

## Stability Critical Graphs and Even Subdivisions of $K_4$

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A graph is stability critical ( $\alpha$ -critical) if the removal of any edge increases the stability number. We give an affirmative answer to a question raised by Chvátal, namely, that every connected, critical graph that is neither  $K_2$  nor an odd cycle contains an even subdivision of  $K_4$ . © 1993 Academic Press, Inc.

All graphs in this paper are assumed to be finite, simple, and undirected. For graph  $G = (V, E)$ , we also denote  $V(G) = V$  and  $E(G) = E$ . A set of mutually nonadjacent nodes in a graph  $G$  is called a *stable (independent) set*. A *maximum stable set (MSS)* is a stable set of maximum cardinality. The *stability number* of  $G$ , denoted by  $\alpha(G)$ , is the cardinality of a maximum stable set in  $G$ . A stable set  $S$  *saturates*  $G$  if  $|S| = \alpha(G)$ . The degree of a node  $v$  in  $G$  is denoted by  $d(G, v)$  (whenever  $G$  is clear from the context, it will be suppressed from the notation). A node with degree equal to zero is said to be *isolated*.  $K_n$  is the complete graph on  $n$  nodes. A graph  $G' = (V', E')$  is a *subgraph* of  $G = (V, E)$ , denoted  $G' \subseteq G$ , if  $V' \subseteq V$  and  $E' \subseteq E$ . If  $W \subseteq V$ , then  $G[W]$  denotes the subgraph *induced* by  $W$ ; i.e.,  $G[W]$  has node set  $W$  and two nodes are adjacent in  $G[W]$  if and only if they are adjacent in  $G$ . If  $v \in V$ , then  $G - v$  will also be used to denote  $G[V \setminus \{v\}]$ . If  $(u, v) \in E$ , then  $G - (u, v)$  denotes the subgraph  $(V, E \setminus \{(u, v)\})$ .

An edge of  $G$  is said to be *critical* if its deletion increases the stability number.  $G$  is  $\alpha$ -critical if every edge of  $G$  is critical. Throughout this paper,

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*critical* will always mean  $\alpha$ -critical. If  $(v, w)$  is a critical edge of  $G$ , then there is a MSS in  $G$  that contains  $v$  and there is a MSS that contains  $w$ . This follows by considering a MSS  $S$  in  $G - (v, w)$ . Since  $(v, w)$  is critical,  $S$  must have cardinality  $\alpha(G) + 1$ , which then implies that  $v, w \in S$ . Thus,  $S \setminus \{v\}$  and  $S \setminus \{w\}$  are MSS's in  $G$  that contain  $w$  and  $v$ , respectively. If  $G$  is a critical graph and  $(v, w) \in E$  with  $d(v) + d(w) > 2$ , then there is a MSS in  $G$  that contains neither  $v$  nor  $w$ . To see this, assume, without loss of generality, that  $(u, v) \in E(G)$  with  $u \neq w$  and let  $S$  be a MSS in  $G - (u, v)$ . Then  $S \setminus \{v\}$  is the desired MSS in  $G$ . Finally, if  $G$  is critical and  $v \in V$  with  $d(v) = 1$ , then  $v$  and its neighbor form a component of  $G$ , since every MSS in  $G$  contains either  $v$  or its neighbor.

The number  $|V| - 2\alpha(G)$ , denoted by  $\delta(G)$ , plays an important role in the study of critical graphs, as demonstrated by the following theorem given by Hajnal in 1965 [4] ( $\delta(\cdot)$  will be used as generic notation whenever the graph has not been specified).

**THEOREM 1 (Hajnal).** *If  $G$  is a critical graph with no isolated nodes, then  $d(v) \leq \delta(G) + 1 \forall v \in V$ .*

This theorem is useful in characterizing critical graphs with small values of  $\delta(\cdot)$ . Let  $\Gamma^\delta$  be the set of all critical graphs with  $\delta(\cdot) = \delta$  and let  $\Gamma_c^\delta$  be the set of all connected graphs in  $\Gamma^\delta$ . If  $G \in \Gamma_c^0$ , then every node of  $G$  has degree at most one, which implies that  $G$  is  $K_2$ . If  $G \in \Gamma_c^1$ , then every node of  $G$  has degree at most two. Since  $G$  is connected,  $G$  must be either a simple path or a cycle. But  $\delta(G) < 1$  for all simple paths and even cycles, so  $G$  must be an odd cycle. A *subdivision* of a graph is obtained by replacing its edges by simple paths, i.e., by inserting new nodes of degree two into the edges. An *even subdivision* results when the number of new nodes inserted into each edge is even. Hence,  $\Gamma_c^1$  consists of even subdivisions of  $K_3$ . The situation for  $\Gamma_c^2$  is more complex, but Andrásfai [1] established the following theorem in 1967.

