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Matroids and Geometric Invariant Theory of torus actions on flag spaces

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Abstract

Let $F//T$ be a Geometric Invariant Theory quotient of a partial flag variety $F = \mathrm{SL}(n, \mathbb{C})/P$ by the action $t \cdot gP = tgP$ of the maximal torus T in $\mathrm{SL}(n, \mathbb{C})$, where P is a parabolic subgroup containing T. The construction of $F//T$ depends upon the choice of a T-linearized line bundle L of F. This note concerns the case $L = L_\lambda$ is a very ample homogeneous line bundle determined by a dominant weight λ, meaning the associated character $\lambda : T \to \mathbb{C}^*$ extends to P and to no larger parabolic subgroup.

If V_λ denotes the irreducible representation of $\mathrm{SL}(n, \mathbb{C})$ with highest weight λ, and $V_\lambda[\mu]$ is the isotypic component corresponding to a weight μ of the torus, then $F//T$ is equal to $\mathrm{Proj}(\bigoplus_{N=0}^{\infty} V_{N\lambda}[N\mu])$. The weight μ is used to twist the canonical T-linearization of L_λ, where the canonical T-linearization of L_λ is obtained by restricting the unique $\mathrm{SL}(n, \mathbb{C})$-linearization of L_λ to T.

We apply a theorem of Gel'fand, Goresky, MacPherson, and Serganova concerning matroid polytopes to show that if $V_\lambda[\mu] \neq 0$ then one gets a well-defined map $F//T \to \mathbb{CP}^{\dim V_\lambda[\mu]-1}$ by taking any basis of $V_\lambda[\mu]$. Equivalently, all the semistable partial flags are detected by degree one T-invariants provided $V_\lambda[\mu]$ is nonzero.

We also show that the closure of any T-orbit in F is projectively normal for the projective embedding $\iota_\lambda : F \to \mathbb{CP}^{\dim V_\lambda - 1}$.

Keywords: Algebraic geometry (Geometric Invariant Theory); Combinatorics

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1. Introduction

Recall that the quotient $\mathrm{SL}(n, \mathbb{C})/P$ may be identified with the space of partial flags

$$F_{\ell_1,\dots,\ell_m}(\mathbb{C}^n) = \{(V_1, \dots, V_m) \mid \text{for each } i,\ V_i \subset V_{i+1},\ \dim(V_i) = \ell_i\},$$

where each V_i is a linear subspace of $\mathbb{C}^n$, and $1 \leqslant \ell_1 < \ell_2 < \cdots < \ell_m \leqslant n-1$. In the case $m = n-1$ we have the space of *full* flags, and P is the Borel subgroup. In the other extreme, if $m = 1$ then P is a maximal parabolic and $F_{\ell_1}(\mathbb{C}^n)$ is the Grassmannian of ℓ_1 dimensional subspaces of $\mathbb{C}^n$. We shall take the Borel subgroup B to be the set of matrices which are upper triangular, the torus T is the set of diagonal matrices, and consequently the parabolics P range over proper subgroups of $\mathrm{SL}(n, \mathbb{C})$ containing B; the elements of P are "staircase" shaped matrices.

The geometry (both symplectic and algebraic) of the quotients $F//T$ have been extensively studied in recent years; Allen Knutson called them "weight varieties"[2] in his thesis [K]. The dependence of the geometry of the quotient on the choice of linearization was studied by Yi Hu [Hu] and the cohomology of nonsingular weight varieties was computed by Rebecca Goldin [Go]. Special cases of weight varieties have been studied since the nineteenth century; for example a G.I.T. quotient $(\mathbb{CP}^{k-1})^n//\mathrm{PGL}(k, \mathbb{C})$ is isomorphic to a G.I.T. quotient $\mathrm{Gr}_k(\mathbb{C}^n)//T$ by the Gel'fand MacPherson correspondence (see [GM]). The projective invariants of n-tuples of points on projective space are still not understood today; we do not know a minimal set of generators for the ring of projective invariants (see p. 8 of [Ha]).

We take one step towards a solution to the generators problem (for $G = \mathrm{SL}(n, \mathbb{C})$) with Theorem 2.3, which implies that the lowest degree T-invariants in the graded ring of F are sufficient to give a well-defined map from $F//T$ to projective space. Consequently these global sections determine a globally generated ample line bundle $\overline{L}_\lambda$ of $F//T$. We are left with the problem of determining which tensor power of $\overline{L}_\lambda$ is very ample.

The proof of Theorem 2.3 is a simple combinatorial argument involving weight polytopes of flags. These weight polytopes are also known as flag matroid polytopes, see [BGW]. Any weight polytope is a Minkowski sum of *matroid* polytopes (see Section 3.1), which are weight

[2] The term "weight variety" actually refers to more general quotients; they are G.I.T. quotients of G/P by a maximal torus T in G, where G is a reductive connected complex Lie group and P is a parabolic subgroup of G containing T.

polytopes corresponding to fundamental weights. We introduce the notion of "root saturated" (see Definition 4.2), and we show this property is preserved under Minkowski sums in the special case $G = \mathrm{SL}(n, \mathbb{C})$; this fact requires that any collection of roots which are linearly independent may be extended to a basis of the root lattice. Secondly we need to know that the set of vertices of a matroid polytope is root saturated; this is where we apply the result of [GGMS] that the edges of a matroid polytope are parallel to roots. This is the only place we use the theory of matroids. Theorem 2.3 follows easily from these facts.

Remark 1.1. If F is a Grassmannian and λ is a fundamental weight, then Theorem 2.3 follows from a theorem of Neil White [W].

An embedded projective variety $V \hookrightarrow \mathbb{CP}^m$ is called *projectively normal* if the associated homogeneous coordinate ring $\mathbb{C}[x_0, \dots, x_m]/I_V$ is integrally closed in its field of fractions (see [Hart]). The tools we develop in proving Theorem 2.3 also allow us to show that the closure of the T-orbit of any $x \in F$ (for the projective embedding ι_λ of F) is a projectively normal toric variety. This extends the result of Neil White [W] who proved the same fact in the case F is a Grassmannian. For more general groups G (for all semisimple complex Lie groups) R. Dabrowski [Dab] proved that projective normality holds for closures of certain *generic* T-orbits in other homogeneous spaces G/P (he covered the case that G is any semisimple complex Lie group).

2. The construction of $F//T$ and statement of main theorem

A weight variety of $G = \mathrm{SL}(n, \mathbb{C})$ is a G.I.T. quotient of a flag manifold $F = G/P$ by the action of the Cartan subgroup T. The construction of such a quotient involves the choice of a T-linearized line bundle L of $F = G/P$. If L is very ample, then its isomorphism class is determined by a choice of dominant weight λ such that P is the largest parabolic subgroup such that the character e^λ defined on the Borel subgroup B of upper triangular matrices extends (uniquely) to P. The T-linearization of L will also depend on a choice of a weight μ, but μ need not be dominant.

