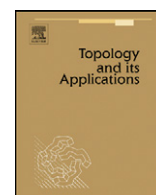Contents lists available at ScienceDirect

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A Boolean algebra and a Banach space obtained by push-out iteration

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A B S T R A C T

Under the assumption that $\mathfrak{c}$ is a regular cardinal, we prove the existence and uniqueness of a Boolean algebra $\mathfrak{B}$ of size $\mathfrak{c}$ defined by sharing the main structural properties that $\mathcal{P}(\omega)/\mathit{fin}$ has under CH and in the $\aleph_2$-Cohen model. We prove a similar result in the category of Banach spaces.

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1. Introduction

In this paper two lines of research converge, one related to the theory of Boolean algebras and the other one to Banach spaces.

In the context of Boolean algebras, the topic goes back to Parovičenko's theorem [12], which establishes that, under CH, $\mathcal{P}(\omega)/\mathit{fin}$ is the unique Boolean algebra of size $\mathfrak{c}$ with the property that given any diagram of embeddings of Boolean algebras like

$$
\begin{array}{c}
S \\
\big\uparrow \\
R \longrightarrow \mathcal{P}(\omega)/\mathit{fin},
\end{array}
$$

where R and S are countable, there exists an embedding $S \to \mathcal{P}(\omega)/\mathit{fin}$ which makes the diagram commutative. This characterization is indeed equivalent to CH [4]. There has been a line of research [5–9,14] showing that many results about $\mathcal{P}(\omega)/\mathit{fin}$ under CH can be generalized to the $\aleph_2$-Cohen model[3] (and to a less extent to any Cohen model). The key point was proven by Steprāns [14] and is that in that case $\mathcal{P}(\omega)/\mathit{fin}$ is tightly σ-filtered. Later, Dow and Hart [5] introduced the notion of Cohen–Parovičenko Boolean algebras. On the one hand, $\mathcal{P}(\omega)/\mathit{fin}$ is Cohen–Parovičenko in Cohen models,

and on the other hand there exists a unique Cohen–Parovičenko Boolean algebra of size $\mathfrak{c}$ whenever $\mathfrak{c} \leqslant \aleph_2$. Therefore this property characterizes $\mathcal{P}(\omega)/\mathit{fin}$ in the $\aleph_2$-Cohen model in the spirit of Parovičenko's theorem.

Our main result states that whenever $\mathfrak{c}$ is a regular cardinal, there exists a unique tightly σ-filtered Cohen–Parovičenko Boolean algebra. Essentially, we are proving that whenever $\mathfrak{c}$ is regular, there exists a canonical Boolean algebra that behaves like $\mathcal{P}(\omega)/\mathit{fin}$ does under CH and in the $\aleph_2$-Cohen model. By a result of Geschke [9], this Boolean algebra cannot be $\mathcal{P}(\omega)/\mathit{fin}$ when $\mathfrak{c} > \aleph_2$. Hence, in the κ-Cohen model ($\kappa > \aleph_2$), $\mathcal{P}(\omega)/\mathit{fin}$ and our Boolean algebra are two non-isomorphic Cohen–Parovičenko Boolean algebras, and this solves a question of Dow and Hart [5, Questions 2 and 3] about uniqueness for this property.

We found that many notions being used in the literature on this topic can be reformulated using the concept of push-out from category theory (see [1]). Apart from aesthetic considerations, this approach has an objective advantage: it allows us to export ideas to other categories like we do with Banach spaces, where we have push-outs with similar properties.

In the context of Banach spaces, an analogue of Parovičenko's theorem has been established by Kubiś [11]: Under CH, there exists a unique Banach space $\mathfrak{X}$ of density $\mathfrak{c}$ with the property that given any diagram of isometric embeddings like

$$
\begin{array}{ccc}
S & & \\
\uparrow & & \\
| & & \\
R & \longrightarrow & \mathfrak{X}
\end{array}
$$
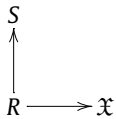
where R and S are separable, there exists an embedding $S \to \mathfrak{X}$ which makes the diagram commutative. The existence of such a space can be proven in ZFC but not its uniqueness [2]. In this paper we shall consider a stronger property than the one stated above which will imply existence and uniqueness of the space $\mathfrak{X}$ under the weaker assumption that $\mathfrak{c}$ is a regular cardinal.

In both situations the method of construction is the same as in [5,2] respectively, by a long chain of push-outs. The difficulty lies in proving that the algebra or the space constructed in this way is indeed unique. It would be nice to have a unified approach for Banach spaces and Boolean algebras in the context of category theory, but our attempts to do so seemed to become too technical and to obscure both subjects rather than to enlighten them. The reader will find, nevertheless, an obvious parallelism.

The paper is structured as follows: In Section 2 we prove the existence and uniqueness of our Boolean algebra $\mathfrak{B}$. In Section 3 we do the same with our Banach space $\mathfrak{X}$. The two sections run parallel but they can be read independently, and for the convenience of the reader they include a self-contained account of the facts that we need about push-outs in each category. In Section 4 we consider the compact space $\mathfrak{K}$, Stone dual of $\mathfrak{B}$, we establish that $\mathfrak{X}$ is isometric to a subspace of the space of continuous functions $C(\mathfrak{K})$ and we also prove that $\mathfrak{K}$ is homogeneous with respect to P-points, generalizing the results of Rudin [13] under CH, Steprāns [14] in the $\aleph_2$-Cohen model, and Geschke [9]. In Section 5 we state several open problems and in Section 6 we point out more general versions of our results for different cardinals.

We would like to thank Wiesław Kubiś for some relevant remarks that helped to improve this paper.

2. Boolean algebras

2.1. Preliminary definitions

The push-out is a general notion of category theory, and the one that we shall use here refers to the category of Boolean algebras. We shall consider only push-outs made of embeddings (one-to-one morphisms), although it is a more general concept. For this reason, we present the subject in a different way than usual, more convenient for us and equivalent for the case of embeddings.

The join and meet operations in a Boolean algebra B are denoted by $\vee$ and $\wedge$, and the complement of $r \in B$ by $\bar{r}$. The subalgebra generated by H is $\langle H \rangle$.

Definition 1. Let B be a Boolean algebra and let S and A be subalgebras of B. We say that B is the internal push-out of S and A if the following conditions hold:

(1) $B = \langle S \cup A \rangle$.
(2) For every $a \in A$ and every $s \in S$, if $a \wedge s = 0$, then there exists $r \in A \cap S$ such that $a \leqslant r$ and $s \leqslant \bar{r}$.

We notice that condition (2) above can be substituted by the following equivalent one:

(2′) For every $a \in A$ and every $s \in S$, if $a \leqslant s$, then there exists $r \in A \cap S$ such that $a \leqslant r \leqslant s$.

The same definition can be found in [9] as "A and S commute".

Suppose that we have a diagram of embeddings of Boolean algebras:

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & A. \end{array}$$
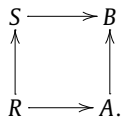
We say that it is a push-out diagram if, when all algebras are viewed as subalgebras of B, we have that $R = S \cap A$ and B is the internal push-out of A and S.

If B is the push-out of S and A, then B is isomorphic to $(S \otimes A)/V$ where $S \otimes A$ denotes the free sum of S and A, and V is the ideal generated by the formal intersections $r \wedge \bar{r}$, where $r \in A \cap S$ (viewing $r \in S$, $\bar{r} \in A$). Also, given a diagram of embeddings

$$\begin{array}{c} S \\ \uparrow \\ R \longrightarrow A \end{array}$$
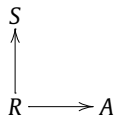
there is a unique way (up to isomorphism) to complete it into a push-out diagram, putting $B = (S \otimes A)/V$ where V is as above.

If B is the push-out of S and A we will write $B = \mathbf{PO}[S, A]$. Sometimes we want to make explicit the intersection space $R = S \cap A$, and then we write $B = \mathbf{PO}_R[S, A]$, meaning that B is the push-out of S and A and $R = S \cap A$.

Definition 2. An embedding of Boolean algebras $A \to B$ is said to be a posex (push-out separable extension) if there exists a push-out diagram of embeddings

$$\begin{array}{ccc} A & \longrightarrow & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$
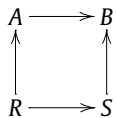
such that S and R are countable.

If $A \subset B$, we can rephrase this definition saying that B is a posex of A if there exists a countable subalgebra $S \subset B$ such that $B = \mathbf{PO}[S, A]$.

Definition 3. We say that a Boolean algebra $\mathfrak{B}$ is tightly σ-filtered if there exists an ordinal λ and a family $\{B_\alpha : \alpha \leqslant \lambda\}$ of subalgebras of $\mathfrak{B}$ such that

(1) $B_\alpha \subset B_\beta$ whenever $\alpha < \beta \leqslant \lambda$,
(2) $B_0 = \{0, 1\}$ and $B_\lambda = \mathfrak{B}$,
(3) $B_{\alpha+1}$ is a posex of B_α for every $\alpha < \lambda$,
(4) $B_\beta = \bigcup_{\alpha < \beta} B_\alpha$ for every limit ordinal $\beta \leqslant \lambda$.

We shall see later that the definition of Koppelberg [10] of tightly σ-filtered Boolean algebra is equivalent to this one.

2.2. The main result

We are now ready to state the main result of this section.

Theorem 4. *($\mathfrak{c}$ is a regular cardinal.) There exists a unique* (*up to isomorphism*) *Boolean algebra $\mathfrak{B}$ with the following properties*:

(1) $|\mathfrak{B}| = \mathfrak{c}$.
(2) $\mathfrak{B}$ *is tightly σ-filtered.*
(3) *For any diagram of embeddings of the form*

$$\begin{array}{c} B \\ \uparrow \\ A \longrightarrow \mathfrak{B}, \end{array}$$
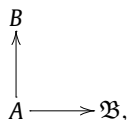
if $|A| < \mathfrak{c}$ and $A \to B$ is posex, then there exists an embedding $B \to \mathfrak{B}$ which makes the diagram commutative.

2.3. Basic properties of push-outs

Before going to the proof of Theorem 4, we collect some elementary properties of push-outs of Boolean algebras.

Proposition 5. *The following facts about push-outs hold*:

(1) *If* $B = \mathbf{PO}[S_\alpha, A]$ *for every* α, *where* $\{S_\alpha \colon \alpha < \xi\}$ *is an increasing chain of Boolean subalgebras of* B, $S_0 \subset S_1 \subset \cdots$, *then* $B = \mathbf{PO}[\bigcup_\alpha S_\alpha, A]$.
(2) *If* $B = \mathbf{PO}[S, A]$ *and we have* $S' \subset S$ *and* $A' \subset A$, *then* $B = \mathbf{PO}[\langle S \cup A' \rangle, \langle A \cup S' \rangle]$.
(3) *If* $B_1 = \mathbf{PO}_{S_0}[S_1, B_0]$ *and* $B_2 = \mathbf{PO}_{S_1}[S_2, B_1]$, *then* $B_2 = \mathbf{PO}_{S_0}[S_2, B_0]$.
(4) *Suppose that we have two increasing sequences of subalgebras of* B, $S_0 \subset S_1 \subset \cdots$ *and* $B_0 \subset B_1 \subset \cdots$, *and* $B_n = \mathbf{PO}_{S_0}[S_n, B_0]$ *for every* n. *Then* $\bigcup_n B_n = \mathbf{PO}_{S_0}[\bigcup_n S_n, B_0]$.
(5) *Suppose that we have two increasing sequences of subalgebras of* B, $S_0 \subset S_1 \subset \cdots$ *and* $B_0 \subset B_1 \subset \cdots$, *and* $B_{n+1} = \mathbf{PO}_{S_n}[S_{n+1}, B_n]$ *for every* n. *Then* $\bigcup_n B_n = \mathbf{PO}_{S_0}[\bigcup_n S_n, B_0]$.
(6) *Suppose that* $A = \bigcup_{i \in I} A_i \subset B$, $S = \bigcup_{j \in J} S_j \subset B$ *and* $\langle A_i \cup S_j \rangle = \mathbf{PO}[S_j, A_i]$ *for every* i *and every* j. *Then* $\langle A \cup S \rangle = \mathbf{PO}[S, A]$.

