 is alternating, by Lemma 3.3, the only possibilities are A_9 and A_{10} with $q = 5$. There is a unique transitive ovoid stabilized by A_9 in $O_8^+(5)$ space and it must be primitive since the point stabilizer is maximal. This ovoid is due to Cooperstein [5].

Next, we consider the cases of classical and exceptional groups. Lemma 3.3 covers the situation where $G_0 \in Lie(p')$. There is only one possibility with $G_0 = PSU_4(3)$ and $q = 5$. This is not possible by Lagrange's Theorem.

Hereafter, we assume that $G_0 \in Lie(p)$. The following argument which substantially shortened the original proof was suggested by an annonymous referee. Since $[G : H] = q^3+1$, H contains a Sylow p-subgroup of G. Since H is maximal in G and H does not contain G_0, by "Tits' Lemma" ([17, 1.6]) H must be a parabolic subgroup of G. Lemma 3.4 gives the possibilities for G. These must be the known 2-transitive representations and hence no possibility occurs. ∎

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