2.1. *Elementary notions from the representation theory of* $\mathrm{SL}(n, \mathbb{C})$

Since $\mathrm{SL}(n, \mathbb{C})$ is simply connected, the set of $\mathrm{SL}(n, \mathbb{C})$ weights are the differentials evaluated at the identity element of characters $\chi : T \to \mathbb{C}^*$, which are holomorphic homomorphisms (that is, the character lattice coincides with the weight lattice). The differential $d\chi$ (evaluated at the identity element of T) of χ lies within the dual Lie algebra $\mathfrak{t}^*$ of T. On the other hand, if $\varpi \in \mathfrak{t}^*$ is a weight, we shall denote e^ϖ as the unique character $e^\varpi : T \to \mathbb{C}^*$ such that $d(e^\varpi) = \varpi$.

A character e^λ applied to $t = \mathrm{diag}(t_1, \dots, t_n) \in T$ must be equal to $\prod_{i=1}^n t_i^{a_i}$ for some fixed integers a_i. Since $\prod_{i=1}^n t_i = 1$ for all $t \in T$, we have that the n-tuple of exponents $(a_1, \dots, a_n)$ and $(a_1 + a, a_2 + a, \dots, a_n + a)$ determine the same character. We may thus view the abelian group of characters of T as $\mathbb{Z}^n/\Delta$ where Δ is the diagonal. On the other hand, the weight $\lambda \in \mathfrak{t}^*$ takes a complex vector $(z_1, \dots, z_n) \in \mathfrak{t}$ (where $z_1 + z_2 + \dots + z_n = 0$) to $\sum_{i=1}^n a_i z_i$. Again, adding a constant to each a_i results in the same function, and so again we have that the additive group of weights is isomorphic to $\mathbb{Z}^n/\Delta$. We shall henceforth identify characters and weights as n-tuple of integers modulo the diagonal Δ.

2.1.1. Dominant weights and construction of very ample line bundles

We say that a weight $\lambda = (a_1, \ldots, a_n)$ is dominant if $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$. Now suppose that λ is dominant, and $P \subset G$ is the largest parabolic subgroup (a subgroup containing all the upper triangular matrices in G) such that e^λ extends to a character $\chi : P \to \mathbb{C}^*$. It is a basic fact that χ is determined by its restriction e^λ to the torus T, so we will abuse notation and identify χ with e^λ.

The dominant weight λ determines a very ample line bundle L_λ of G/P. The total space of L_λ is the set of equivalence classes of pairs (g, z) for $g \in G$ and $z \in \mathbb{C}$, where $(g, z) \sim (gp, e^\lambda(p)z)$ for all $p \in P$. The map π from the total space to G/P is given by $\pi : (g, z) \mapsto gP$. Each global section of L_λ is given by $s_f(gP) = (g, f(g))$ where $f : G \to \mathbb{C}$ is a holomorphic function such that $f(gp) = e^\lambda(p) f(g)$ for all $p \in P$ and $g \in G$.

There is a natural action of G on the total space of L_λ, given by $g \cdot (g', z) = (gg', z)$. This defines an action on sections by

$$(g \cdot s)(g'P) = g \cdot s\big(g^{-1}g'P\big) = g \cdot \big(g^{-1}g', f\big(g^{-1}g'\big)\big) = \big(g'P, f\big(g^{-1}g'\big)\big).$$

The vector space V_λ of global sections is an irreducible representation of G; the action of $g \in G$ on s_f is $(g \cdot s_f)(g'P) = g \cdot s_f(g^{-1}g'P)$.

The Nth tensor power $L_\lambda^{\otimes N}$ is isomorphic to $L_{N\lambda}$.

2.1.2. Choosing a T-linearization of L_λ

There is a canonical T-linearization of L_λ, given by restricting the action of G on L_λ to T. We shall call this the "democratic" linearization. A weight μ may be used to twist the democratic linearization;

$$t \cdot (g, z) = \big(tg, \mu(t)z\big).$$

We shall call this the "μ-linearization." Indeed the set of all T-linearizations are given by the characters μ of T.

The μ-twisted action of T on a section s_f is given by the formula,

$$(t \cdot s_f)(gP) = \big(gP, e^\mu(t) f\big(t^{-1}g\big)\big).$$

Hence s_f is T-invariant iff $\mu(t) f(t^{-1}g) = f(g)$ for all $t \in T$. Equivalently, we have that s_f is T-invariant iff for all $t \in T$ and $g \in G$,

$$f(tg) = e^\mu(t) f(g).$$

The action on a section s_f of $L_\lambda^{\otimes N}$ is given by $(t \cdot s_f)(gP) = (gP, e^{N\mu}(t) f(g))$, and so the T-invariant sections s_f of the Nth tensor power of L_λ are those which satisfy

$$f(t \cdot g) = e^{N\mu}(t) f(g).$$

2.2. The G.I.T. construction

The G.I.T. quotient $F//T$ associated to the pair (λ, μ) is the projective variety,

$$F//T = \mathrm{Proj}\left(\bigoplus_{N=0}^{\infty} \Gamma\left(F, L_\lambda^{\otimes N}\right)^T\right),$$

where T acts on L_λ via the μ-linearization.

Definition 2.1. The set of semistable points $F^{ss} \subset F$ is defined by $p \in F^{ss}$ iff there exists some positive integer N and a T-invariant global section s of $L_\lambda^{\otimes N}$ such that $s(p) \neq 0$. (Normally there is the additional requirement that $X_s = \{p \in F \mid s(p) \neq 0\}$ is affine but this is automatic since F is a projective variety.) If we take the μ-linearization of L_λ, then we shall say that a semistable point is μ-semistable.

There is a surjective map $\pi : F^{ss} \to F//T$, where $\pi(x) = \pi(y)$ iff the closures of the T-orbits of x and y (Zariski closure) have nonempty intersection in F^{ss}; $\overline{T \cdot x} \cap \overline{T \cdot y} \cap F^{ss} \neq \emptyset$.

The following proposition is taken from the proof of Theorem 8.1 of [Do] and is a foundational result in Geometric Invariant Theory.