Proof. (1) is trivial.

For item (2) it is enough to consider the case where $S' = \emptyset$, that is to prove $B = \mathbf{PO}[\langle S \cup A' \rangle, A]$, the general case being obtained by the successive application of the cases $S' = \emptyset$ and its symmetric one $A' = \emptyset$. Clearly $B = \langle S \cup A \rangle = \langle (S \cup A') \cup A \rangle$. Let $s \in S \cup A'$ and $a \in A$ be such that $a \leqslant s$. If $s \in S$, our hypothesis guarantees that there is $r \in A \cap S \subseteq A \cap (S \cup A')$ such that $a \leqslant r \leqslant s$. If $s \in A' \subseteq A \cap (S \cup A')$, take $r = s$ and we are done.

For item (3), it is clear that $S_2 \cap B_0 = S_0$ and $B_2 = \langle S_2 \cup B_0 \rangle$. On the other hand assume that $a \leqslant s_2$ for some $a \in B_0$ and $s \in S_2$. Then, since $a \in B_1$ and $B_2 = \mathbf{PO}_{S_1}[S_2, B_1]$, there exists $r_1 \in S_1$ such that $a \leqslant r_1 \leqslant s$. Since we also have $B_1 = \mathbf{PO}_{S_0}[S_1, B_0]$, we find $r_0 \in S_0$ with $a \leqslant r_0 \leqslant r_1 \leqslant s$.

Item (4) is evident (it holds even for transfinite sequences though we do not need to use that) and item (5) follows from combining (3) and (4). Item (6) is also easy to see. □

We remark that some of the properties of push-outs of Banach spaces in Proposition 17 do not hold in the context of Boolean algebras. Namely, suppose that A is freely generated by $\{x_i \colon i \in I\}$ and B is freely generated by $\{x_i \colon i \in I\} \cup \{y\}$. If $S = \langle y \rangle$, then $B = \mathbf{PO}[S, A]$. However, if we consider $D = \langle x_i, x_i \wedge y \colon i \in I \rangle$, then $A \subset D \subset B$ but $B \neq \mathbf{PO}[S, D]$ and $D \neq \mathbf{PO}[S \cap D, A]$. So, we don't have an analogue of items (1) and (2) of Proposition 17.

Proposition 6. B *is a posex of* A *if and only if the following two conditions hold*:

(1) B *is countably generated over* A (*that is,* $B = \langle A \cup Q \rangle$ *for some countable set* Q), *and*
(2) *for every* $b \in B$, *the set* $\{a \in A \colon a \leqslant b\}$ *is a countably generated ideal of* A.

Proof. We suppose first that B is a posex of A, $B = \mathbf{PO}_R[S, A]$ for some countable S. It is obvious that B is countably generated over A so we concentrate in proving the other property. It is clear that property (2) holds whenever $b \in A$ (in that case the ideal is just generated by b) and also when $b \in S$, since by the push-out property the ideal $I_b = \{a \in A \colon a \leqslant b\}$ is generated by $I_b \cap R$, which is countable. Let us consider now an arbitrary element $b \in B$. We can express it in the form $b = (s_1 \wedge a_1) \vee \cdots \vee (s_n \wedge a_n)$ where $a_1, \ldots, a_n$ form a partition in A (that is, they are disjoint and their join is 1) and $s_1, \ldots, s_n \in S$. Pick now $a \in A$ such that $a \leqslant b$. Then $a \wedge a_i \leqslant s_i$ for every i, so since $a \wedge a_i \in A$ and $s_i \in S$, there exists $r_i \in R$ with $a \wedge a_i \leqslant r_i \leqslant s_i$. We have that

$$a = (a \wedge a_1) \vee \cdots \vee (a \wedge a_n) \leqslant (r_1 \wedge a_1) \vee \cdots \vee (r_n \wedge a_n) \leqslant (s_1 \wedge a_1) \vee \cdots \vee (s_n \wedge a_n) = b.$$

Notice that $(r_1 \wedge a_1) \vee \cdots \vee (r_n \wedge a_n) \in A$. It follows that the ideal $\{a \in A \colon a \leqslant b\}$ is generated by the countable set

$$\left\{ r = (r_1 \wedge a_1) \vee \cdots \vee (r_n \wedge a_n) \colon r_i \in R, \ r \leqslant b \right\}.$$

We prove now the converse implication. So we assume that $B = \langle A \cup Q \rangle$ with Q countable, and for every $b \in B$, we fix a countable set $G_b \subset A$ that generates the ideal $\{a \in A \colon a \leqslant b\}$. We define an increasing sequence of subalgebras of B making $S_0 = \langle Q \rangle$ and $S_{n+1} = \langle S_n \cup \bigcup_{b \in S_n} G_b \rangle$. All these are countably generated – hence countable – Boolean algebras. Taking $S = \bigcup_{n < \omega} S_n$ we get that $B = \mathbf{PO}[S, A]$. Namely, if $a \leqslant s$ with $s \in S$ and $a \in A$, then $s \in S_n$ for some n, and then there exists $r \in G_s$ with $a \leqslant r \leqslant s$. Just observe that $r \in G_s \subset A \cap S_{n+1} \subset A \cap S$. □

Condition (2) of Proposition 6 is found in the literature under the following names: A is a good subalgebra of B [14]; A is an $\aleph_0$-ideal subalgebra of B [5]; A is a σ-subalgebra of B [8,9]. We keep the latter terminology. So, B is a posex of A if and only if A is a σ-subalgebra of B and B is countably generated over A. Conversely, A is a σ-subalgebra of B if and only if every intermediate algebra $A \subset C \subset B$ which is countably generated over A is a posex of A. After these equivalences,

it becomes obvious that Koppelberg's definition [10] of tightly σ-filtered algebra and our own are the same, cf. also [9, Theorem 2.5].

Proposition 7. *The following facts hold*:

(1) *If $A \subset B$ and B is countable, then B is a posex of A.*
(2) *If B is a posex of A, $A \subset B' \subset B$, and B' is countably generated over A, then B' is a posex of A.*
(3) *If B_1 is a posex of B_0 and B_2 is a posex of B_1, then B_2 is a posex of B_0.*
(4) *If we have $B_n \subset B_{n+1}$ and B_n is a posex of B_0 for every $n < \omega$, then $\bigcup_n B_n$ is a posex of B_0.*
(5) *If B_{n+1} is a posex of B_n for every $n \in \mathbb{N}$, then $\bigcup_n B_n$ is a posex of B_0.*
(6) *If B is a posex of A and $S_0 \subset B$ is countable, then there exists a countable subalgebra $S_0 \subset S \subset B$ with $B = \mathbf{PO}[S, A]$.*
(7) *If we have $B_0 \subset B_1 \subset B_2$, B_2 is a posex of B_0 and B_1 is countably generated over B_0, then B_2 is a posex of B_1.*

Proof. (1) is trivial, (2) follows from Proposition 6. We prove now (3). Suppose that $B_1 = \mathbf{PO}_{R_0}[S_0, B_0]$ and $B_2 = \mathbf{PO}_{T_0}[U_0, B_1]$. Our objective is to apply Proposition 5(3), so we need to overcome the difficulty that $S_0 \neq T_0$. Inductively on n, we will define countable subalgebras R_n, S_n, T_n, U_n forming four increasing sequences so that $B_1 = \mathbf{PO}_{R_n}[S_n, B_0]$ and $B_2 = \mathbf{PO}_{T_n}[U_n, B_1]$ for every n. The inductive procedure is as follows: pick a countable set $Q_n \subset B_0$ such that $T_n \subset \langle S_n \cup Q_n \rangle$, then define $S_{n+1} = \langle S_n \cup Q_n \rangle$, $R_{n+1} = S_{n+1} \cap B_0$, $U_{n+1} = \langle U_n \cup S_{n+1} \rangle$ and $T_{n+1} = U_{n+1} \cap B_1$. The push-out relations $B_1 = \mathbf{PO}_{R_n}[S_n, B_0]$ and $B_2 = \mathbf{PO}_{T_n}[U_n, B_1]$ follow from Proposition 5(2). Also notice that $T_n \subset S_{n+1} \subset T_{n+1}$ for every n. Hence $\bigcup_n S_n = \bigcup_n T_n$. On the other hand, by Proposition 5(4) we have that $B_1 = \mathbf{PO}_{\bigcup_n R_n}[\bigcup_n S_n, B_0]$ and $B_2 = \mathbf{PO}_{\bigcup_n T_n}[\bigcup_n U_n, B_1]$. By Proposition 5(3), $B_2 = \mathbf{PO}_{\bigcup_n R_n}[\bigcup_n U_n, B_0]$, which proves that B_2 is a posex of B_0, since $\bigcup_n U_n$ is countable.

Item (4) is proven easily using Proposition 6, and (5) is a consequence of (3) and (4). For (6), consider first a countable subalgebra $S_1 \subset B$ such that $B = \mathbf{PO}[S_1, A]$. Then find a countable set $Q \subset A$ such that $S_0 \subset \langle Q \cup S_1 \rangle$. From Proposition 5(2) we get that $B = \mathbf{PO}[\langle Q \cup S_1 \rangle, A]$. To prove (7), use (2) to get that B_1 is a posex of B_0 hence $B_1 = \mathbf{PO}[S_1, B_0]$ for some countable subalgebra S_1. By (6) there exists a countable subalgebra S_2 such that $B_2 = \mathbf{PO}[S_2, B_0]$ and $S_1 \subset S_2$. By Proposition 5(2), $B_2 = \mathbf{PO}[S_2, \langle B_0 \cup S_1 \rangle]$ but $\langle B_0 \cup S_1 \rangle = B_1$. $\square$

Consider again the example where A is freely generated by $\{x_i: i \in I\}$ with I uncountable, B is freely generated by $\{x_i: i \in I\} \cup \{y\}$ and $D = \langle x_i, x_i \wedge y: i \in I \rangle$. Then $B = \mathbf{PO}[\langle y \rangle, A]$, so B is a posex of A. On the other hand, $A \subset D \subset B$ but neither D is a posex of A nor B is a posex of D. Indeed D is not countably generated over A, and $\{a \in D: a \leqslant y\}$ is not a countably generated ideal of D.

2.4. Proof of Theorem 4

The following definition, as well as Lemma 9 are due to Geschke [9]. A proof of Lemma 9 can also be obtained by imitating the proof of Lemma 20.