**THEOREM 2 (Andrásfai).** *If  $G \in \Gamma_c^2$ , then  $G$  is an even subdivision of  $K_4$ .*

In 1978 Lovász [6] established that for each fixed value of  $\delta$  there is a finite set of graphs (a finite "basis") such that every graph in  $\Gamma_c^\delta$  is an even subdivision of one of these basis graphs. The preceding discussion together with Theorem 2 imply that  $K_2$  is the basis for  $\Gamma_c^0$  (in fact,  $K_2$  is the only graph in  $\Gamma_c^0$ ),  $K_3$  is the basis for  $\Gamma_c^1$  and  $K_4$  is the basis for  $\Gamma_c^2$ . Furthermore, in [7] it is shown that there is a finite basis for  $\Gamma_c^\delta$  using the more general operation defined in the following theorem. (The basis for  $\Gamma_c^3$  is given explicitly in [7].)

**THEOREM 3 (Lovász and Plummer).** *Let  $G$  be a critical graph and  $x$  a node of degree two in  $G$ . Let  $y$  and  $z$  be the neighbors of  $x$ . If  $y$  and  $z$  are*

adjacent, then  $\{x, y, z\}$  forms a component of  $G$ . If  $y$  and  $z$  are not adjacent, then no node different from  $x$  is adjacent to both of them and, furthermore, if the edges  $(x, y)$  and  $(x, z)$  are contracted, the resulting graph  $G'$  is critical with  $\alpha(G') = \alpha(G) - 1$  and  $\delta(G') = \delta(G)$ . Conversely, suppose  $G'$  is a critical graph and  $w$  is any node of  $G'$ . Split  $w$  into two nodes  $y$  and  $z$ , each of degree at least one, create a new node  $x$  and connect it to both  $y$  and  $z$ . Then the resulting graph  $G$  is critical with  $\delta(G) = \delta(G')$ .

A subgraph  $H$  of  $G$  is said to be a  $\delta$ -subgraph of  $G$  if  $H$  is critical,  $V(H) = V(G)$ ,  $\alpha(H) = \alpha(G)$  (hence  $\delta(H) = \delta(G)$ ) and  $H$  does not contain any isolated nodes. In 1975 Surányi [11] proved the following two results concerning  $\delta$ -subgraphs.

**LEMMA 4** (Surányi). *Let  $G$  be a critical graph and  $(v, w) \in E(G)$ . If  $H$  is a  $\delta$ -subgraph of  $G - v$ , then  $d(H, w) = d(G, w) - 1$ .*

**THEOREM 5** (Surányi). *If  $G$  is a critical graph without isolated nodes and  $v \in V$  with  $d(v) > 1$ , then there exists a  $\delta$ -subgraph of  $G - v$ .*

Harary and Plummer [5] showed that every critical graph with  $\delta(\cdot) \geq 1$  contains an odd cycle, i.e., an even subdivision of the basis graph for  $\Gamma_c^1$ . In 1975 Chvátal [3] proved that every connected, critical graph with  $\delta(\cdot) \geq 2$  contains a subdivision of  $K_4$  and he posed the question of whether every connected, critical graph with  $\delta(\cdot) \geq 2$  must contain an even subdivision of  $K_4$ , i.e., an even subdivision of the basis graph for  $\Gamma_c^2$ . The following theorem establishes an affirmative answer to this question.

**THEOREM 6.** *If  $G = (V, E)$  is a connected, critical graph with  $\delta(G) \geq 2$ , then  $G$  contains an even subdivision of  $K_4$ .*

*Proof.* We first note that we may assume with no loss of generality that  $d(G, v) \geq 3, \forall v \in V$ . To see this, note that  $G$  is connected, so  $G$  contains no isolated nodes. Furthermore, if  $G$  had a node of degree one, then  $G$  could only consist of a single edge, contradicting  $\delta(G) \geq 2$ . Finally, if  $d(G, x) = 2$ , suppose  $(y, x), (z, x) \in E$  with  $y \neq z$ . Then Theorem 3 implies  $(y, z) \notin E$ . Thus, again by Theorem 3, we can remove  $x$  and identify  $y$  and  $z$  to obtain a connected, critical graph  $G'$  with  $\delta(G') \geq 2$ . It is not difficult to see that  $G$  contains an even subdivision of  $K_4$  provided  $G'$  does, so we replace  $G$  by  $G'$ . Repeated application of this argument allows us to assume  $d(G, v) \geq 3, \forall v \in V$ .