Proposition 2.2. (See [Do, Theorem 8.1].) *Suppose that $N > 0$ is sufficiently large so that the supports $X_s = \{x \in F \mid s(x) \neq 0\}$ of all T-invariant global sections s of $L^{\otimes N}$ cover F^{ss}. Then there exists an ample line bundle $\overline{L}_{N\lambda}$ of $F//T$ such that $\pi^*(\overline{L}_{N\lambda}) = L_\lambda^{\otimes N}$ restricted to the semistable points F^{ss}. Furthermore $\overline{L}_{N\lambda}$ is globally generated, meaning that for each $x \in F//T$ there exists a global section $s \in \Gamma(F//T, \overline{L}_{N\lambda})$ such that $s(x) \neq 0$.*

The proof of Theorem 2.3 below will be given in Section 4. By Proposition 2.2 it implies that L_λ itself ($N = 1$) descends to a globally generated ample line bundle of $F//T$.

Theorem 2.3 *(Main Theorem). Suppose that λ is a dominant weight and μ is any weight, such that $\lambda - \mu$ lies in the root lattice of* $\mathrm{SL}(n, \mathbb{C})$*. If $p \in F$ is μ-semistable, then there is a global T-invariant section s of L_λ such that $s(p) \neq 0$.*

Remark 2.4. It is well known that $\Gamma(F, L_\lambda)^T = V_\lambda[\mu]$ is nonzero if and only if $\lambda - \mu$ is in the root lattice and μ lies within the convex hull of the Weyl orbit of λ.

3. Weight polytopes and matroid polytopes

Suppose that V is a finite dimensional complex representation of a torus T. Then V is a direct sum of weight spaces,

$$V = \bigoplus_{\mu} V[\mu],$$

where $V[\mu] = \{v \in V \mid t \cdot v = e^{\mu}(t)v \text{ for all } t \in T\}$. Note that a section $s \in V_\lambda = \Gamma(F, L_\lambda)$ is T-invariant for the μ-linearization of L_λ if and only if $s \in V_\lambda[\mu]$.

Given a dominant weight λ let P_λ denote the associated parabolic subgroup; that is P_λ is the largest parabolic subgroup P such that the character $e^\lambda : T \to \mathbb{C}^*$ extends to P_λ. For each $g \in G$, let

$$wt_\lambda(g) = \{\mu \mid (\exists s \in V_\lambda[\mu]) \, (s(gP_\lambda) \neq 0)\}.$$

Let the *weight polytope* $\overline{wt}_\lambda(g)$ be the convex hull of $wt_\lambda(g)$ in $\mathfrak{t}_0^*$.

Lemma 3.1. *For any two dominant weights λ_1 and λ_2, we have*

$$wt_{\lambda_1}(g) + wt_{\lambda_2}(g) = wt_{\lambda_1+\lambda_2}(g),$$

where the summation denotes the Minkowski sum, $A + B = \{a + b \mid a \in A, \ b \in B\}$.

Proof. Suppose that $\mu_1 \in wt_{\lambda_1}(g)$ and $\mu_2 \in wt_{\lambda_2}(g)$. Let $s_1 \in V_{\lambda_1}[\mu_1]$ and $s_2 \in V_{\lambda_2}[\mu_2]$ such that $s_1(gP_{\lambda_1}) \neq 0$ and $s_2(gP_{\lambda_2}) \neq 0$. Recall there are functions $f_1, f_2 : G \to \mathbb{C}$ such that $s_1 = s_{f_1}$ and $s_2 = s_{f_2}$. We have that $f_1(g) \neq 0$ and $f_2(g) \neq 0$. Hence, $f_1(g)f_2(g) \neq 0$. The section $s_{f_1 f_2}$ lies in $V_{\lambda_1+\lambda_2}[\mu_1 + \mu_2]$, and is nonzero at $gP_{\lambda_1+\lambda_2}$.

Now suppose that $\mu \in wt_{\lambda_1+\lambda_2}(g)$. We may identify the irreducible representation V_λ as the space of global sections of $\pi^*(L_\lambda)$ of G/B where B is the Borel subgroup of G and $\pi : G/B \to G/P_\lambda$. This is justified since the pullback $\pi^* : \Gamma(G/P_\lambda, L_\lambda) \to \Gamma(G/B, \pi^*(L_\lambda))$ is an isomorphism of vector spaces. We shall also abuse notation and identify L_λ with the pullback $\pi^* L_\lambda$.

The tensor product $V_{\lambda_1} \otimes V_{\lambda_2}$ is the vector space of sections of the outer tensor product $L_{\lambda_1} \boxtimes L_{\lambda_2}$ of $G/B \times G/B$, where B is the Borel subgroup. The irreducible representation $V_{\lambda_1+\lambda_2}$ is a direct summand of $V_{\lambda_1} \otimes V_{\lambda_2}$, and the projection $V_{\lambda_1} \otimes V_{\lambda_2} \to V_{\lambda_1+\lambda_2}$ is realized by pulling back $L_{\lambda_1} \boxtimes L_{\lambda_2}$ to the diagonal $\Delta \subset G/B \times G/B$. We have assumed there is a section $s \in V_{\lambda_1+\lambda_2}[\mu]$ such that $s(gB) \neq 0$. Clearly $(V_{\lambda_1} \otimes V_{\lambda_2})[\mu]$ surjects onto $V_{\lambda_1+\lambda_2}[\mu]$. Furthermore,

$$(V_{\lambda_1} \otimes V_{\lambda_2})[\mu] = \sum_{\mu_1+\mu_2=\mu} V_{\lambda_1}[\mu_1] \otimes V_{\lambda_2}[\mu_2].$$

Hence there must exist weights μ_1, μ_2 such that $\mu_1 + \mu_2 = \mu$ and some component $s' = s_1 s_2$ of s such that $s_1(gB)s_2(gB) \neq 0$ and $s_1 \in V_{\lambda_1}[\mu_1]$ and $s_2 \in V_{\lambda_2}[\mu_2]$. $\square$

Corollary 3.2. *Suppose that $\lambda = \sum_{i=1}^{n-1} a_i \varpi_i$ is dominant, i.e. each a_i is nonnegative and ϖ_i denotes the ith fundamental weight connected to the Grassmannian $\mathrm{Gr}_i(\mathbb{C}^n)$. Then for any $g \in G$,*

$$wt_\lambda(g) = \sum_{i=1}^{n-1} a_i \cdot wt_{\varpi_i}(g),$$

where the sum indicates Minkowski sum and $a_i \cdot wt_{\varpi_i}(g)$ denotes the a_i-fold Minkowski sum of $wt_{\varpi_i}(g)$.

The weight polytope $\overline{wt}_\lambda(g)$ is a *flag matroid polytope* within the more general setting of Coxeter matroid polytopes, see [BGW]. Now we will study the special case of matroid polytopes $\overline{wt}_{\varpi_i}(g)$, which are the building blocks of more general weight polytopes by the above corollary.