Definition 8. Let $\mathfrak{B}$ be a (uncountable) Boolean algebra. An additive σ-skeleton of $\mathfrak{B}$ is a family $\mathcal{F}$ of subalgebras of $\mathfrak{B}$ with the following properties:

(1) $\{0, 1\} \in \mathcal{F}$.
(2) For every subfamily $\mathcal{G} \subset \mathcal{F}$, we have $\langle \bigcup \mathcal{G} \rangle \in \mathcal{F}$.
(3) For every infinite subalgebra $A \subset \mathfrak{B}$, there exists $A' \in \mathcal{F}$ such that with $A \subset A' \subset \mathfrak{B}$ and $|A| = |A'|$.
(4) Every $A \in \mathcal{F}$ is a σ-subalgebra of $\mathfrak{B}$.

We will often use the following property of an additive σ-skeleton: If $A \in \mathcal{F}$, $A \subset A' \subset \mathfrak{B}$ and A' is countably generated over A, then there exists $B \in \mathcal{F}$ such that $A' \subset B$ and B is countably generated over A'. This is a direct consequence of properties (2) and (3): Suppose that $A' = \langle A \cup S \rangle$ with S countable Boolean algebra. Then there exists a countable $S_1 \in \mathcal{F}$ with $S \subset S_1$, and we can take $B = \langle A \cup S_1 \rangle$.

Lemma 9 *(Geschke). For a Boolean algebra $\mathfrak{B}$ the following are equivalent:*

(1) $\mathfrak{B}$ *is tightly σ-filtered.*
(2) *There exists an additive σ-skeleton $\mathcal{F}$ of $\mathfrak{B}$.*

We can now prove Theorem 4. First we prove existence. We consider $\mathfrak{c} = \bigcup_{\alpha < \mathfrak{c}} \Phi_\alpha$ a decomposition of the continuum into $\mathfrak{c}$ many subsets of cardinality $\mathfrak{c}$ such that $\alpha \leqslant \min(\Phi_\alpha)$ for every α. We define recursively an increasing chain of Boolean algebras $\{B_\alpha: \alpha < \mathfrak{c}\}$, so that at the end $\mathfrak{B} = \bigcup_{\alpha < \mathfrak{c}} B_\alpha$. We start with $B_0 = \{0, 1\}$. After B_α is defined, we consider a family $\{(R_\gamma, S_\gamma): \gamma \in \Phi_\alpha\}$ such that:

- For every $\gamma \in \Phi_\alpha$, S_γ is a countable Boolean algebra and $R_\gamma = B_\alpha \cap S_\gamma$.
- For every countable subalgebra $R \subset B_\alpha$ and every countable superalgebra $S \supset R$ there exists $\gamma \in \Phi_\alpha$ and a Boolean isomorphism $j : S \to S_\gamma$ such that $R = R_\gamma$ and $j(x) = x$ for $x \in R$.

For limit ordinals β we define $B_\beta = \bigcup_{\alpha < \beta} B_\alpha$. At successor stages we define $B_{\alpha+1} = \mathbf{PO}_{R_\alpha}[S_\alpha, B_\alpha]$. By construction, it is clear that $\mathfrak{B}$ is a tightly σ-filtered Boolean algebra of cardinality $\mathfrak{c}$. We check property (3) in the statement of the theorem. Suppose that we have $A \subset \mathfrak{B}$ with $|A| < \mathfrak{c}$, and $B = \mathbf{PO}_R[S, A]$ with S countable. By the regularity of $\mathfrak{c}$, we can find $\alpha < \mathfrak{c}$ such that $A \subset B_\alpha$. Then, there exists $\gamma \in \Phi_\alpha$ such that $R = R_\gamma$ and (modulo an isomorphism) $S = S_\gamma$, so that $B_{\gamma+1} = \mathbf{PO}_R[S, B_\gamma]$. Consider $\tilde{B} = \langle S \cup A \rangle \subset B_{\gamma+1}$, so that $\tilde{B} = \mathbf{PO}_R[S, A]$. Since push-out is unique up to isomorphism, we can find an isomorphism $\tilde{u} : B \to \tilde{B} \subset \mathfrak{B}$ such that $\tilde{u}(a) = a$ for all $a \in A$.

We prove now uniqueness. Suppose that we have two Boolean algebras like this, $\mathfrak{B}$ and $\mathfrak{B}'$. We consider their respective additive σ-skeletons $\mathcal{F}$ and $\mathcal{F}'$ that witness that they are tightly σ-filtered. Let us suppose that $\mathfrak{B} = \{x_\alpha : \alpha < \mathfrak{c}\}$ and $\mathfrak{B}' = \{y_\alpha : \alpha < \mathfrak{c}\}$. We shall construct recursively two increasing chains of subalgebras $\{B_\alpha : \alpha < \mathfrak{c}\}$ and $\{B'_\alpha : \alpha < \mathfrak{c}\}$ and a family of Boolean isomorphisms $f_\alpha : B_\alpha \to B'_\alpha$ with the following properties:

(1) The isomorphisms are coherent, that is $f_\beta|_{B_\alpha} = f_\alpha$ whenever $\alpha < \beta$.
(2) For every α, $B_\alpha \in \mathcal{F}$ and $B'_\alpha \in \mathcal{F}'$.
(3) $x_\alpha \in B_{\alpha+1}$ and $y_\alpha \in B'_{\alpha+1}$. In this way we make sure that $\mathfrak{B} = \bigcup_{\alpha < \mathfrak{c}} B_\alpha$ and $\mathfrak{B}' = \bigcup_{\alpha < \mathfrak{c}} B'_\alpha$.
(4) $B_{\alpha+1}$ is countably generated over B_α for every α. This implies that $|B_\alpha| = |\alpha|$ for every $\alpha \geqslant \omega$.

After this, the isomorphisms f_α induce a global isomorphism $f : \mathfrak{B} \to \mathfrak{B}'$. We proceed to the inductive construction. We start with $B_0 = \{0, 1\}$ and $B'_0 = \{0, 1\}$. If β is a limit ordinal, we simply put $B_\beta = \bigcup_{\alpha < \beta} B_\alpha$, $B'_\beta = \bigcup_{\alpha < \beta} B'_\alpha$ and the isomorphism f_β is induced by the previous ones. Now we show how to construct $B_{\alpha+1}$, $B'_{\alpha+1}$ and $f_{\alpha+1}$ from the previous ones. We construct inductively on n, sequences of subalgebras $B_\alpha[n] \subset \mathfrak{B}$, $B'_\alpha[n] \subset \mathfrak{B}'$ and coherent isomorphisms $f_\alpha[n] : B_\alpha[n] \to B'_\alpha[n]$ as in the picture:

$$B_\alpha = B_\alpha[0] \subset B_\alpha[1] \subset B_\alpha[2] \subset B_\alpha[3] \subset \cdots \subset \mathfrak{B}$$
$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$
$$B'_\alpha = B'_\alpha[0] \subset B'_\alpha[1] \subset B'_\alpha[2] \subset B'_\alpha[3] \subset \cdots \subset \mathfrak{B}'$$

and we will make $B_{\alpha+1} = \bigcup_{n < \omega} B_\alpha[n]$, $B'_{\alpha+1} = \bigcup_{n < \omega} B'_\alpha[n]$ and $f_{\alpha+1}$ induced by the isomorphisms $f_\alpha[n]$. All the algebras $B_\alpha[n+1]$ and $B'_\alpha[n+1]$ will be countably generated over $B_\alpha[n]$ and $B'_\alpha[n]$ respectively. The inductive procedure is as follows. There are two cases:

Case 1. n is even. Then, we define $B_\alpha[n+1]$ to be such that $x_\alpha \in B_\alpha[n+1]$, $B_\alpha[n+1]$ is countably generated over $B_\alpha[n]$, and $B_\alpha[n+1] \in \mathcal{F}$. Since $B_\alpha \in \mathcal{F}$ which is an additive σ-skeleton, $B_\alpha[n+1]$ is a posex of B_α, and by Proposition 7(7), also $B_\alpha[n+1]$ is a posex of $B_\alpha[n]$. Hence, since $\mathcal{B}'$ satisfies the statement of our theorem, we can find a Boolean embedding $f_\alpha[n+1] : B_\alpha[n+1] \to \mathcal{B}'$ such that $f_\alpha[n+1]|_{B_\alpha[n]} = f_\alpha[n]$. We define finally $B'_\alpha[n+1] = f_\alpha[n+1](B_\alpha[n+1])$.

Case 2. n is odd. Then, we define $B'_\alpha[n+1]$ to be such that $y_\alpha \in B'_\alpha[n+1]$, $B'_\alpha[n+1]$ is countably generated over $B'_\alpha[n]$ and $B'_\alpha[n+1] \in \mathcal{F}'$. Since $B'_\alpha \in \mathcal{F}'$ which is an additive σ-skeleton, $B'_\alpha[n+1]$ is a posex of B_α, hence also a posex of $B'_\alpha[n]$, so since $\mathfrak{B}$ satisfies the statement of our theorem, we can find an embedding $g_\alpha[n+1] : B'_\alpha[n+1] \to \mathfrak{B}$ such that $g_\alpha[n+1]|_{B_\alpha[n]} = f_\alpha^{-1}[n]$. We define finally $B_\alpha[n+1] = g_\alpha[n+1](B_\alpha[n+1])$ and $f_\alpha[n+1] = g_\alpha^{-1}[n+1]$.

Proceeding this way, we have that $B_\alpha[n] \in \mathcal{F}$ for n odd, while $B'_\alpha[n] \in \mathcal{F}'$ for n even. At the end, $B_{\alpha+1} = \bigcup_{n < \omega} B_\alpha[2n+1] \in \mathcal{F}$ and $B'_{\alpha+1} = \bigcup_{n < \omega} B'_\alpha[2n] \in \mathcal{F}'$, which concludes the proof. $\quad\square$

2.5. Remarks

Dow and Hart [5] say that B is $(*, \aleph_0)$-ideal, if for every $\kappa < \mathfrak{c}$ there exists κ-cub of σ-subalgebras of B. If we have an additive σ-skeleton $\mathcal{F}$ for B, then the algebras in $\mathcal{F}$ of cardinality κ form a κ-cub of σ-subalgebras of B. Hence, every tightly σ-filtered Boolean algebra of cardinality $\mathfrak{c}$ is $(*, \aleph_0)$-ideal.

Also, they say that a subalgebra $A \subset B$ is $\aleph_0$-ideal complete if for every two orthogonal countably generated ideals I, J of A, there exists $c \in B$ such that $I = \{a \in A : a \leqslant c\}$ and $J = \{a \in A : a \leqslant \bar{c}\}$. We have the following fact:

Proposition 10. *A is an $\aleph_0$-ideal complete subalgebra of B if and only if for every posex $A \to C$ there exists an embedding $g : C \to B$ with $g|_A = 1_A$.*

Proof. If the statement in the right holds, and we take two orthogonal countably generated ideals I, $J \subset A$, we can consider R a countable subalgebra of A generated by the union of countable sets of generators of I and J. Take a superalgebra of the

form $S = \langle R \cup \{c\} \rangle$ where $x \leqslant c$ for every $x \in I$ and $x \leqslant \bar{c}$ for every $x \in J$, and let $C = \mathbf{PO}_R[S, A]$. Our assumption provides an embedding $g : C \to B$ and the element $g(c)$ is the desired one.