We may further assume that no proper subgraph of  $G$  satisfies the assumptions of the theorem, i.e., that  $G$  is *minimal* with respect to the stipulations connected, critical, and  $\delta(G) \geq 2$  (else we could replace  $G$  by such a subgraph and proceed with the proof). It follows that we need only consider cubic graphs, for if  $d(G, w) > 3$  with  $(v, w) \in E$ , then let  $H$  be a  $\delta$ -subgraph of  $G - v$  (see Theorem 5). By Lemma 4,  $d(H, w) \geq 3$ . If  $H'$  is the

component of  $H$  containing  $w$ , then  $H'$  is connected and critical with  $\delta(H') \geq d(H', w) - 1 \geq 2$  (see Theorem 1), contradicting the minimality of  $G$ .

The proof thus reduces to showing that any connected, critical, cubic, minimal graph  $G = (V, E)$  contains an even subdivision of  $K_4$ . We denote  $n = |V|$ ,  $\alpha = \alpha(G)$ , and  $\delta = \delta(G)$ .

CLAIM 1. *Suppose  $v \in V$  and  $H_v$  is a  $\delta$ -subgraph of  $G - v$ . Let  $E_v = E \setminus E(H_v)$ . Then:*

- (i)  $\alpha(H_v) = \alpha(G)$  and  $\delta(H_v) = \delta(G) - 1$ .
- (ii)  $H_v$  consists of isolated edges and  $\delta - 1$  odd cycles.
- (iii) If  $(v, w) \in E$ , then  $w$  is contained in an odd cycle in  $H_v$ .
- (iv)  $\alpha(G - v - e) = \alpha \forall e \in E_v$ .
- (v) If  $(v, w) \in E$ , then  $E_v \cap E_w = \{(v, w)\}$ .
- (vi)  $\forall w \in V$ , some edge incident to  $w$  is in  $E_v$ .
- (vii) Let  $C_1, \dots, C_{\delta-1}$  be the odd cycles in  $H_v$  and  $e_1, \dots, e_s$  be the isolated edges. If  $I$  is a stable set in  $G$  with  $|I| = \alpha$  and  $v \notin I$ , then  $I$  saturates  $C_1, \dots, C_{\delta-1}$  and  $e_1, \dots, e_s$ .

*Proof.* (i) Since  $G$  is critical, there exists a MSS which does not include  $v$ . Therefore,  $\alpha(H_v) = \alpha(G - v) = \alpha$  and  $\delta(H_v) = (n - 1) - 2\alpha(H_v) = n - 1 - 2\alpha = \delta - 1$ .

(ii)  $H_v$  is a  $\delta$ -subgraph, so there are no isolated nodes. Each component of  $H_v$  must be connected and critical; moreover, minimality of  $G$  forces  $\delta(\cdot) \leq 1$  for each component of  $H_v$ . Therefore, each component is an isolated edge or an odd cycle (see the discussion preceding Theorem 2). Furthermore,  $\delta(H_v)$  equals the sum of  $\delta(\cdot)$  for each component, so  $H_v$  contains  $\delta - 1$  odd cycles.

(iii) This follows from Lemma 4 and (ii).

(iv) Suppose  $e \in E_v$ . Then  $\alpha(G - v - e) \geq \alpha(G - v) = \alpha$  and  $\alpha(G - v - e) \leq \alpha(H_v) = \alpha$ .

(v) Suppose  $(v, w) \in E$  and  $e \in E_v \cap E_w$ . It is easy to see that  $(v, w) \in E_v \cap E_w$ , so suppose that  $e \neq (v, w)$ . Let  $I$  be a stable set in  $G - e$  with  $|I| = \alpha + 1$ . Clearly,  $I$  cannot contain both  $v$  and  $w$ , so without loss of generality assume  $v \notin I$ . Then  $I$  is a stable set in  $G - v - e$  with  $|I| = \alpha + 1$ , which is a contradiction to (iv).

(vi) Let  $w \in V$ . If no edge incident to  $w$  is in  $E_v$ , then  $w$  has degree three in  $H_v$ , contradicting (ii).