3.1. Matroid polytopes

A matroid is a pair $M = (E, \mathcal{B})$ where E is a finite set called the ground set of M, and $\mathcal{B}$ is a nonempty collection of subsets of E called bases that satisfy the exchange condition. The exchange condition states that for any two bases $B_1, B_2 \in \mathcal{B}$, if $x \in B_1 \setminus B_2$ then there is an element $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$ is a basis. Necessarily it follows that all bases $B \in \mathcal{B}$ have the same cardinality, which is called the rank of M. Matroids are a generalization of finite configurations of vectors $v_1, \ldots, v_n$, where the only data known about the set of vectors is which subsets are independent. The collection of maximal independent subsets satisfies the exchange condition.

Definition 3.3. An element x of the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ determines a matroid $M(x)$ of rank k, given by the vector configuration $v_i = \pi_x(e_i)$, $1 \leqslant i \leqslant n$, where π_x is orthogonal projection onto the dimension k subspace of $\mathbb{C}^n$ corresponding to x (for the standard Hermitian form) and the e_i's are the standard basis vectors of $\mathbb{C}^n$.

It should be noted that some matroids are not equal to any $M(x)$. The $M(x)$ matroids form a proper subclass; they are the matroids which are representable over the field $\mathbb{C}$.

Definition 3.4. (See [GGMS].) Suppose that $M = (E, \mathcal{B})$ is a matroid, and $E = \{1, 2, 3, \ldots, n\}$. For each basis $B \in \mathcal{B}$ let $v^B \in \mathbb{R}^n$ be given by $v_i^B = 0$ if $i \notin B$ and $v_i^B = 1$ if $i \in B$. Let P_M be the convex hull of $\{v^B \mid B \in \mathcal{B}\}$. We call P_M a matroid polytope.

It is easy to see that each point v^B is a vertex of P_M and so M may be recovered from P_M. Therefore the map $M \mapsto P_M$ is one-to-one on the class of matroids, and so the theory of the special polytopes P_M is essentially the same as the theory of matroids.

Theorem 3.5. (Gel'fand, Goresky, MacPherson, Serganova [GGMS]) *Two vertices v^{B_1}, v^{B_2} of P_M form an edge of P_M iff $v^{B_1} - v^{B_2} = e_i - e_j$ for some $i \neq j$, where $e_1, \ldots, e_n$ are the standard basis vectors of $\mathbb{R}^n$.*

Conversely, if P is a polytope where all vertices are $0/1$ vectors (each component is either 0 or 1), and each edge of P is parallel to $e_i - e_j$ for some $i \neq j$, then there exists a (unique) matroid M such that $P = P_M$.

If we identify the root lattice of $\mathrm{SL}(n, \mathbb{C})$ with integral vectors $(a_1, \ldots, a_n)$ such that $\sum_i a_i = 0$, then the edges of a matroid polytope are all roots of $\mathrm{SL}(n, \mathbb{C})$. Now suppose that $\Delta \subset \mathbb{R}^n$ denotes the diagonal line $\{(x, x, \ldots, x) \in \mathbb{R}^n \mid x \in \mathbb{R}\}$. Let R denote the set of roots of $\mathrm{SL}(n, \mathbb{C})$, and let $Q(R)$ (respectively $P(R)$) be the root lattice (respectively weight lattice). Let T_0 denote the maximal compact subgroup of T, and let $\mathfrak{t}_0^*$ denote the dual of the Lie algebra of T_0. We shall identify

$$\mathfrak{t}_0^* = Q(R) \otimes \mathbb{R} = P(R) \otimes \mathbb{R} = \mathbb{R}^n / \Delta.$$

With this in mind we shall diverge a little from the definition of matroid polytope given above, and we will identify the image $\pi(P_M) \subset \mathbb{R}^n/\Delta$ with P_M itself, where $\pi : \mathbb{R}^n \to \mathbb{R}^n/\Delta$. No essential information is lost here, since P_M lies entirely in the affine hyperplane $H_r = \{v \in \mathbb{R}^n \mid \sum_{i=1}^n v_i = r\}$ (where $r = \mathrm{rank}(M)$), and H_r is mapped isomorphically onto $\mathbb{R}^n/\Delta$.

Another reason for taking P_M to lie within $\mathbb{R}^n/\Delta$ rather than $\mathbb{R}^n$ is that if we take the momentum mapping $\rho : \mathrm{Gr}_k(\mathbb{C}^n) \to \mathfrak{t}_0^*$ for the action of T_0, then $\rho(\overline{T \cdot x})$ is equal to $\pi(P_{M(x)})$; see [GGMS] or [BGW].

Proposition 3.6. *Suppose that ϖ_k is the kth fundamental weight. Then $\overline{wt}_{\varpi_k}(g)$ is a matroid polytope for any $g \in G$.*

Proof. A basis for the sections of L_{ϖ_k} is given by bracket functions $[i_1, i_2, \ldots, i_k]$ where $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$. The section $s = [i_1, i_2, \ldots, i_k]$ is equal to s_f, where $f : G \to \mathbb{C}$ assigns the determinant of the k by k submatrix given by columns $1, 2, \ldots, k$ and rows $i_1, i_2, \ldots, i_k$ of $g \in G$. The bracket $[i_1, i_2, \ldots, i_k]$ belongs to the weight space $V_{\varpi_k}[\mu]$ where $e^\mu = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n/\Delta$ is given by $a_i = 1$ if $i = i_t$ for some t, $1 \leqslant t \leqslant k$, otherwise $a_i = 0$. Now suppose that $gP_{\varpi_k} \in G/P_{\varpi_k} = \mathrm{Gr}_k(\mathbb{C}^n)$. The linear subspace defined by gP_{ϖ_k} is the span of the first k columns of g. We have that $\mu \in \overline{wt}_{\varpi_k}(g)$ iff μ is a 0/1 vector (mod Δ) with k ones (occurring at $I = (i_1, i_2, \ldots, i_k)$) and $n - k$ zeros such that the Ith minor of g (the minor consisting of the first k columns and rows indexed by $i \in I$) is nonzero.

Let $M(g)$ be the matroid with ground set $\{1, 2, \ldots, n\}$ of the vector configuration $r_1, r_2, \ldots, r_n \in \mathbb{C}^k$ where r_i is the ith row of g restricted to the first k columns, i.e. $r_i = (g_{i,1}, g_{i,2}, \ldots, g_{i,k})$. It is clear that the matroid polytope of $M(g)$ is the weight polytope $\overline{wt}_{\varpi_k}(g)$. □

In fact, one could show for any dominant weight λ that

$$\rho_\lambda(\overline{T \cdot gP_\lambda}) = \overline{wt}_\lambda(g),$$

where ρ_λ is the momentum mapping for the action of T_0 on G/P_λ, where G/P_λ gets has the natural symplectic form realized as the coadjoint orbit through λ. However, we will not need this here; we shall prove all our results using the fundamental weights ϖ_i as "building blocks."