Assume now that A is $\aleph_0$-ideal complete and consider $f : A \to C$ posex. It is enough to consider the case when C is finitely generated over $f(A)$, so suppose that $C = \langle f(A) \cup \{c_1, c_2, \ldots, c_n\} \rangle$ where $\{c_1, \ldots, c_n\}$ form a partition. For every $i < n$ find $d_i \in B$ such that

$$\{a \in A \colon f(a) \leqslant c_i\} = \{a \in A \colon a \leqslant d_i\},$$
$$\{a \in A \colon f(a) \leqslant \overline{c_i}\} = \{a \in A \colon a \leqslant \overline{d_i}\}.$$

Define $d_n = 1$. Convert the d_i's into a partition by setting $d'_k = d_k \setminus (\bigvee_{i<n} d_i)$ for $k \leqslant n$. Then it is possible to define the desired extension g by declaring $g(c_k) = d'_k$. $\quad\square$

Finally, Dow and Hart call a Boolean algebra $\mathfrak{B}$ to be Cohen–Parovičenko if:

(1) $|\mathfrak{B}| = \mathfrak{c}$.
(2) $\mathfrak{B}$ is $(*, \aleph_0)$-ideal.
(3) All subalgebras of $\mathfrak{B}$ of cardinality less than $\mathfrak{c}$ are $\aleph_0$-ideal complete.

Dow and Hart prove that, if $\mathfrak{c} \leqslant \aleph_2$, there is a unique Cohen–Parovičenko Boolean algebra of size $\mathfrak{c}$ and they ask if this is true in ZFC. The answer is negative. On the one hand, $\mathcal{P}(\omega)/fin$ is Cohen–Parovičenko in any Cohen model, as they show based on results of Steprāns [14]. On the other hand, as it follows from the comments above, items (1) and (3) are equivalent to the corresponding items of Theorem 4, while (2) is weaker, which guarantees that the Boolean algebra of Theorem 4 is Cohen–Parovičenko. As a consequence of a result of Geschke we get the following result, which guarantees that the Boolean algebra of Theorem 4 is not $\mathcal{P}(\omega)/fin$, whenever $\mathfrak{c} > \aleph_2$:

Theorem 11. *If $\mathfrak{B} = \mathcal{P}(\omega)/fin$ is the algebra of Theorem 4, then $\mathfrak{c} \leqslant \aleph_2$.*

Proof. Geschke [9] proves that a complete Boolean algebra of size greater than $\aleph_2$ cannot be tightly σ-filtered. Hence, if $\mathfrak{c} > \aleph_2$, $\mathcal{P}(\omega)$ is not tightly σ-filtered. This implies that neither $\mathcal{P}(\omega)/fin$ is such, because a tower of subalgebras witnessing tight σ-filtration could be lifted to $\mathcal{P}(\omega)$. $\quad\square$

Another property of the algebra $\mathfrak{B}$ of Theorem 4 is:

Proposition 12. *If A is a tightly σ-filtered Boolean algebra with $|A| \leqslant \mathfrak{c}$, then A is isomorphic to a subalgebra of $\mathfrak{B}$.*

Proof. Let $\{A_\alpha \colon \alpha < \kappa\}$ be subalgebras that witness that A is tightly σ-filtered. We can suppose that κ is the cardinality of A, cf. [9]. Then, inductively we can extend a given embedding $A_\alpha \to \mathfrak{B}$ to $A_{\alpha+1} \to \mathfrak{B}$, by the properties of $\mathfrak{B}$. $\quad\square$

3. Banach spaces

3.1. Preliminary definitions

The push-out is a general notion of category theory, and the one that we shall use here refers to the category Ban_1 of Banach spaces, together with operators of norm at most 1. We shall consider only push-outs made of isometric embeddings of Banach spaces, although it is a more general concept. For this reason, we present the subject in a different way than usual, more convenient for us but equivalent for the case of isometric embeddings.

Definition 13. Let Y be a Banach space and let S and X be subspaces of Y. We say that Y is the internal push-out of S and X if the following conditions hold:

(1) $Y = \overline{S + X}$,
(2) $\|x + s\| = \inf\{\|x + r\| + \|s - r\| \colon r \in S \cap X\}$ for every $x \in X$ and every $s \in S$.

Suppose that we have a diagram of isometric embeddings of Banach spaces like:

$$
\begin{array}{ccc}
S & \longrightarrow & Y \\
\uparrow & & \uparrow \\
\vert & & \vert \\
R & \longrightarrow & X.
\end{array}
$$
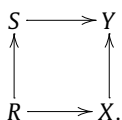
We say that it is a push-out diagram if, when all spaces are seen as subspaces of Y, we have that $R = S \cap X$ and Y is the internal push-out of X and S.

Note that Y being the push-out of S and X means that Y is isometric to the quotient space $(S \oplus_{\ell_1} X)/V$ where $V = \{(r, -r): r \in X \cap S\}$. In particular, $Y = S + X = \{s + x: s \in S,\ x \in X\}$. Also, given a diagram of isometric embeddings

$$
\begin{array}{c}
S \\
\uparrow u \\
R \xrightarrow{\ v\ } X,
\end{array}
$$

there is a unique way (up to isometries) to complete it into a push-out diagram, precisely by making $Y = (S \oplus_{\ell_1} X)/\tilde{R}$ and putting the obvious arrows, where $\tilde{R} = \{(u(r), v(-r)): r \in R\}$.

If Y is the push-out of S and X as above, we will write $Y = \mathbf{PO}[S, X]$. Sometimes we want to make explicit the intersection space $R = S \cap X$, and then we write $Y = \mathbf{PO}_R[S, X]$.

Definition 14. An isometric embedding of Banach spaces $X \to Y$ is said to be a posex if there exists a push-out diagram of isometric embeddings

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\uparrow & & \uparrow \\
R & \longrightarrow & S
\end{array}
$$

such that S and R are separable.

If $X \subset Y$, we can rephrase this definition saying that there exists a separable subspace $S \subset Y$ such that $Y = \mathbf{PO}[S, X]$.

Definition 15. We say that a Banach space $\mathfrak{X}$ is tightly σ-filtered if there exists an ordinal λ and a family $\{X_\alpha: \alpha \leqslant \lambda\}$ of subspaces of $\mathfrak{X}$ such that

(1) $X_\alpha \subset X_\beta$ whenever $\alpha < \beta \leqslant \lambda$,
(2) $X_0 = 0$ and $X_\lambda = X$,
(3) $X_{\alpha+1}$ is a posex of X_α for every $\alpha < \lambda$,
(4) $X_\beta = \overline{\bigcup_{\alpha < \beta} X_\alpha}$ for every limit ordinal $\beta \leqslant \lambda$.

3.2. The main result

Theorem 16. *($\mathfrak{c}$ is a regular cardinal.) There exists a unique (up to isometry) Banach space $\mathfrak{X}$ with the following properties*:

(1) $dens(\mathfrak{X}) = \mathfrak{c}$.
(2) $\mathfrak{X}$ *is tightly σ-filtered.*
(3) *For any diagram of isometric embeddings of the form*

$$
\begin{array}{c}
Y \\
\uparrow \\
X \longrightarrow \mathfrak{X},
\end{array}
$$

if $dens(X) < \mathfrak{c}$ and $X \to Y$ is posex, then there exists an isometric embedding $Y \to \mathfrak{X}$ which makes the diagram commutative.

3.3. Basic properties of push-outs

Before entering the proof of Theorem 16, we collect some elementary properties of push-outs that we shall use. For the sake of completeness, we include their proofs.

Proposition 17. *The following facts about push-outs hold*:

(1) *Suppose $Y = \mathbf{PO}[S, X]$, $S \subset S' \subset Y$ and $X \subset X' \subset Y$. Then $Y = \mathbf{PO}[S', X']$.*
(2) *Suppose $Y = \mathbf{PO}[S, X]$, and $X \subset Y' \subset Y$. Then $Y' = \mathbf{PO}[S \cap Y', X]$.*
(3) *Suppose $Y = \mathbf{PO}[S, X]$, and $S \subset Y' \subset Y$. Then $Y' = \mathbf{PO}[S, X \cap Y']$.*
(4) *Let $X \subset Y \subset Z$ and $S_0 \subset S_1 \subset S_2 \subset Z$ be such that $Y = \mathbf{PO}_{S_0}[S_1, X]$ and $Z = \mathbf{PO}_{S_1}[S_2, Y]$. Then $Z = \mathbf{PO}_{S_0}[S_2, X]$.*

(5) *Let $X_0 \subset X_1 \subset \cdots$ and $S_0 \subset S_1 \subset \cdots$ be Banach spaces such that for every $n \in \mathbb{N}$ we have that $X_{n+1} = \mathbf{PO}_{S_n}[S_{n+1}, X_n]$. Let $X = \overline{\bigcup_n X_n}$ and $S = \overline{\bigcup_n S_n}$. Then $X = \mathbf{PO}_{S_0}[S, X_0]$.*

(6) *Suppose $X = \overline{\bigcup_{i \in I} X_i} \subset Y$, $S = \overline{\bigcup_{j \in J} S_j} \subset Y$ and $\overline{X_i + S_j} = \mathbf{PO}[S_j, X_i]$ for every i and every j. Then $\overline{X + S} = \mathbf{PO}[S, X]$.*

Proof. For (1) it is enough to consider the case when $X = X'$. The case $S = S'$ is just the same, and the general fact follows from the application of those two. So we have to prove that given $x \in X$, $s' \in S'$ and $\varepsilon > 0$ there exists $t \in X \cap S'$ such that

$$\|x + s'\| + \varepsilon > \|x + t\| + \|s' - t\|.$$

Write $s' = s + x'$ where $s \in S$ and $x' \in X$. Then there exists $r \in X \cap S$ such that

$$\|x + s'\| = \|x + x' + s\| > \|x + x' + r\| + \|s - r\| - \varepsilon = \|x + (x' + r)\| + \|s' - (x' + r)\| - \varepsilon$$

so just take $t = x' + r$.

Item (2) is immediate, simply notice that $Y' = X + (Y' \cap S)$, because if $y \in Y'$ and $y = x + s$ with $x \in X$ and $s \in S$, then $s = y - x \in Y' \cap S$. Item (3) is symmetric to item (2).