(vii) Suppose  $I$  is a stable set in  $G$  with  $|I| = \alpha$  and  $v \notin I$ . Then  $I$  is a stable set in  $H_v$  with  $|I| = \alpha(H_v)$ . Therefore,  $I$  induces a MSS in each component of  $H_v$ ; i.e.,  $I$  saturates  $C_1, \dots, C_{\delta-1}$  and  $e_1, \dots, e_s$ . ■

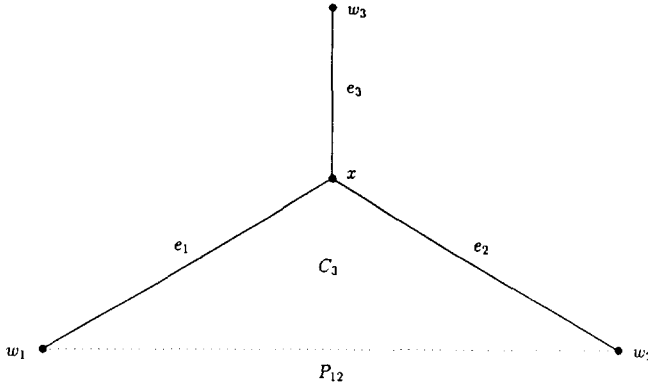


FIG. 1. The structure of  $C_3$ .

Now let  $x \in V$  and let  $e_1 = (x, w_1)$ ,  $e_2 = (x, w_2)$ ,  $e_3 = (x, w_3)$  be the three edges incident to  $x$ . Let  $H_x, H_1, H_2$ , and  $H_3$  be  $\delta$ -subgraphs of  $G - x, G - w_1, G - w_2$ , and  $G - w_3$ , respectively. Let  $C_i$  be the odd cycle in  $H_i$  that contains  $x$ , for  $i = 1, 2, 3$ . Let  $P_{ij}$  be the path from  $w_i$  to  $w_j$  on  $C_k \setminus \{x\}$ , where  $1 \leq i, j, k \leq 3$  are distinct indices. Obviously,  $P_{ij}$  has odd edge-length. Note that  $P_{ij}$  and  $P_{ji}$  have the same underlying undirected path. Now we want to show that  $C_1 \cup C_2 \cup C_3$  contains an even subdivision of  $K_4$ . The main difficulty is that the  $P_{ij}$ 's may intersect. The remainder of the proof concentrates on finding nonintersecting subpaths of the  $P_{ij}$ 's that are of the correct parity to form an even subdivision of  $K_4$ . See Fig. 1.

In our subsequent development we let  $I_i$  be a stable set in  $G - e_i$  with  $|I_i| = \alpha + 1$ , for  $i = 1, 2, 3$ .

CLAIM 2. *The nodes of  $P_{ij}$  alternate between  $I_i$  and  $I_j$  (see Fig. 2).*

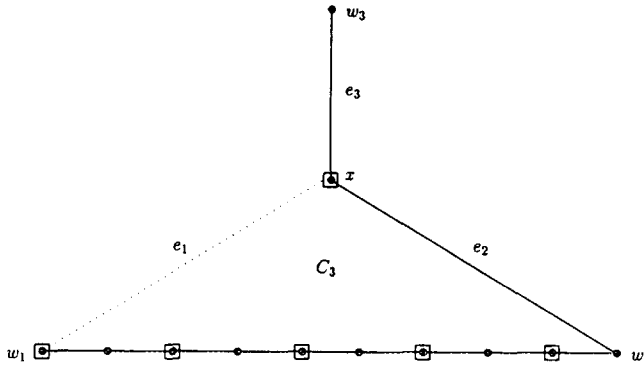


FIG. 2. Distribution of  $I_i$  on  $C_3$ .

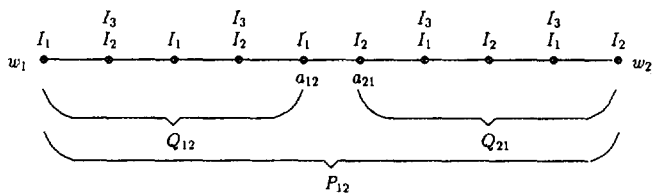


FIG. 3. Alternation of the  $I_i$ 's on  $Q_{12}$  and  $Q_{21}$ .

*Proof.* It is easy to see that  $|I_i| = \alpha + 1$ ,  $w_k \notin I_i$ , and  $\alpha(H_k) = \alpha$  imply that  $|I_i \cap V(C_k)| = (|V(C_k)| + 1)/2$ . This implies the nodes on  $P_{ij}$  which are of even distance from  $w_i$  are in  $I_i$ . Similar reasoning holds for  $w_j$  and  $I_j$ . ■

**CLAIM 3.**  $P_{ij}$  contains exactly one edge, call it  $e_{ij}$ , such that neither of its endnodes is in  $I_k$  (see Fig. 3).