4. Saturation properties of weight polytopes

We shall prove Lemma 4.1 below by a combinatorial argument. Our main theorem (Theorem 2.3) follows quite easily from Lemma 4.1. Neil White proved in [W] that Lemma 4.1 holds for $\lambda = \varpi_k$ using a theorem of Edmonds in matroid theory concerning when the ground set of a matroid partitions into bases.

Lemma 4.1. *Suppose $g \in G = \mathrm{SL}(n, \mathbb{C})$ and λ is a dominant weight. Suppose μ is a weight such that $\lambda - \mu$ is in the root lattice. Then for all $N > 0$, if $N\mu \in wt_{N\lambda}(g)$ then $\mu \in wt_\lambda(g)$.*

Recall that R is the set of $\mathrm{SL}(n, \mathbb{C})$ roots, and $Q(R)$ (respectively $P(R)$) denote the root lattice (respectively weight lattice). Convex hulls of subsets of the weight lattice, denoted by an overline, should take place in $\mathfrak{t}_0^* = P(R) \otimes \mathbb{R} = \overline{P(R)} = \mathbb{R}^n/\Delta$. The map $\epsilon : P(R) \to \mathbb{Z}/n\mathbb{Z}$ given by $\epsilon(a_1, \ldots, a_n) = \sum_i a_i \bmod n$ is a homomorphism of abelian groups, and $Q(R) = \ker(\epsilon)$.

Definition 4.2. A finite subset A of $Q(R)$ is called *root-saturated* if

(1) the convex hull $\overline{A}$ is such that each edge e_i is an integral multiple of some root γ_i in R (i.e. $\overline{A}$ is a flag matroid polytope, see [BGW]).
(2) $A = \overline{A} \cap Q(R)$.

We will eventually prove (Corollary 4.7) that $wt_\lambda(g) - \lambda = \{\mu - \lambda \mid \mu \in wt_\lambda(g)\}$ is root-saturated for any dominant weight λ.

Lemma 4.3. *Suppose that $\alpha_1, \dots, \alpha_{n-1} \in R$ are independent over $\mathbb{Q}$. Then they are a basis for the root lattice $Q(R)$. (R is unimodular.)*

Proof. The proof goes by induction on n. If $n = 2$ there are only two roots α, $-\alpha$ and they generate the same lattice. Now suppose that $n > 2$. Let $\mathbb{Z}[\alpha_1, \dots, \alpha_{n-1}]$ be the $\mathbb{Z}$-span of $\alpha_1, \dots, \alpha_{n-1}$. Without loss of generality we may assume that each α_i is a positive root since negating α_i does not change the span over $\mathbb{Z}$. Let $\sigma_1, \dots, \sigma_{n-1}$ be the standard simple roots of $\mathrm{SL}(n)$. That is, $\sigma_i = e_i - e_{i+1}$. Note that any positive root $e_i - e_j = \sum_{t=i}^{j-1} \sigma_t$ is a sum of consecutive simple roots. Conversely any consecutive sum of simple roots is a positive root. We may choose some $w \in W$ (where W is the Weyl group) such that $w(\alpha_{n-1}) = \sigma_{n-1}$. In particular if $\alpha_{n-1} = e_i - e_j$ let w be the product of two cycles $(n-1 \quad i)(n \quad j)$. Since elements of W induce isomorphisms of the lattice $Q(R)$, we have that $w(\alpha_1), \dots, w(\alpha_{n-1})$ is a basis of $Q(R)$ if and only if $\alpha_1, \dots, \alpha_{n-1}$ is a basis of $Q(R)$. Reassign $\alpha_i := w(\alpha_i)$. For each $i \leqslant n-2$, if $\alpha_i = e_s - e_n = \sigma_s + \dots + \sigma_{n-1}$ replace α_i with $\alpha_i - \sigma_{n-1} = \sigma_s + \dots + \sigma_{n-2} = e_s - e_{n-1}$. Now the roots $\alpha_1, \dots, \alpha_{n-2}$ may be identified with roots of $\mathrm{SL}(n-1)$. By the induction hypothesis $\mathbb{Z}[\alpha_1, \dots, \alpha_{n-2}] = \mathbb{Z}[\sigma_1, \dots, \sigma_{n-2}]$. Since $\alpha_{n-1} = \sigma_{n-1}$ we have that $\mathbb{Z}[\alpha_1, \dots, \alpha_{n-1}] = Q(R)$. $\square$

Lemma 4.4. *Suppose that A and B are root-saturated, and $\overline{A} \cap \overline{B}$ is nonempty. Then $A \cap B$ is nonempty.*

Proof. The proof is by induction on the dimension of $\overline{A}$. If $\dim \overline{A} = 0$ then $A = \{a\}$ for some $a \in Q(R)$. Then $\overline{A} \cap \overline{B} = A \cap B = \{a\}$. Now suppose that $\dim \overline{A} \geqslant 1$.

We have two cases, the first case is that $\overline{B}$ contains a boundary point of $\overline{A}$. Let $\partial \overline{A}$ denote the boundary of $\overline{A}$.

Case I. ($\overline{B} \cap \partial \overline{A} \neq \emptyset$.) There is some facet F of $\overline{A}$ such that $F \cap \overline{B}$ is nonempty. We claim $F \cap A$ is root-saturated. The vertices of F are within $F \cap A$ so $\overline{F \cap A} \supset F$. On the other hand, $F \subset F \cap A$ so $F \subset \overline{F \cap A}$; therefore $F = \overline{F \cap A}$. The edges of F are also edges of $\overline{A}$ hence they are parallel to roots. Furthermore, for any $x \in F \cap A$, we have $\overline{F \cap A} \cap Q(R) \subset A$ since A is root-saturated, and it follows that $\overline{F \cap A} \cap Q(R) = F \cap A$ since $F \cap A \subset A \subset Q(R)$. Since $\dim F < \dim \overline{A}$ we may apply the induction hypothesis to get that $F \cap A \cap B$ is nonempty and hence $A \cap B$ is nonempty.