For (4), consider $\varepsilon > 0$, $x \in X$ and $s_2 \in S_2$. Then since $Z = \mathbf{PO}_{S_1}[S_2, Y]$, there exists $s_1 \in S_1$ such that

$$\|x + s_2\| > \|x + s_1\| + \|s_2 - s_1\| - \varepsilon/2$$

and since $Y = \mathbf{PO}_{S_0}[S_1, X]$ there exists $s_0 \in S_0$ with

$$\|x + s_1\| > \|x + s_0\| + \|s_1 - s_0\| - \varepsilon/2$$

so finally

$$\|x + s_2\| > \|x + s_0\| + \|s_1 - s_0\| + \|s_2 - s_1\| - \varepsilon \geqslant \|x + s_0\| + \|s_2 - s_0\| - \varepsilon.$$

For (5), we know by repeated application of item (4) that $X_{n+1} = \mathbf{PO}_{S_0}[S_n, X_0]$. Consider $\varepsilon > 0$, $x_0 \in X_0$ and $s \in S$. Then there exists n and $s_n \in S_n$ such that $\|s_n - s\| < \varepsilon/3$. Then, we can find $r \in S_0$ such that

$$\|x_0 + s\| > \|x_0 + s_n\| - \varepsilon/3 > \|x_0 + r\| + \|s_n - r\| - 2\varepsilon/3 > \|x_0 + r\| + \|s - r\| - \varepsilon.$$

Item (6) is proven similarly by approximation. $\quad\square$

Corollary 18. *The following facts about posexes hold*:

(1) *If Y is a posex of X, then Y/X is separable.*
(2) *$X \subset Y$ and Y is separable, then Y is a posex of X.*
(3) *If Y is a posex of X and $X \subset Y' \subset Y$, then Y' is a posex of X.*
(4) *If Y is a posex of X and $X \subset X' \subset Y$, then Y is a posex of X'.*
(5) *If Y is a posex of X and Z is a posex of Y, then Z is a posex of X.*
(6) *If X_{n+1} is a posex of X_n for every $n \in \mathbb{N}$, then $\overline{\bigcup_n X_n}$ is a posex of X_0.*
(7) *If we have $X_n \subset X_{n+1}$ and X_n is a posex of X_0 for every $n \in \mathbb{N}$, then $\overline{\bigcup_n X_n}$ is a posex of X_0.*

A subspace X of a Banach space $\mathfrak{X}$ will be called a σ-subspace of $\mathfrak{X}$, if for every $X \subset Y \subset \mathfrak{X}$, if Y/X is separable, then Y is a posex of X.

Definition 19. Let $\mathfrak{X}$ be a (nonseparable) Banach space. An additive σ-skeleton of $\mathfrak{X}$ is a family $\mathcal{F}$ of subspaces of $\mathfrak{X}$ with the following properties:

(1) $0 \in \mathcal{F}$.
(2) For every subfamily $\mathcal{G} \subset \mathcal{F}$, we have $\overline{span}(\bigcup \mathcal{G}) \in \mathcal{F}$.
(3) For every infinite-dimensional $X \subset \mathfrak{X}$ there exists $Y \in \mathcal{F}$ with $X \subset Y \subset \mathfrak{X}$ and $dens(Y) = dens(X)$.
(4) Each $X \in \mathcal{F}$ is a σ-subspace of $\mathfrak{X}$.

We will often use the following property of an additive σ-skeleton: If $X \in \mathcal{F}$, $X \subset Y \subset \mathfrak{X}$ and Y/X is separable, then there exists $Z \in \mathcal{F}$ such that $Y \subset Z$ and Z/Y is separable. The proof is a direct consequence of properties (2) and (3): Suppose that $Y = \overline{X + S}$ with S separable. Then there exists a separable $S_1 \in \mathcal{F}$ with $S \subset S_1$, and we can take $Z = \overline{X + S_1}$.

We prove now a result for Banach spaces corresponding to Lemma 9.

Lemma 20. *For a Banach space $\mathfrak{X}$ the following are equivalent*:

(1) $\mathfrak{X}$ *is tightly σ-filtered.*
(2) $\mathfrak{X}$ *has an additive σ-skeleton.*

Proof. That (2) implies (1) is evident: it is enough to define the subspaces X_α inductively just taking care that $X_\alpha \in \mathcal{F}$ for every α. Now, we suppose that we have a tower of subspaces $\{X_\alpha: \alpha \leqslant \lambda\}$ like in (2). For every $\alpha < \lambda$ we consider separable subspaces $R_\alpha \subset S_\alpha$ such that $X_{\alpha+1} = \mathbf{PO}_{R_\alpha}[S_\alpha, X_\alpha]$.

Given a set of ordinals $\Gamma \subset \lambda$, we define $E(\Gamma) = \overline{span}(\bigcup_{\gamma \in \Gamma} S_\gamma)$. We say that the set Γ is saturated if for every $\alpha \in \Gamma$ we have that $R_\alpha \subset E(\Gamma \cap \alpha)$. We shall prove that the family $\mathcal{F} = \{E(\Gamma): \Gamma \subset \lambda \text{ is saturated}\}$ is the additive σ-skeleton that we are looking for.

It is clear that $0 = E(\emptyset)$. Also, if we have a family $\{\Gamma_i: i \in I\}$ of saturated sets, then $\overline{span}(\bigcup_i E(\Gamma_i)) = E(\bigcup_i \Gamma_i)$, so the union of saturated sets is saturated.

Given any countable set $\Gamma \subseteq \lambda$, it is possible to find a countable saturated set Δ such that $\Gamma \subset \Delta$: We define $\Delta = \{\delta_s: s \in \omega^{<\omega}\}$ where $(\delta_s)_{s \in \omega^{<\omega}}$ is defined inductively on the length of $s \in \omega^{<\omega}$ as follows. Let $\Gamma = \{\delta_{(n)}: n \in \omega\}$ and given $s \in \omega^{<\omega}$, let $\{\delta_{s \frown n}: n \in \omega\}$ be such that $\delta_{s \frown n} < \delta_s$ and $R_{\delta_s} \subset E(\{\delta_{s \frown n}: n \in \omega\})$. Notice that Δ is a countable and saturated set which contains Γ and also $\sup(\delta + 1: \delta \in \Gamma) = \sup(\delta + 1: \delta \in \Delta)$.

Now, suppose $X \subseteq \mathfrak{X}$ and let us find a saturated set Δ such that $X \subseteq E(\Delta)$ and $dens(X) = dens(E(\Delta))$. Since the union of saturated sets is saturated, we can assume without loss of generality that X is separable and find a countable set Γ such that $X \subset E(\Gamma)$. Take Δ as in the previous paragraph and notice that $X \subset E(\Delta) \in \mathcal{F}$ and $E(\Delta)$ is separable since Δ is countable.

It remains to prove that if $X \in \mathcal{F}$ and we have $X \subset Y \subset \mathfrak{X}$ with Y/X separable, then Y is a posex of X. It is enough to prove the following statement, and we shall do it by induction on $\delta_1 = \sup(\delta + 1: \delta \in \Delta)$:

"For every saturated set $\Gamma \subset \lambda$ and every countable set $\Delta \subset \lambda$, $E(\Gamma \cup \Delta)$ is a posex of $E(\Gamma)$".

We fix $\delta_1 < \lambda$ and we assume that the statement above holds for all saturated sets $\Gamma \subset \lambda$ and for all countable sets $\Delta' \subset \lambda$ with $\sup(\delta + 1: \delta \in \Delta') < \delta_1$.

Case 1. δ_1 is a limit ordinal. This follows immediately from the inductive hypothesis using Corollary 18(7).

Case 2. $\delta_1 = \delta_0 + 1$ for some δ_0 and $\Delta = \{\delta_0\}$. We distinguish two subcases:

Case 2a. $\sup(\Gamma) \leqslant \delta_0$. Consider the chain of subspaces

$$E(\Gamma) \subset \overline{E(\Gamma) + R_{\delta_0}} \subset E(\Gamma \cup \{\delta_0\}).$$

The left hand extension is a posex extension by the inductive hypothesis, because there exists a countable set Δ' such that $R_{\delta_0} \subset E(\Delta')$ and $\sup(\delta + 1, \delta \in \Delta') \leqslant \delta_0 < \delta_1$. The right hand extension is also posex, because we have the push-out diagram

$$
\begin{array}{ccc}
S_{\delta_0} & \longrightarrow & X_{\delta_1} \\
\uparrow & & \uparrow \\
\\
R_{\delta_0} & \longrightarrow & X_{\delta_0}
\end{array}
$$

and since $\sup(\Gamma) \leqslant \delta_0$ (and $\delta_0 \notin \Gamma$, otherwise it is trivial), we can interpolate

$$
\begin{array}{ccccc}
S_{\delta_0} & \longrightarrow & E(\Gamma \cup \{\delta_0\}) & \longrightarrow & X_{\delta_1} \\
\uparrow & & \uparrow & & \uparrow \\
\\
R_{\delta_0} & \longrightarrow & \overline{E(\Gamma) + R_{\delta_0}} & \longrightarrow & X_{\delta_0}
\end{array}
$$

where clearly $E(\Gamma \cup \{\delta_0\}) = \overline{E(\Gamma) + R_{\delta_0} + S_{\delta_0}}$, so that the left hand square is a push-out diagram.

Case 2b. $\sup(\Gamma) > \delta_0$. For every $\xi \leqslant \lambda$, we call $\Gamma_\xi = \Gamma \cap \xi$. By Case 2a, there exists a separable space S such that

$$E(\Gamma_{\delta_1} \cup \{\delta_0\}) = \mathbf{PO}[S, E(\Gamma_{\delta_1})].$$

We can suppose that $S_{\delta_0} \subset S$. We shall prove by induction on ξ that

$$E(\Gamma_\xi \cup \{\delta_0\}) = \mathbf{PO}[S, E(\Gamma_\xi)] \quad \text{for } \delta_1 \leqslant \xi \leqslant \lambda.$$

If ξ is a limit ordinal, then $E(\Gamma_\xi) = \overline{\bigcup_{\eta < \xi} E(\Gamma_\eta)}$, and we just need Proposition 17(6). Otherwise suppose that $\xi = \eta + 1$. In the nontrivial case, $\eta \in \Gamma$ and $\Gamma_\xi = \Gamma_\eta \cup \{\eta\}$. By the inductive hypothesis we have that

$$E(\Gamma_\eta \cup \{\delta_0\}) = \mathbf{PO}[S, E(\Gamma_\eta)]$$

and on the other hand $X_\xi = X_{\eta+1} = \mathbf{PO}[S_\eta, X_\eta]$.

Suppose that we are given $x \in E(\Gamma_\xi)$, $s \in S$ and $\varepsilon > 0$ and we have to find $r \in E(\Gamma_\xi) \cap S$ such that $\|x + s\| \geqslant \|x + r\| + \|s - r\| - \varepsilon$. We write $x = s_\eta + x_\eta$ where $s_\eta \in S_\eta$ and $x_\eta \in E(\Gamma_\eta)$. Notice that $x_\eta + s \in E(\Gamma_\eta) + S \subset E(\Gamma_\eta \cup \{\delta_0\}) \subset X_\eta$ since $S_{\delta_0} \subset X_\eta$ ($\delta_0 < \eta$ as $\delta_1 < \xi$). Hence, using that $X_\xi = \mathbf{PO}[S_\eta, X_\eta]$ we find $r_\eta \in S_\eta \cap X_\eta$ such that

$$\|s_\eta + x_\eta + s\| \geqslant \|x_\eta + s + r_\eta\| + \|s_\eta - r_\eta\| - \varepsilon/2.$$

Since $\eta \in \Gamma$ and Γ is saturated, we have that $R_\eta \subset E(\Gamma_\eta)$, therefore $x_\eta + r_\eta \in E(\Gamma_\eta)$ and $s \in S$. Hence, using that $E(\Gamma_\eta \cup \{\delta_0\}) = \mathbf{PO}[S, E(\Gamma_\eta)]$ we get $r \in S \cap E(\Gamma_\eta)$ such that

$$\|x_\eta + s + r_\eta\| \geqslant \|s + r\| + \|x_\eta + r_\eta - r\| - \varepsilon/2.$$

Combining both inequalities,

$$\|s_\eta + x_\eta + s\| \geqslant \|s + r\| + \|x_\eta + r_\eta - r\| + \|s_\eta - r_\eta\| - \varepsilon \geqslant \|s + r\| + \|x_\eta + s_\eta - r\| - \varepsilon,$$

as desired.