*Proof.* The stable set  $I_k \setminus \{w_k\}$  saturates  $C_k$ , so  $C_k$  contains exactly one edge such that both of its endnodes are not in  $I_k$ . This edge must lie on  $P_{ij}$  because  $x \in I_k \setminus \{w_k\}$ . ■

Let  $a_{ij}$  be the first node of  $e_{ij}$  encountered when traversing  $P_{ij}$  from  $w_i$  to  $w_j$  and let  $Q_{ij}$  be the subpath of  $P_{ij}$  from  $w_i$  to  $a_{ij}$ . Note that  $Q_{ij}$  and  $Q_{ji}$  are node-disjoint, whereas  $P_{ij}$  and  $P_{ji}$  specify the same underlying undirected path. Note also that  $Q_{ij}$  or  $Q_{ji}$  may consist of a single node.

**CLAIM 4.**  $I_j \cap Q_{ij} = I_k \cap Q_{ij}$  and  $I_i \cap Q_{ji} = I_k \cap Q_{ji}$ ; i.e., the nodes of  $Q_{ij}$  alternate between  $I_i$  and  $I_j \cap I_k$  with the first and last node of  $Q_{ij}$  in  $I_i$  (and the nodes of  $Q_{ji}$  alternate between  $I_j$  and  $I_i \cap I_k$ ) (see Fig. 3).

*Proof.* Let  $P$  be the path from  $a_{ij}$  to  $a_{ji}$  on  $C_k \setminus \{e_{ij}\}$ . Claim 3 implies that alternate nodes of  $P$  must be in  $I_k$ . Now,  $x \in I_k$  implies  $w_i \notin I_k$  and  $w_j \notin I_k$ . Furthermore,  $a_{ij} \notin I_k$  and  $a_{ji} \notin I_k$ , by definition. Finally, by Claim 2,  $Q_{ij}$  alternates between  $I_i$  and  $I_j$ , and the result follows. ■

We observe that the proof of Claim 4 also demonstrates that  $a_{ij} \in I_i$ , for any distinct index pair.

**CLAIM 5.** The path  $P_{ik}$  contains no node of  $Q_{ji}$  (nor of  $Q_{jk}$ ) (see Fig. 4).

*Proof.* Since  $P_{ik}$  alternates between nodes of  $I_i$  and  $I_k$ , it contains no node of  $I_i \cap I_k$  nor any node of  $I_j \setminus (I_i \cup I_k)$ . But  $Q_{ji}$  alternates between nodes in  $I_j$  and  $I_i \cap I_k$ . ■

We now denote  $E_x = E \setminus E(H_x)$  and  $E_i = E \setminus E(H_i)$ , for  $i = 1, 2, 3$ .

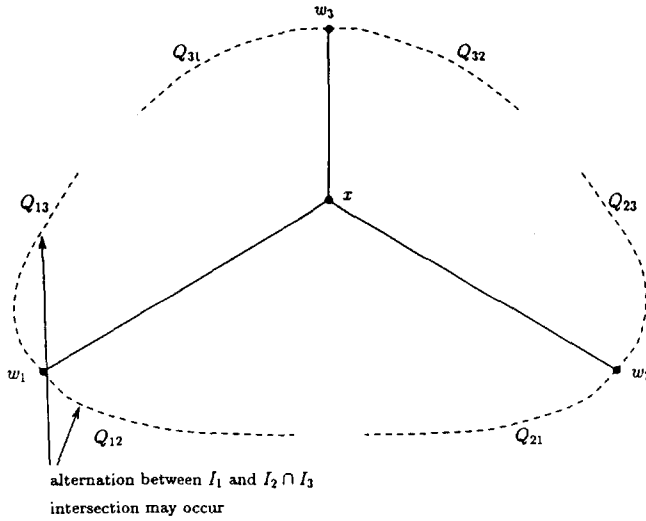


FIG. 4. The structure of the  $Q_{ij}$ 's.

CLAIM 6. Let the edges of  $P_{ij}$  be labeled  $d_1, \dots, d_r$ . Then the edges with odd index are not in  $E_x$  and the edges with even index are in  $E_x$ ; i.e., the edges of  $P_{ij}$  are alternately in and not in  $E_x$ , with the first and last edge not in  $E_x$ .