Case II. ($\overline{B} \cap \partial \overline{A} = \emptyset$.) Now suppose that $\overline{A} \cap \overline{B}$ contains no boundary point of $\overline{A}$. Let $L_A(R)$ be the sublattice of $Q(R)$ spanned by the roots which are parallel to some edge of $\overline{A}$. Let $a_0 \in A$ be a vertex of $\overline{A}$. Note that the affine space $H_A = a_0 + \overline{L_A(R)}$ is the smallest affine space containing $\overline{A}$. We claim $H_A \cap \overline{B} = \overline{A} \cap \overline{B}$. Suppose that $z \in H_A \cap \overline{B}$. Let $a \in \overline{A} \cap \overline{B}$. Since H_A has the same dimension as $\overline{A}$, there are linear inequalities $\eta_i(x) \leqslant f_i$ where the interior of $\overline{A}$ consists of points $x \in H_A$ where the inequalities are strict; that is, $\eta_i(x) < f_i$ for all i if and only if x is an interior point of $\overline{A}$. The boundary points of $\overline{A}$ are those points $x \in \overline{A}$ such that $\eta_i(x) = f_i$ for some i. Let $c(t) = (1-t)a + tz$ for $0 \leqslant t \leqslant 1$. Suppose that $z \notin \overline{A}$. Then there is some i such that $\eta_i(z) > f_i$. However a is an interior point of $\overline{A}$ and so $\eta_i(a) < f_i$. Hence there is some t_0 such that $\eta(c(t_0)) = f_i$ in which case $c(t_0)$ is a boundary point of $\overline{A}$. But $c(t) \in \overline{B}$ for each t by convexity of $\overline{B}$. This contradicts that $\overline{A} \cap \overline{B}$ is disjoint from the boundary of A. Hence $H_A \cap \overline{B} = \overline{A} \cap \overline{B}$. Therefore $(H_A \cap Q(R)) \cap B = A \cap B$ since $\overline{A} \cap Q(R) = A$ and $\overline{B} \cap Q(R) = B$.

Replace A with the affine sublattice $H_A \cap Q(R)$: We now show by induction on $\dim \overline{B}$, that for any B which is root-saturated, that $H_A \cap \overline{B}$ is nonempty implies $(H_A \cap Q(R)) \cap B$ is nonempty. Suppose that $\dim \overline{B} = 0$. Then $B = \{b\}$ for some $b \in Q(R)$, and so $b \in (H_A \cap Q(R)) \cap B$. Now suppose that $\dim \overline{B} \geqslant 1$. We have two cases.

Case IIa. ($H_A \cap \partial \overline{B} \neq \emptyset$.) First suppose that H_A intersects the boundary of $\overline{B}$ nontrivially. Then there is a face F of $\overline{B}$ such that $H_A \cap F$ is nonempty. Since $F \cap B$ is root-saturated, $\overline{F \cap B} = F$, $H_A \cap F$ is nonempty, and $\dim F < \dim B$, we may apply the induction hypothesis and we are finished.

Case IIb. ($H_A \cap \partial \overline{B} = \emptyset$.) Now suppose that H_A is disjoint from the boundary of $\overline{B}$. Let $L_B(R)$ be the sublattice of $Q(R)$ spanned by the roots which are parallel to some edge of $\overline{B}$. Let $b_0 \in B$ be a vertex of $\overline{B}$.

Replace B with the affine sublattice $H_B \cap Q(R)$: The affine space $H_B = b_0 + \overline{L_B(R)}$ is the smallest affine space containing $\overline{B}$. As above, we have that $H_A \cap H_B = H_A \cap \overline{B}$ and so $(H_A \cap Q(R)) \cap (H_B \cap Q(R)) = A \cap B$.

The intersection $H_A \cap H_B$ is a unique point $z_0 \in Q(R)$: Since H_A does not intersect the boundary of $\overline{B}$, we have that $H_A \cap H_B$ is a single point z_0, since if the dimension of the intersection $H_A \cap H_B = H_A \cap \overline{B}$ is greater than zero then $H_A \cap \overline{B}$ is unbounded. But $\overline{B}$ is compact since B is finite and this cannot happen. We now show that $z_0 \in Q(R)$. We have that $z_0 = a_0 + v_A = b_0 + v_B$ where $a_0 \in A$, $b_0 \in B$, $v_A \in \overline{L_A(R)}$, $v_B \in \overline{L_B(R)}$. Let $\{\alpha_1, \dots, \alpha_p\} \subset R$ be a basis of $L_A(R)$ and let $\{\beta_1, \dots, \beta_q\} \subset R$ be a basis of $L_B(R)$. Since the intersection of H_A and H_B is a point, we have that $\overline{L_A(R)} \cap \overline{L_B(R)} = \{0\}$. Hence the set $\{\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q\}$ is linearly independent in $\overline{Q(R)}$. Choose $\{\gamma_1, \dots, \gamma_r\} \subset R$ so that $\{\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q, \gamma_1, \dots, \gamma_r\}$ is a basis for $\overline{Q(R)}$. By Lemma 4.3 this is also a basis for the lattice $Q(R)$. Now $v_A = \sum_i c_i \alpha_i$ and $v_B = \sum_j d_j \beta_j$ are unique expressions for v_A, v_B. But also the difference $a_0 - b_0 = v_B - v_A = (\sum_j d_j \beta_j) - (\sum_i c_i \alpha_i)$ lies within the lattice $Q(R)$, and so the coefficients c_i, d_j must be integers. Hence, z_0 is a lattice point and we have finished the proof of the lemma. □

Theorem 4.5. *Suppose that A and B are root-saturated. Then the Minkowski sum $A + B = \{a + b \mid a \in A,\ b \in B\}$ is root-saturated.*

Proof. We show that the Minkowski sum $A + B$ is root-saturated if A and B are each root-saturated. Clearly $A + B$ is finite, and the elements are within $Q(R)$ since $Q(R)$ is closed under addition. First we show that the edges of $\overline{A + B}$ are parallel to roots. Clearly $\overline{A} + \overline{B} = \overline{A + B}$. The Minkowski sum of two polytopes P, Q has edges of the following types:

- (vertex of P) + (edge of Q).
- (edge of P) + (vertex of Q).
- (edge of P) + (edge of Q), providing these edges are parallel.

We leave the proof to the reader (the proof is easily obtained by observing that the fan (see [Z]) of $P + Q$ is the meet of the fan of P with the fan of Q). In all three cases, the resulting edge is parallel to an edge of either P or Q or both, and hence it is parallel to some root in R.

Next we must show that $A + B = (\overline{A + B}) \cap Q(R)$. Suppose that $z \in (\overline{A + B}) \cap Q(R)$. Hence there exists $x \in \overline{A}$ and $y \in \overline{B}$ such that $x + y = z$. Hence $x \in (z + \overline{-B}) \cap \overline{A}$, where $-B = \{-b\colon b \in B\}$. Clearly $z + (-B)$ is root-saturated. Hence, we may apply the lemma above to get a lattice point x_0 in the intersection. Since A is saturated, we have that $x_0 \in A$. Now we have that $z = x_0 + y_0$ where $y_0 \in \overline{B}$. But since $z, x_0 \in Q(R)$ we have that $y_0 = z - x_0 \in Q(R)$, and so $y_0 \in B$ since B is root-saturated, and we are finished. □

Lemma 4.6. *If ϖ_k is a fundamental weight and $g \in G$ then the translation $wt_{\varpi_k}(g) - \varpi_k$ is root-saturated.*

Proof. The representation theorists will see this immediately follows from the fact that ϖ_k is a miniscule representation and $\overline{wt}_{\varpi_k}(g)$ is a matroid polytope.