Case 3. $\delta_1 = \delta_0 + 1$ for some $\delta_0 \in \Delta$ and $|\Delta| > 1$. We consider the set $\Delta \setminus \{\delta_0\}$. We have proven few paragraphs above in this proof that we can find a countable saturated set $\Delta' \supset \Delta \setminus \{\delta_0\}$ such that $\sup\{\delta + 1: \delta \in \Delta'\} \leqslant \delta_0$. By the inductive hypothesis $E(\Gamma \cup \Delta')$ is a posex of $E(\Gamma)$. Since $\Gamma \cup \Delta'$ is saturated, the already proven Case 2 provides that $E(\Gamma \cup \Delta' \cup \{\delta_0\})$ is a posex of $E(\Gamma \cup \Delta')$. Composing both posex extensions we get that $E(\Gamma \cup \Delta' \cup \{\delta_0\})$ is a posex of $E(\Gamma)$. Since $E(\Gamma) \subset E(\Gamma \cup \Delta) \subset E(\Gamma \cup \Delta' \cup \{\delta_0\})$ we finally get that $E(\Gamma \cup \Delta)$ is a posex of $E(\Gamma)$. $\quad\square$

We finally prove Theorem 16. First we prove existence. We consider $\mathfrak{c} = \bigcup_{\alpha < \mathfrak{c}} \Phi_\alpha$ a decomposition of the continuum into $\mathfrak{c}$ many subsets of cardinality $\mathfrak{c}$ such that $\alpha \leqslant \min(\Phi_\alpha)$ for every α. We define recursively an increasing chain of Banach spaces $\{X_\alpha: \alpha < \mathfrak{c}\}$, so that at the end $\mathfrak{X} = \bigcup_{\alpha < \mathfrak{c}} X_\alpha$. We start with $X_0 = 0$. After X_α is defined, we consider a family $\{(R_\gamma, S_\gamma): \gamma \in \Phi_\alpha\}$ where:

- For every $\gamma \in \Phi_\alpha$, S_γ is a separable Banach space and $R_\gamma = X_\alpha \cap S_\gamma$.
- For every separable subspace $R \subset X_\alpha$ and every separable superspace $S \supset R$ there exists $\gamma \in \Gamma$ and an isometry $j: S \to S_\gamma$ such that $R = R_\gamma$ and $j(x) = x$ for $x \in R$.

For limit ordinals β we define $X_\beta = \overline{\bigcup_{\alpha < \beta} X_\alpha}$. At successor stages we define $X_{\alpha+1} = \mathbf{PO}_{R_\alpha}[S_\alpha, X_\alpha]$. By construction, it is clear that $\mathfrak{X}$ is a tightly σ-filtered Banach space of density $\mathfrak{c}$. We check property (3) in the statement of the theorem. Suppose that we have $X \subset \mathfrak{X}$ with $dens(X) < \mathfrak{c}$, and $Y = \mathbf{PO}_R[S, X]$ with S separable. By the regularity of $\mathfrak{c}$, we can find $\alpha < \mathfrak{c}$ such that $X \subset X_\alpha$. Then, there exists $\gamma \in \Phi_\alpha$ such that $R = R_\gamma$ and (modulo an isometry) $S = S_\gamma$. Then $X_{\gamma+1} = \mathbf{PO}_R[S, X_\gamma]$. Consider $\tilde{Y} = S + X \subset X_{\gamma+1}$, so that $\tilde{Y} = \mathbf{PO}_R[S, X]$. Since push-out is unique up to isometry, we can find an isometry $\tilde{u}: Y \to \tilde{Y} \subset \mathfrak{X}$ such that $\tilde{u}(x) = x$ for all $x \in X$.

We prove now uniqueness. Suppose that we have two spaces like this, $\mathfrak{X}$ and $\mathfrak{X}'$. We consider their respective additive σ-skeletons $\mathcal{F}$ and $\mathcal{F}'$ that witness that they are tightly σ-filtered. Let us suppose that $\mathfrak{X} = \overline{span}\{x_\alpha: \alpha < \mathfrak{c}\}$ and $\mathfrak{X}' = \overline{span}\{y_\alpha: \alpha < \mathfrak{c}\}$. We shall construct recursively two increasing chains of subspaces $\{X_\alpha: \alpha < \mathfrak{c}\}$ and $\{X'_\alpha: \alpha < \mathfrak{c}\}$ and a family of bijective isometries $f_\alpha: X_\alpha \to X'_\alpha$ with the following properties:

(1) The isometries are coherent, that is $f_\beta|_{X_\alpha} = f_\alpha$ whenever $\alpha < \beta$.
(2) For every α, $X_\alpha \in \mathcal{F}$ and $X'_\alpha \in \mathcal{F}'$.
(3) $x_\alpha \in X_{\alpha+1}$ and $y_\alpha \in X'_{\alpha+1}$. In this way we make sure that $\mathfrak{X} = \bigcup_{\alpha < \mathfrak{c}} X_\alpha$ and $\mathfrak{X}' = \bigcup_{\alpha < \mathfrak{c}} X'_\alpha$.
(4) Each quotient $X_{\alpha+1}/X_\alpha$ is separable. This implies that $dens(X_\alpha) = |\alpha|$ for every $\alpha \geqslant \omega$.

After this, the isometries f_α induce a global isometry $f: \mathfrak{X} \to \mathfrak{X}'$. We proceed to the inductive construction. We start with $X_0 = 0$ and $X'_0 = 0$. If β is a limit ordinal, we simply put $X_\beta = \overline{\bigcup_{\alpha < \beta} X_\alpha}$, $X'_\beta = \overline{\bigcup_{\alpha < \beta} X'_\alpha}$ and the isometry f_β is induced by the previous isometries. Now, we show how to construct $X_{\alpha+1}$, $X'_{\alpha+1}$ and $f_{\alpha+1}$ from the previous ones. We construct inductively on n, sequences of subspaces $X_\alpha[n] \subset \mathfrak{X}$, $X'_\alpha[n] \subset \mathfrak{X}'$ and coherent isometries $f_\alpha[n]: X_\alpha[n] \to X'_\alpha[n]$ as in the picture:

$$
\begin{array}{ccccccccc}
X_\alpha & = & X_\alpha[0] & \subset & X_\alpha[1] & \subset & X_\alpha[2] & \subset & X_\alpha[3] & \subset \cdots \subset & \mathfrak{X} \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
X'_\alpha & = & X'_\alpha[0] & \subset & X'_\alpha[1] & \subset & X'_\alpha[2] & \subset & X'_\alpha[3] & \subset \cdots \subset & \mathfrak{X}'
\end{array}
$$

and we will make $X_{\alpha+1} = \overline{\bigcup_{n<\omega} X_\alpha[n]}$, $X'_{\alpha+1} = \overline{\bigcup_{n<\omega} X'_\alpha[n]}$ and $f_{\alpha+1}$ induced by the isometries $f_\alpha[n]$. All the quotient spaces $X_\alpha[n+1]/X_\alpha[n]$ and $X'_\alpha[n+1]/X'_\alpha[n]$ will be separable. The inductive procedure is as follows. There are two cases:

Case 1. n is even. Then, we define $X_\alpha[n+1]$ to be such that $x_\alpha \in X_\alpha[n+1]$, $X_\alpha[n+1]/X_\alpha[n]$ is separable, and $X_\alpha[n+1] \in \mathcal{F}$. Since $X_\alpha \in \mathcal{F}$ which is an additive σ-skeleton, $X_\alpha[n+1]$ is a posex of $X_\alpha[n]$, so since $\mathfrak{X}'$ satisfies the statement of our theorem, we can find an into isometry $f_\alpha[n+1] : X_\alpha[n+1] \to \mathfrak{X}'$ such that $f_\alpha[n+1]|_{X_\alpha[n]} = f_\alpha[n]$. We define finally $X'_\alpha[n+1] = f_\alpha[n+1](X_\alpha[n+1])$.

Case 2. n is odd. Then, we define $X'_\alpha[n+1]$ to be such that $y_\alpha \in X'_\alpha[n+1]$, $X'_\alpha[n+1]/X'_\alpha[n]$ is separable, and $X'_\alpha[n+1] \in \mathcal{F}'$. Since $X'_\alpha \in \mathcal{F}'$ which is an additive σ-skeleton, $X'_\alpha[n+1]$ is a posex of $X'_\alpha[n]$, so since $\mathfrak{X}$ satisfies the statement of our theorem, we can find an into isometry $g_\alpha[n+1] : X'_\alpha[n+1] \to \mathfrak{X}$ such that $g_\alpha[n+1]|_{X_\alpha[n]} = f_\alpha^{-1}[n]$. We define finally $X_\alpha[n+1] = g_\alpha[n+1](X_\alpha[n+1])$ and $f_\alpha[n+1] = g_\alpha^{-1}[n+1]$.

Proceeding this way, we have that $X_\alpha[n] \in \mathcal{F}$ for n odd, while $X'_\alpha[n] \in \mathcal{F}'$ for n even. At the end, $X_{\alpha+1} = \overline{\bigcup_{n<\omega} X_\alpha[2n+1]} \in \mathcal{F}$ and $X'_{\alpha+1} = \overline{\bigcup_{n<\omega} X'_\alpha[2n]} \in \mathcal{F}'$. $\quad\square$

3.4. Universality property

Let $\mathfrak{X}$ denote the space in Theorem 16.

Theorem 21. *If X is a tightly σ-filtered Banach space with $dens(X) \leqslant \mathfrak{c}$, then X is isometric to a subspace of the space $\mathfrak{X}$.*

Proof. Let $\{X_\alpha : \alpha \leqslant \kappa\}$ be subspaces that witness that X is tightly σ-filtered. By the proof of Lemma 20, we can suppose that κ is the cardinal $dens(X)$. Then, inductively we can extend a given isometric embedding $X_\alpha \to \mathfrak{X}$ to $X_{\alpha+1} \to \mathfrak{X}$, by the properties of $\mathfrak{X}$. $\quad\square$

4. Compact spaces

Along this section and the subsequent ones, $\mathfrak{X}$ will always denote the Banach space in Theorem 16 and $\mathfrak{B}$ the Boolean algebra in Theorem 4.

4.1. The compact space $\mathfrak{K}$

Definition 22. Suppose that we have a commutative diagram of continuous surjections between compact spaces,

$$
\begin{array}{ccc}
K & \xrightarrow{\ f\ } & L \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle v} \\
S & \xrightarrow{\ u\ } & R.
\end{array}
$$

We say that this is a pull-back diagram if the following conditions hold:

(1) For every $x, y \in K$, if $x \neq y$, then either $f(x) \neq f(y)$ or $g(x) \neq g(y)$.
(2) If we are given $x \in S$ and $y \in L$ such that $u(x) = v(y)$, then there exists $z \in K$ such that $f(z) = y$ and $g(z) = x$.

Again, the notion of pull-back is more general in category theory, and in particular pull-back diagrams of continuous functions which are not surjective can be defined, but for our purposes we restrict to the case defined above.