Proof. Let  $d_m = (u, v)$  be an edge of  $P_{ij}$  with  $m$  odd (see Fig. 5a). Suppose  $d_m \in E_x$  and  $I$  is a stable set in  $G - d_m$  with  $|I| = \alpha + 1$ . Then, by Claim 1(iv),  $d_m \in E_x$  implies  $x \in I$ . Thus  $w_k \notin I$ , so Claim 1(vii) implies both  $I \setminus \{u\}$  and  $I \setminus \{v\}$  saturate  $C_k$ . A simple parity argument shows that this cannot happen, so  $d_m \notin E_x$ . Now suppose  $m$  is even and consider the three edges adjacent to  $v$  (see Fig. 5b). One of these edges is  $d_{m+1}$ , which is not in  $E_x$  by the above argument. Another one of the edges is in  $E_k$  and therefore not in  $E_x$ , by Claim 1(v). Hence, the remaining edge,  $d_m$ , must be in  $E_x$  by Claim 1(vi). ■

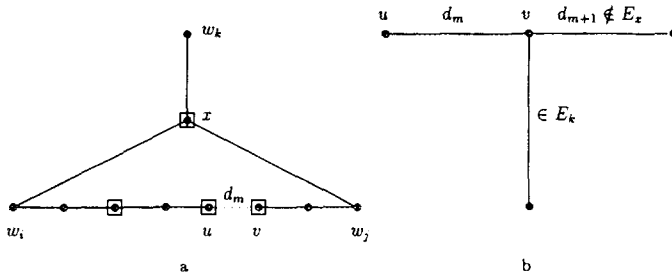


FIG. 5. Distribution of edges in  $E_x$  on  $P_{ij}$ .

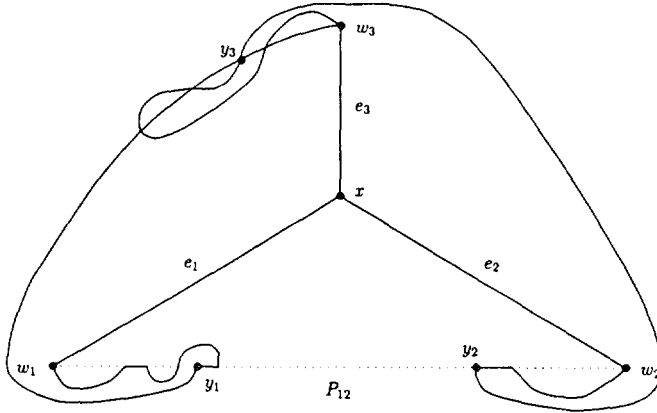


FIG. 6.  $P_{ij}$ 's partitioned by the  $y_i$ 's.

Now let  $P_{12} = v_0, d_1, v_1, d_2, \dots, d_r, v_r$ , where  $v_0 = w_1$  and  $v_r = w_2$ . For  $i = 1, 2, 3$ , let  $y_i$  be the last node of  $P_{i,i+1}$  encountered when traversing  $P_{i,i+2}$  from  $w_i$  to  $w_{i+2}$ , where addition is performed modulo 3 (see Fig. 6).

CLAIM 7. *The number of edges between  $w_i$  and  $y_i$  on  $P_{i,i+1}$  is even.*

*Proof.* By symmetry it suffices to show this for the case of  $w_1, y_1$ , and  $P_{12}$ . From Claim 5,  $P_{13}$  cannot use any node on  $Q_{21}$  and  $P_{12}$  cannot use any node of  $Q_{31}$ , so  $P_{13} \cap P_{12} \subseteq Q_{12} \cap Q_{13}$ . According to Claim 4, the nodes of both  $Q_{12}$  and  $Q_{13}$  alternate between  $I_1$  and  $I_2 \cap I_3$ . Therefore, if a node lies on both  $P_{12}$  and  $P_{13}$ , then its distance from  $w_1$  along  $P_{12}$  must be of the same parity as along  $P_{13}$ . Suppose that  $y_1$  is of odd (edge-) distance from  $w_1$  on  $P_{12}$ ; i.e.,  $y_1 = v_m$  with  $m$  odd (see Fig. 7). Then  $y_1$  is of odd distance from  $w_1$  on  $P_{13}$  and, by Claim 6, the next edge on  $P_{13}$  must be in  $E_x$ . (Note that  $v_m = w_2$  is impossible, since  $w_2 \in Q_{21}$ ; thus  $d_{m+1}$  exists.) However, the only edge incident to  $v_m$  that is in  $E_x$  is  $d_{m+1}$  (since  $d_m \notin E_x$  and  $f \notin E_x$ , where  $f$  is the edge of  $G$  that is incident to  $y_1$  and is not on  $C_3$ ). Therefore,  $v_{m+1}$  is the next node on  $P_{13}$ , which contradicts the choice of  $y_1$  as the last node on  $P_{12}$  when following  $P_{13}$  from  $w_1$  to  $w_3$ . ■