Note that all elements of $wt_{\varpi_k}(g)$ are $0/1$ vectors (mod Δ) having k ones and $n - k$ zeros. Translating by $-\varpi_k$ results in vectors whose first k components may be either 0 or -1 and last $n - k$ components are 0 or $+1$, and the sum of all components is zero. Hence the first k components define a vertex of the negated unit k-cube $[0, 1]^k$, and the last $n - k$ components are vertices of the $(n - k)$-cube. Therefore, there can be no additional lattice points in the convex hull. We already showed that the convex hull of $wt_{\varpi_k}(g)$ is a matroid polytope, so the edges are parallel to roots. This property is preserved by translations. □

Corollary 4.7. *For any dominant weight λ and $g \in G$, the set $wt_\lambda(g) - \lambda$ is root-saturated.*

Proof. We have that $\lambda = \sum_{k=1}^{n-1} a_k \varpi_k$, where the a_k's are nonnegative integers. Also, $wt_\lambda(g) = \sum_{k=1}^{n-1} a_k \cdot wt_{\varpi_k}(g)$ (Minkowski sum). Hence,

$$wt_\lambda(g) - \lambda = \sum_{k=1}^{n-1} a_k \cdot \bigl(wt_{\varpi_k}(g) - \varpi_k\bigr).$$

Since the root-saturated property is preserved under Minkowski sums, we have that $wt_\lambda(g) - \lambda$ is root-saturated. □

Proof of Lemma 4.1. Suppose that $N\mu \in wt_{N\lambda}(g)$. Then $N(\mu - \lambda) \in wt_{N\lambda}(g) - N\lambda$. The convex hull of $wt_{N\lambda}(g) - N\lambda$ scaled by $1/N$ is equal to the convex hull of $wt_\lambda(g) - \lambda$ since $N \cdot wt_\lambda(g) = wt_{N\lambda}(g)$. Therefore $\mu - \lambda$ is in the convex hull of $wt_\lambda(g) - \lambda$. But since $\mu - \lambda \in Q(R)$ and $wt_\lambda(g) - \lambda$ is root-saturated, we have that $\mu - \lambda \in wt_\lambda(g) - \lambda$, so $\mu \in wt_\lambda(g)$. □

Proof of Theorem 2.3. Suppose that gP_λ is semistable relative to the μ-linearization of the line bundle L_λ. This means there is some $N > 0$ and a section $s \in \Gamma(G/P_\lambda, L_\lambda^{\otimes N})^T$ such that $s(gP_\lambda) \neq 0$. This means that $N\mu \in wt_{N\lambda}(g)$. By Lemma 4.1 we have that $\mu \in wt_\lambda(g)$. So there must exist a section $s' \in \Gamma(G/P_\lambda, L_\lambda)^T$ such that $s'(gP_\lambda) \neq 0$. □

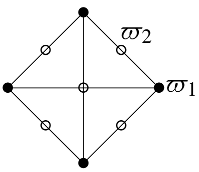
4.1. Failure of main theorem for $G = \mathrm{SO}(5, \mathbb{C})$

Let $B(z, w)$ be the bilinear form on $\mathbb{C}^5$ given by

$$B(z, w) = z_1w_5 + z_2w_4 + z_3w_3 + z_4w_2 + z_5w_1 = 2z_1w_5 + 2z_2w_4 + z_3w_3.$$

Now $SO(5,\mathbb{C}) \subset SL(5,\mathbb{C})$ is the subgroup preserving B. The maximal torus T may be taken to the diagonal matrices in $SO(5,\mathbb{C})$. Elements of T have the form $\mathrm{diag}(t_1, t_2, 1, 1/t_2, 1/t_1)$ for $t_1, t_2 \in \mathbb{C}^*$. Let ϖ_1 denote the first fundamental weight of $SO(5,\mathbb{C})$. We have $e^{\varpi_1}(t_1, t_2, 1, 1/t_1, 1/t_2) = t_1$, but the second fundamental weight does not lift to a character of $SO(5,\mathbb{C})$—one needs to go the universal cover to find such a character. Let $P_{\varpi_1} \subset SO(5,\mathbb{C})$ be the associated parabolic subgroup. The quotient space $SO(5,\mathbb{C})/P_{\varpi_1}$ may be identified with the space of isotropic lines in $\mathbb{C}^5$.

Let x be the (isotropic) line through $(1, \sqrt{-1}, 0, \sqrt{-1}, 1)$. Let $g_x \in SO(5,\mathbb{C})$ be such that $g_x P_{\varpi_1} = x$. The set $wt_{\varpi_1}(g_x)$ is equal to $\{\varpi_1, 2\varpi_2 - \varpi_1, -2\varpi_2 + \varpi_1, -\varpi_1\}$. Depiction:

$wt_{\varpi_1}(g_x) =$ 

This set is missing the origin, although $V_{\varpi_1}[0] \neq 0$ and $\varpi_1 \in Q(SO(5,\mathbb{C}))$, so $wt_{\varpi_1}(g_x) - \varpi_1$ is not root-saturated. Note the origin does belong to $wt_{2\varpi_1}(g_x) = wt_{\varpi_1}(g_x) + wt_{\varpi_1}(g_x)$. Therefore x is semistable for the democratic linearization of L_{ϖ_1}. It follows that for the democratic linearization of L_{ϖ_1}, one requires a T-invariant section of $L_{\varpi_1}^{\otimes 2}$ to detect the semistable point x.

5. Projective normality

Let H be the group of diagonal matrices in $GL(n,\mathbb{C})$. Hence $T \subset H$ is the set of diagonal matrices with determinant one. Let $\chi_1, \ldots, \chi_m$ be m characters of H. That is, each $\chi_i : H \to \mathbb{C}^*$ is an algebraic homomorphism of groups. Each χ_i is given by a point $\mathbf{a}_i = (a_{i,1}, \ldots, a_{i,n}) \in \mathbb{Z}^n$, where

$$\chi_i(h_1, \ldots, h_n) = \prod_{j=1}^{n} h_j^{a_{i,j}}.$$

These characters determine an action of H on $\mathbb{A}^m$ by

$$h \cdot (z_1, z_2, \ldots, z_m) = \big(\chi_1(h)z_1, \chi_2(h)z_2, \ldots, \chi_m(h)z_m\big).$$

Now take any point $z \in \mathbb{A}^m$, and let $X(z)$ be the Zariski closure of the H-orbit of z. That is, $X(z) = cl(H \cdot z)$. Certainly $X(z)$ contains a dense torus and there is a natural action of this torus on $X(z)$; so $X(z)$ is a (possibly nonnormal) toric variety.