If we are given two continuous surjections $u : S \to R$ and $v : L \to R$ between compact spaces, we can always construct a pull-back diagram as above making $K = \{(x, y) \in S \times L : u(x) = v(y)\}$ and taking f and g to be the coordinate projections. Moreover any other pull-back diagram
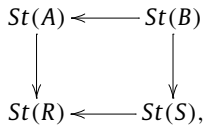
$$
\begin{array}{ccc}
K' & \xrightarrow{\ f'\ } & L \\
{\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle v} \\
S & \xrightarrow{\ u\ } & R
\end{array}
$$

is homeomorphic to the canonical one by a homeomorphism $h : K \to K'$ with $f'h = f$ and $g'h = g$.

**Proposition 23.** *A diagram of embeddings of Boolean algebras*

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\uparrow & & \uparrow \\
R & \longrightarrow & S
\end{array}
$$
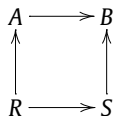
is a push-out diagram if and only if the diagram of compact spaces obtained by Stone duality,

$$
\begin{array}{ccc}
St(A) & \longleftarrow & St(B) \\
\downarrow & & \downarrow \\
St(R) & \longleftarrow & St(S),
\end{array}
$$

is a pull-back diagram.

**Proof.** Left to the reader. □

Proposition 24. *Let Y be a Banach space and X, S, R subspaces of Y such that $Y = \mathbf{PO}_R[S, X]$. Then the diagram obtained by duality between the dual balls endowed with the weak* topology,*

$$
\begin{array}{ccc}
B_{Y^*} & \longrightarrow & B_{X^*} \\
\downarrow & & \downarrow \\
B_{S^*} & \longrightarrow & B_{R^*}
\end{array}
$$
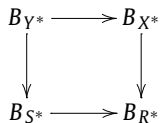
is a pull-back diagram.

Proof. We can suppose that $Y = (X \oplus_{\ell_1} S)/V$ where $V = \{(r, -r): r \in R\}$, and then $B_{Y^*} \subset B_{(X \oplus_{\ell_1} S)^*} = B_{Y^*} \times B_{S^*}$ and it is precisely the set of pairs which agree on R. □

Definition 25. A continuous surjection between compact spaces $f : K \to L$ is called posex if there exists a pull-back diagram of continuous surjections

$$
\begin{array}{ccc}
K & \xrightarrow{\ f\ } & L \\
\downarrow & & \downarrow \\
S & \longrightarrow & R
\end{array}
$$
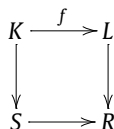
with R and S metrizable compact spaces.

Definition 26. A compact space K is called pull-back generated if there exists a family $\{K_\alpha : \alpha \leqslant \xi\}$ of compact spaces and continuous surjections $\{f_\alpha^\beta : K_\beta \to K_\alpha : \alpha \leqslant \beta \leqslant \xi\}$ such that:

(1) K_0 is a singleton and $K_\xi = K$.
(2) f_α^α is the identity map on K_α.
(3) $f_\alpha^\beta f_\beta^\gamma = f_\alpha^\gamma$ for $\alpha \leqslant \beta \leqslant \gamma \leqslant \xi$.
(4) $f_\alpha^{\alpha+1} : K_{\alpha+1} \to K_\alpha$ is posex for every $\alpha < \xi$.
(5) If $\gamma \leqslant \xi$ is a limit ordinal and $x, y \in K_\gamma$ with $x \neq y$, then there exists $\beta < \gamma$ such that $f_\beta^\gamma(x) \neq f_\beta^\gamma(y)$ (this means that K_γ is the inverse limit of the system below γ).

We can prove again a theorem about existence and uniqueness of a compact space in a similar way as we did for Banach spaces and Boolean algebras. But it is not worth to repeat the procedure because we would obtain just the Stone compact space of the Boolean algebra $\mathfrak{B}$ of Theorem 4. Hence, we just denote this Stone space by $\mathfrak{K} = St(\mathfrak{B})$.

Lemma 27. *Let K be a pull-back generated compact space. Then there exists a zero-dimensional pull-back generated compact space L of the same weight as K and such that there is a continuous surjection from L onto K.*

Proof. Assume that we have an inverse system $\{f_\alpha^\beta : K_\beta \to K_\alpha\}_{\alpha \leqslant \beta \leqslant \xi}$ as above witnessing that K is pull-back generated. We produce our compact space L and the continuous surjection $h : L \to K$ by constructing inductively a similar inverse system $\{g_\alpha^\beta : L_\beta \to L_\alpha\}_{\alpha \leqslant \beta \leqslant \xi}$ together with continuous surjections $h_\alpha : L_\alpha \to K_\alpha$ satisfying $h_\alpha g_\alpha^\beta = f_\alpha^\beta h_\beta$ for $\alpha \leqslant \beta$. We

need that each L_α is zero-dimensional and the weight of L_α equals the weight of K_α. The key step is providing $L_{\alpha+1}$, $f_\alpha^{\alpha+1}$ and $h_{\alpha+1}$ from L_α and h_α. Since $f_\alpha^{\alpha+1}$ is posex there exist metrizable compact spaces S and R and a pull back diagram

$$
\begin{array}{ccc}
K_{\alpha+1} & \longrightarrow & K_\alpha \\
\downarrow & & \downarrow \\
S & \longrightarrow & R.
\end{array}
$$

Consider S' a metrizable zero-dimensional compact space and $u : S' \to S$ a continuous surjection. We have a larger diagram

$$
\begin{array}{ccc}
& & L_\alpha \\
& & \downarrow{\scriptstyle h_\alpha} \\
K_{\alpha+1} & \longrightarrow & K_\alpha \\
\downarrow & & \downarrow \\
S' \xrightarrow{u} S & \longrightarrow & R.
\end{array}
$$

We can define $L_{\alpha+1}$ and $g_\alpha^{\alpha+1}$ by making the pull back of the larger square above,

$$
\begin{array}{ccc}
L_{\alpha+1} & \xrightarrow{g_\alpha^{\alpha+1}} & L_\alpha \\
\downarrow & & \downarrow \\
S' & \longrightarrow & R.
\end{array}
$$

The continuous surjection $h_{\alpha+1} : L_{\alpha+1} \to K_{\alpha+1}$ can be obtained by applying the so-called universal property of pull-back. In this case, the pull-back $K_{\alpha+1}$ can be seen as a subspace of $S \times K_\alpha$ and similarly $L_{\alpha+1} \subset S' \times L_\alpha$. One can define simply $h_{\alpha+1}(s, x) = (u(s), h_\alpha(x))$. $\quad\square$

Proposition 28. *For any diagram of continuous surjections between compact spaces*

$$
\begin{array}{c}
K \\
\downarrow \\
L \longleftarrow \mathfrak{K},
\end{array}
$$

if $f : K \to L$ is posex and weight$(L) < \mathfrak{c}$, then there exists a continuous surjection $\mathfrak{K} \to K$ that makes the diagram commutative.

Proof. Without loss of generality, we suppose that we have a pull-back diagram

$$
\begin{array}{ccc}
K & \longrightarrow & L \\
\downarrow & & \downarrow \\
S & \longrightarrow & R
\end{array}
$$

where S is metrizable and zero-dimensional. We can factorize into continuous surjections $\mathfrak{K} \to L' \to L$ such that L' is zero-dimensional and $weight(L') = weight(L)$. Consider then K' the pull-back of K, L and L', which is zero-dimensional because it is also the pull-back of S, R and L',

$$
\begin{array}{ccccc}
S & \longleftarrow & K & \longleftarrow & K' \\
\downarrow & & \downarrow & & \downarrow \\
R & \longleftarrow & L & \longleftarrow & L' \longleftarrow \mathfrak{K}.
\end{array}
$$

Then we have a similar diagram as in the statement of the theorem but all compact spaces are zero-dimensional. By Stone duality and property (3) of $\mathfrak{B}$ in Theorem 4 there is a continuous surjection $\mathfrak{K} \to K'$ that completes the diagram, and this provides the desired $\mathfrak{K} \to K$. $\quad\square$

4.2. The relation between $\mathfrak{X}$ and $C(\mathfrak{K})$

Theorem 29. *$\mathfrak{X}$ is isometric to a subspace of $C(\mathfrak{K})$.*

Proof. Let $\{X_\alpha \colon \alpha \leqslant \xi\}$ be an increasing chain of subspaces of $\mathfrak{X}$ witnessing the fact that $\mathfrak{X}$ is a push-generated Banach space. Then, by Proposition 24 the dual balls $\{B_{X_\alpha^*} \colon \alpha \leqslant \xi\}$ form an inverse system that witness the fact that $B_{\mathfrak{X}^*}$ is a pull-back generated compact space. By Lemma 27 there exists a zero-dimensional pull-back generated compact space L of weight $\mathfrak{c}$ that maps onto $B_{\mathfrak{X}^*}$. By Stone duality, using Proposition 23, we get that the Boolean algebra B of clopens of L is a tightly σ-filtered Boolean algebra of cardinality $\mathfrak{c}$. Hence, by Proposition 12, we can write $B \subset \mathfrak{B}$. Therefore $\mathfrak{K}$ maps continuously onto L, which maps continuously onto $B_{\mathfrak{X}^*}$. And this implies that $\mathfrak{X} \subset C(B_{\mathfrak{X}^*}) \subset C(L) \subset C(\mathfrak{K})$. $\quad\square$

4.3. P-points

Remember that a point p of a topological space K is called a P-point if the intersection of countably many neighborhoods of p contains a neighborhood of p. The following result was proven by Rudin [13] under CH and by Steprāns [14] in the $\aleph_2$-Cohen model: for every two P-points $p, q \in \omega^*$ there exists a homeomorphism $f : \omega^* \to \omega^*$ such that $f(p) = q$. Geschke [9] proves that it is sufficient to assume that $\mathcal{P}(\omega)$ is tightly σ-filtered. In this section we prove that this is a property of our compact space $\mathfrak{K}$.

Theorem 30. *Let $p, q \in \mathfrak{K}$ be P-points. Then there exists a homeomorphism $f : \mathfrak{K} \to \mathfrak{K}$ such that $f(p) = q$.*

In what follows, points of the Stone space of a Boolean algebra B are considered as ultrafilters on B. Given $Q_1, Q_2 \subset B$ we write $Q_1 \leqslant Q_2$ if $q_1 \leqslant q_2$ for every $q_1 \in Q_1$ and $q_2 \in Q_2$. If $b \in B$ and $Q \subset B$, then $b \leqslant Q$ means $\{b\} \leqslant Q$ and $Q \leqslant b$ means $Q \leqslant \{b\}$.

Definition 31. If A is a subalgebra of B, $Q \subset A$ and $b \in B$, then we write $b \leqslant_A Q$ if $\{a \in A \colon b \leqslant a\}$ equals the filter generated by Q in A. Similarly, we write $Q \leqslant_A b$ if $\{a \in A \colon a \leqslant b\}$ equals the ideal generated by Q in A.

With this notation, the notion of push-out of Boolean algebras can be rephrased as follows: given Boolean algebras $R \subset A$, $S \subset B$, then $B = \mathbf{PO}_R[S, A]$ if and only if for every $b \in S$, $\{a \in R \colon a \leqslant b\} \leqslant_A b \leqslant_A \{a \in R \colon b \leqslant a\}$.