CLAIM 8. *Let  $R_1$  be the subpath of  $P_{12}$  from  $y_1$  to  $y_2$ . Let  $R_2$  be the subpath of  $P_{23}$  from  $y_2$  to  $y_3$  and let  $R_3$  be the subpath of  $P_{31}$  from  $y_3$  to  $y_1$ . Let  $C$  be formed by adjoining  $R_1, R_2$ , and  $R_3$ . Then  $C$  is an odd cycle.*

*Proof.* The paths  $R_1, R_2$ , and  $R_3$  are disjoint by construction (see Fig. 8), except for their endnodes  $y_1, y_2$ , and  $y_3$ , so  $C$  is a simple cycle. Since  $y_1$  is of even distance from  $w_1$  on  $P_{12}$ ,  $y_2$  is of even distance from  $w_2$



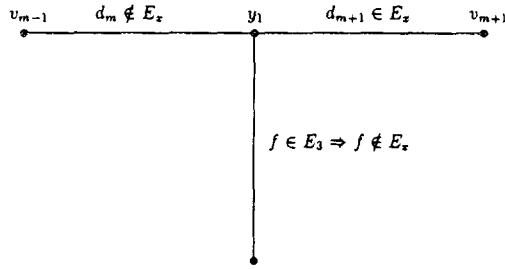


FIG. 7. The status of the edges adjacent to  $y_1$ .

on  $P_{12}$  and  $P_{12}$  is an odd path, then  $R_1$  must be an odd path. The same holds for  $R_2$  and  $R_3$ . Thus,  $C$  has an odd number of edges. ■

*CLAIM 9.* Let path  $S_1$  be formed by adjoining  $(x, w_1)$  to the subpath of  $P_{12}$  from  $w_1$  to  $y_1$ . Let  $S_2$  be formed by adjoining  $(x, w_2)$  to the subpath of  $P_{23}$  from  $w_2$  to  $y_2$ . Let  $S_3$  be formed by adjoining  $(x, w_3)$  to the subpath of  $P_{31}$  from  $w_3$  to  $y_3$ . Then  $S_1, S_2, S_3$ , and  $C$  form an even subdivision of  $K_4$ .

*Proof.* Since  $y_i$  is of even distance from  $w_i$  on  $S_i$ ,  $S_i$  must be an odd path,  $i = 1, 2, 3$ . By the choice of the  $y_i$ 's, the  $S_i$ 's and  $C$  are mutually disjoint, except for  $y_1, y_2$ , and  $y_3$ . From Claim 8,  $C$  is an odd cycle and  $y_1, y_2, y_3$  divide it into arcs of odd length. ■

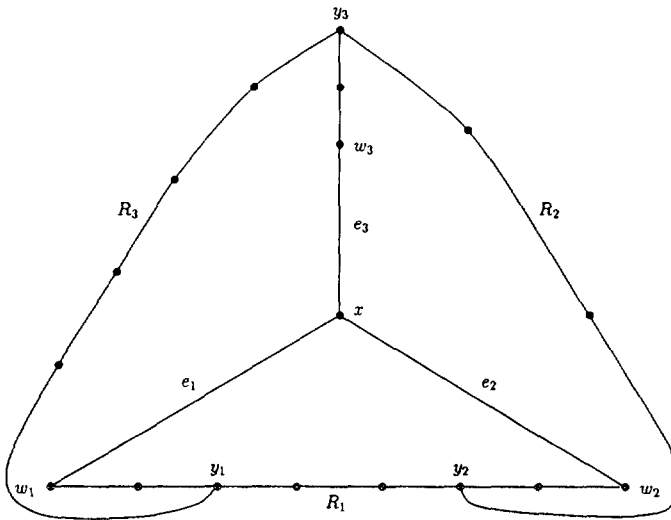
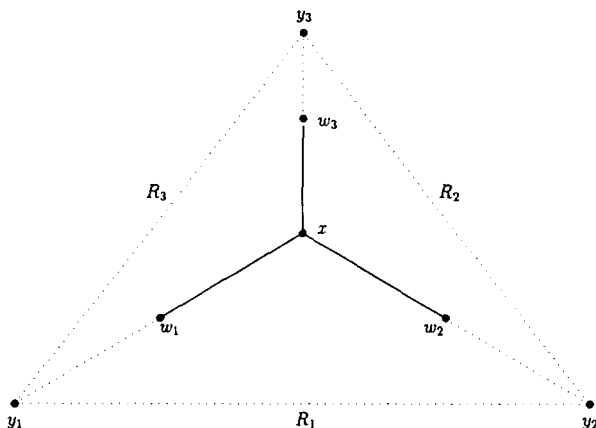


FIG. 8. The subpaths  $R_1, R_2$ , and  $R_3$ .