Lemma 32. *Let $p \in \mathfrak{K}$ be a P-point, let $A \subset \mathfrak{B}$ be a subalgebra with $|A| < \mathfrak{c}$ and let $Q \subset A \cap p$ be countable. Then, there exists $b \in p$ such that $\{0\} \leqslant_A b \leqslant_A Q$.*

Proof. Because p is a P-point and Q is countable, there exists $b_0 \in p$ such that $b_0 \leqslant Q$. Since $|A| < \mathfrak{c}$, any posex extension of A can be represented inside $\mathfrak{B}$. In particular, we can find $b_1 \in \mathfrak{B}$ such that $\{0\} \leqslant_A b_1 \leqslant_A Q$ in such a way that $B_1 = \langle A \cup \{b_1\} \rangle$ is a posex of A. Observe that $b_0 \cup b_1 \leqslant_A Q$ and $b_0 \cup b_1 \in p$. By the same reason as before, we can find $b_2 \in \mathfrak{B}$ such that $\{0\} \leqslant_{B_1} b_2 \leqslant_{B_1} \{1\}$. Notice that we have $\{0\} \leqslant_A (b_1 \cup b_0) \cap b_2 \leqslant_A Q$ and $\{0\} \leqslant_A (b_1 \cup b_0) \setminus b_2 \leqslant_A Q$ and since $b_1 \cup b_0 \in p$ and p is an ultrafilter, either $(b_1 \cup b_0) \cap b_2$ or $(b_1 \cup b_0) \setminus b_2$ belong to p. $\quad\square$

Lemma 33. *Let A be a Boolean algebra with $|A| < \mathfrak{c}$, and $u : A \to \mathfrak{B}$ and $v : A \to B$ be Boolean embeddings with v being posex. Fix P-points, p_0, p and q of A, $\mathfrak{B}$ and B respectively such that $u(p_0) \subset p$ and $v(p_0) \subset q$. Then there exists an embedding $\tilde{u} : B \to \mathfrak{B}$ such that $\tilde{u}v = u$ and $\tilde{u}(q) \subset p$.*

Proof. Suppose that $B = \mathbf{PO}_R[S, v(A)]$ where S is a countable subalgebra of B. We can produce a further posex superalgebra $B_0 \supset B$ of the form $B_0 = \langle B \cup \{b_0\} \rangle$ such that $\{0\} \leqslant_B b_0 \leqslant_B S \cap q$. By Lemma 32 we can find $b_1 \in p$ such that $\{0\} \leqslant_{u(A)} b_1 \leqslant_{u(A)} u(R \cap p_0)$. Notice that $\{0\} \leqslant_{v(A)} b_0 \leqslant_{v(A)} v(R \cap p_0)$, and this allows to define a Boolean embedding $w : \langle v(A) \cup \{b_0\} \rangle \to \mathfrak{B}$ such that $wv = u$ and $w(b_0) = b_1$. Since B_0 is a posex of $v(A)$ and $\langle v(A) \cup \{b_0\} \rangle$ is countably generated over $v(A)$, we have that B_0 is a posex of $\langle v(A) \cup \{b_0\} \rangle$, so using the second property stated in Theorem 4, we find $\tilde{w} : B_0 \to \mathfrak{B}$ such that $\tilde{w}|_{\langle v(A) \cup \{b_0\} \rangle} = w$. We consider finally $\tilde{u} = \tilde{w}|_B$. It is clear that $\tilde{u}v = u$. On the other hand, since $\tilde{u}(b_0) = w(b_0) = b_1 \in p$ and $b_0 \leqslant S \cap q$, we have that $\tilde{u}(S \cap q) \subset p$. It is also clear that $\tilde{u}(q \cap v(A)) = u(p_0) \subset p$. Since $B = \langle S \cup v(A) \rangle$ the ultrafilter q is the filter generated by $(q \cap S) \cup (q \cap v(A))$, so we finally get that $\tilde{u}(q) \subset p$. $\quad\square$

We can rephrase the statement of Theorem 30 as follows: "If $\mathfrak{B}$ and $\mathfrak{B}'$ are Boolean algebras satisfying the conditions of Theorem 4 and p and q are P-points in $\mathfrak{B}$ and $\mathfrak{B}'$ respectively, then there exists an isomorphism $f : \mathfrak{B} \to \mathfrak{B}'$ such that $f(p) = q$". In order to prove this, we just have to follow the proof of uniqueness in Theorem 4 and make sure that at each step it is possible to choose the partial isomorphisms $f_\alpha : B_\alpha \to B'_\alpha$ in such a way that $f(p \cap B_\alpha) = q \cap B'_\alpha$. And what we need for that is exactly Lemma 33 applied to $\mathfrak{B}$ and $\mathfrak{B}'$.

5. Open problems

5.1. When $\mathfrak{c}$ is singular

Problem 1. Do Theorem 16 and Theorem 4 hold when $\mathfrak{c}$ is singular?

Here the point is that regularity looks essential to control all substructures of size less than $\mathfrak{c}$ in the *existence* part of the proof. Perhaps these theorems do not hold for singular $\mathfrak{c}$ as they are stated, but it would be satisfactory any variation that would allow us to speak about unique objects $\mathfrak{X}$ and $\mathfrak{B}$ defined by some properties in ZFC. We remark that if we restrict below a given cardinal, *existence* can be proven. We state it for Boolean algebras, and leave the Banach space version to the reader.

Proposition 34. *Fix a cardinal* $\lambda < \mathfrak{c}$. *There exists a Boolean algebra* $\mathfrak{B}$ *with the following properties*:

(1) $|\mathfrak{B}| = \mathfrak{c}$.
(2) $\mathfrak{B}$ *is tightly* σ*-filtered.*
(3) *For any diagram of embeddings of the form*

$$
\begin{array}{c}
B \\
\uparrow \\
\mid \\
A \longrightarrow \mathfrak{B},
\end{array}
$$
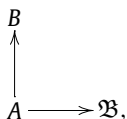
if $|A| \leqslant \lambda$ *and* $A \to B$ *is posex, then there exists an isometric embedding* $B \to \mathfrak{B}$ *which makes the diagram commutative.*

Proof. Let λ^+ be the successor cardinal of κ. Construct $\mathfrak{B}$ in the same way as in the proof of Theorem 4 or Theorem 16, but do it with a tower of length $\mathfrak{c} \cdot \lambda^+$ instead of length $\mathfrak{c}$. □

5.2. Properties of $\mathfrak{B}$ *reflected on Banach spaces*

We know much more about $\mathfrak{B}$ than about its Banach space relative $\mathfrak{X}$. So it is natural to ask whether certain facts that hold for $\mathfrak{B}$ in some models are reflected by analogous properties for $\mathfrak{X}$ or for $C(\mathfrak{K})$.

For example, we mentioned that in the $\aleph_2$-Cohen model, $\mathfrak{B} = \mathcal{P}(\omega)/\mathit{fin}$ [5]. In this model, therefore, $\mathfrak{B}$ has some additional properties of extensions of morphisms: given any diagram of arbitrary morphisms

$$
\begin{array}{c}
S \\
\uparrow \\
\mid \\
R \longrightarrow \mathfrak{B}
\end{array}
$$
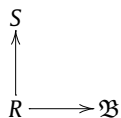
where R is countable and S is arbitrary, there exists a morphism $S \to \mathfrak{B}$ which makes the diagram commutative. The analogous property for Banach spaces is called (1-)universally separably injectivity (we add a 1 if the operator $S \to \mathfrak{X}$ can be found with the same norm as $R \to \mathfrak{X}$), cf. [2]. This property implies that the space contains ℓ_∞.

Problem 2. In the $\aleph_2$-Cohen model, is the space $\mathfrak{X}$ universally separably injective? Does it contain ℓ_∞?

Another observation is that in this model, $\mathcal{P}(\omega)/\mathit{fin}$ does not contain any ω_2-chain. Indeed, Dow and Hart [5] prove in ZFC that no Cohen–Parovičenko Boolean algebra (in particular $\mathfrak{B}$) can contain ω_2-chains. On the other hand, Brech and Koszmider [3] prove that in the $\aleph_2$-Cohen model, ℓ_∞/c_0 does not contain the space $C[0, \omega_2]$ of continuous functions on the ordinal interval $[0, \omega_2]$. So the natural question is:

Problem 3. Is it true in general that $C(\mathfrak{K})$ (or at least $\mathfrak{X}$) does not contain $C[0, \omega_2]$?

Problem 4. Can spaces like $\ell_2(\omega_2)$ or $c_0(\omega_2)$ be subspaces of $\mathfrak{X}$?

Under (MA $+ \mathfrak{c} = \aleph_2$), Dow and Hart [5] prove that $\mathfrak{B}$ does not contain $\mathcal{P}(\omega)$ as a subalgebra. So we may formulate

Problem 5 (MA $+ \mathfrak{c} = \aleph_2$). Does $C(\mathfrak{K})$ contain a copy of ℓ_∞?

This question is already posed in [2]. A negative answer would solve a problem by Rosenthal by providing an F-space K such that $C(K)$ does not contain ℓ_∞. In [2] it is proven that, under MA $+ \mathfrak{c} = \aleph_2$, there is an isometric embedding $c_0 \to C(\mathfrak{K})$ which cannot be extended to an embedding $\ell_\infty \to C(\mathfrak{K})$.

The space $\mathfrak{X}$ plays the role of $\mathfrak{B}$ in the category of Banach spaces. But we do not have a Banach space playing the role of $\mathcal{P}(\omega)/\mathit{fin}$, when it is different from $\mathfrak{B}$. This is related to the general question of finding intrinsic characterizations of $\mathcal{P}(\omega)/\mathit{fin}$ out of CH and $\aleph_2$-Cohen models, that we could translate into other categories.

Problem 6. Is there a Banach space counterpart of $\mathcal{P}(\omega)/\mathit{fin}$?

6. Other cardinals

We comment that a more general version of some of our results can be stated if we let arbitrary cardinals to play the role of $\mathfrak{c}$ and countability. We state it for Banach spaces, and leave the Boolean version to the reader. For an uncountable cardinal τ, we say that an isometric embedding $X \to Y$ is τ-posex if $Y = \mathbf{PO}[S, X]$ for some S with $dens(S) < \tau$. We say that X is tightly τ-filtered if X is the union of a continuous tower of subspaces starting at 0 and such that each $X_{\alpha+1}$ is a τ-posex of X_α (in the case of Boolean algebras, such a definition is equivalent to the one given in [9]).

Theorem 35. *Let κ be a regular cardinal, and τ an uncountable cardinal with $\kappa^{<\tau} = \kappa$. There exists a unique (up to isometry) Banach space $\mathfrak{X} = \mathfrak{X}(\kappa, \tau)$ with the following properties*:

(1) $dens(\mathfrak{X}) = \kappa$.
(2) $\mathfrak{X}$ is tightly τ-filtered.
(3) *For any diagram of isometric embeddings of the form*

$$
\begin{array}{c}
Y \\
\uparrow \\
\vert \\
X \longrightarrow \mathfrak{X},
\end{array}
$$

if $dens(X) < \kappa$ and $X \to Y$ is τ-posex, then there exists an isometric embedding $w : Y \to \mathfrak{X}$ which completes the diagram.

The proof would be just the same. Our original space corresponds to $\mathfrak{X}(\mathfrak{c}, \aleph_1)$.

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