FIG. 9. An even subdivision of  $K_4$  centered at  $x$ .

The determination of the even subdivision of  $K_4$  depicted in Fig. 9 completes the proof of the theorem. ■

With the aid of Theorem 6, it is easy to see that if  $G \in \Gamma_c^\delta$  with  $\delta \geq 2$ , then  $G$  contains a graph in  $\Gamma_c^0$ ,  $\Gamma_c^1$ , and  $\Gamma_c^2$ . An interesting open question is whether or not this can be generalized to the following: If  $G \in \Gamma_c^\delta$  with  $\delta \geq 1$ , then  $G$  contains a graph in each of  $\Gamma_c^0, \Gamma_c^1, \dots, \Gamma_c^{\delta-1}$ . Of course, an affirmative answer would imply that  $G$  contains an even subdivision of a basis graph from each of  $\Gamma_c^0, \Gamma_c^1, \dots, \Gamma_c^{\delta-1}$ .

Berge [2] proved that every pair of adjacent edges in an  $\alpha$ -critical graph is contained in a chordless odd cycle. This result can be restated as every pair of adjacent edges in an  $\alpha$ -critical graph is contained in a subgraph which is in  $\Gamma_c^1$ . The above development shows that those graphs which are minimal in the sense required in the proof of Theorem 6 satisfy the more general stipulation that every triple of edges which share a common endnode is contained in a subgraph which is in  $\Gamma_c^2$ , i.e., an even subdivision of  $K_4$ . We conjecture that this property remains valid for all  $\alpha$ -critical graphs.

We close by mentioning that the characterization of  $\alpha$ -critical graphs given in Theorem 6 is used in [10, 8] to show that every rank facet of the stable set polytope, other than those derived from edges and odd cycles, contains an even subdivision of  $K_4$ . This leads to a polynomial time algorithm to find a maximum cardinality stable set for the class of graphs which do not contain an even subdivision of  $K_4$ . While we are unaware of a polynomial time algorithm to recognize graphs which do not contain an even subdivision of  $K_4$ , the algorithm from [10, 8] can also take an arbitrary graph as input and in polynomial time either produce a

maximum cardinality stable set or prove that the graph contains an even subdivision of  $K_4$  (without actually finding the even subdivision of  $K_4$ ). We also mention that in [8, 9] the concept of  $\alpha$ -criticality is generalized to the case where weights are assigned to the nodes, and a characterization analogous to Andrásfai's theorem (Theorem 2) for graphs in  $\Gamma_c^2$  is obtained.

## REFERENCES

1. B. ANDRÁSFAI, On critical graphs, in "Theory of Graphs" (P. Rosenstiehl, Ed.), pp. 9–19, Gordon & Breach, New York, 1967.
2. C. BERGE, Alternating chain methods: A survey, in "Graph Theory and Computing" (R. Read, Ed.), pp. 1–13, Academic Press, New York, 1972.
3. V. CHVÁTAL, On certain polytopes associated with graphs, *J. Combin. Theory Ser. B* **18** (1975), 138–154.
4. A. HAJNAL, A theorem on  $k$ -saturated graphs, *Canad. Math. J.* **17** (1965), 720–724.
5. F. HARARY AND M. D. PLUMMER, On indecomposable graphs, *Canad. J. Math.* **19** (1967), 800–809.
6. L. LOVÁSZ, Some finite basis theorems in graph theory, in "Combinatorics II" (A. Hajnal and V. T. Sós, Eds.), pp. 717–729, Colloq. Math. Soc. Janos Bolyai, Vol. 18, North-Holland, Amsterdam, 1978.
7. L. LOVÁSZ AND M. D. PLUMMER, "Matching Theory," Akademiai Kiado, Budapest; North-Holland, Amsterdam, 1986.
8. E. C. SEWELL, "Stability Critical Graphs and the Stable Set Polytope," Ph.D. thesis, Cornell University, School of OR&IE, Ithaca, New York, 1990.
9. E. C. SEWELL, "Stability Critical Graphs and the Stable Set Polytope," Technical Report, John M. Olin School of Business, Washington University, 1992.
10. E. C. SEWELL AND L. E. TROTTER, JR., "Stability Critical Graphs and Rank Facets of the Stable Set Polytope," Technical Report, John M. Olin School of Business, Washington University, 1992.
11. L. SURÁNYI, On line critical graphs, in "Infinite and Finite Sets (Colloq. Keszthely, Hungary, 1973), III" (A. Hajnal, R. Rado, and V. T. Sós, Eds.), pp. 1411–1444, Colloq. Math. Soc. Janos Bolyai, North-Holland, Amsterdam, 1975.