But when is $X(z)$ a *normal* toric variety, i.e. when is the coordinate ring of $X(z)$ integrally closed in its field of fractions? Some notation: if A is a finite subset of $\mathbb{Z}^d$ then let $\mathbb{Z}(A)$ be the sublattice generated by A, let $\mathbb{N}(A)$ be the semigroup of all nonnegative integral combinations of elements of A, and let $\mathbb{Q}_0^+(A)$ be the rational cone in $\mathbb{Q}^d$ given by all nonnegative rational combinations of elements of A. According to Proposition 13.5 of [St] we have that the semigroup algebra $\mathbb{C}[\mathbb{N}(A)]$ is normal iff $\mathbb{N}(A) = \mathbb{Z}(A) \cap \mathbb{Q}_0^+(A)$.

The following proposition is likely well known but we give a proof for lack of reference.

Proposition 5.1. *Let* $\mathrm{supp}(z) = \{i \mid z_i \neq 0\}$. *Let* $A(z) = \{\chi_i \mid i \in \mathrm{supp}(z)\}$. *Then* $X(z)$ *is isomorphic to the affine toric variety defined by* $A(z) \subset \mathbb{Z}^n$. *That is,* $X(z)$ *is isomorphic to the affine variety* $V \subset \mathbb{C}^{\#A(n)}$ *of the semigroup algebra* $\mathbb{C}[\mathbb{N}(A(z))]$, *where* $\mathbb{N}(A(z))$ *is the semigroup in* $\mathbb{Z}^n$ *generated by* $A(z)$. *Hence* $X(z)$ *is normal if and only if* $\mathbb{Z}(A(z)) \cap \mathbb{Q}_0^+(A(z)) = \mathbb{N}(A(z))$.

Proof. Let $\bar{z} \in \mathbb{C}^m$ be given by $\bar{z}_i = 1$ if $i \in \mathrm{supp}(z)$ and $\bar{z}_i = 0$ otherwise. Let $s_i = 1/z_i$ if $z_i \neq 0$ and $s_i = 1$ if $z_i = 0$. Then the matrix $\mathrm{diag}(s_1, \ldots, s_m)$ defines an algebraic automorphism of $\mathbb{A}^m$ which takes $X(z)$ to $X(\bar{z})$, so $X(\bar{z})$ is isomorphic to $X(z)$. Hence we may assume that all components of z are either 0 or 1. Additionally, $X(z)$ lives entirely within the components i where z_i is nonzero. Hence, we may project $X(z)$ onto the linear subspace given by the components in $\mathrm{supp}(z)$, which defines an isomorphism of $X(z)$ onto its image. Thus, we may assume that each component of z is equal to one. If $\chi_i = \chi_j$ for some i, j, we may also project away one of these. Hence we have reduced to the case that the χ_i's are distinct, and z is the vector of all ones. The coordinate ring of $X(z)$ is now easily seen to be the semigroup algebra $\mathbb{C}[\mathbb{N}(A(z))]$. □

A dominant weight λ of $\mathrm{SL}(n, \mathbb{C})$ may be lifted to a dominant weight $\tilde{\lambda}$ of $\mathrm{GL}(n, \mathbb{C})$ by normalizing λ so that the last component is zero. That is, the image of $\tilde{\lambda} \in \mathbb{Z}^n$ in $\mathbb{Z}^n/\Delta$ is λ, and $\tilde{\lambda}_n = 0$. Let

$$|\tilde{\lambda}| = \sum_{i=1}^{n} \tilde{\lambda}_i.$$

Now V_λ is also an irreducible representation of $\mathrm{GL}(n, \mathbb{C})$, where $zI_n \in \mathrm{GL}(n, \mathbb{C})$ acts by scaling each vector $s \in V_\lambda$ by $z^{|\tilde{\lambda}|}$, and so if $\tilde{\lambda} g = zg$ where $z \in \mathbb{C}^*$ and $g \in \mathrm{SL}(n, \mathbb{C})$ then the action of $\tilde{g}$ is defined by $\tilde{g} \cdot s = z^{|\tilde{\lambda}|}(g \cdot s)$. A basis for the representation V_λ is given by semistandard tableaux τ of shape $\tilde{\lambda}$ (with total number of slots equal to $|\tilde{\lambda}|$), filled with indices from 1 to n. A section $s_\tau \in V_\lambda[\mu]$ iff the number of times the index i appears in τ is equal to μ_i. Here we are treating μ as a weight of $\mathrm{GL}(n, \mathbb{C})$. Note that if $V_\lambda[\mu] \neq 0$ then $|\mu| = \sum_{i=1}^n \mu_i = |\tilde{\lambda}|$ since $|\mu|$ must equal the total number of slots in τ, where $s_\tau \in V_\lambda[\mu]$.

Recall that $H = \mathbb{C}^*(T)$ is the maximal torus in $\mathrm{GL}(n, \mathbb{C})$ consisting of diagonal matrices. For each $g \in \mathrm{GL}(n, \mathbb{C})$ let

$$wt_{\tilde{\lambda}}(g) = \bigl\{\mu \bigm| \bigl(\exists s \in V_\lambda[\mu]\bigr) \bigl(s(gP_{\tilde{\lambda}}) \neq 0\bigr)\bigr\},$$

where $P_{\tilde{\lambda}} \subset \mathrm{GL}(n, \mathbb{C})$ is the parabolic subgroup $\mathbb{C}^*(P_\lambda)$ associated to $\tilde{\lambda}$. Each $\mu \in wt_{\tilde{\lambda}}(g) \subset \mathbb{Z}^n$ satisfies $|\mu| = |\tilde{\lambda}|$.

Note that the root lattice of $\mathrm{SL}(n, \mathbb{C})$ may be identified with integral vectors $v \in \mathbb{Z}^n$ whose components sum to zero. Hence, for any $g \in \mathrm{SL}(n, \mathbb{C})$ we have an identification of $wt_\lambda(g) - \lambda$ with $wt_{\tilde{\lambda}}(g) - \tilde{\lambda}$. In particular, $wt_{\tilde{\lambda}}(g) - \tilde{\lambda}$ is root-saturated.

Let $N_{\tilde{\lambda}}$ be the sublattice of $\mathbb{Z}^n$ given by

$$N_{\tilde{\lambda}} = \left\{ v \in \mathbb{Z}^n \;\middle|\; |v| = \sum_{i=1}^{n} v_i \equiv 0 \bmod |\tilde{\lambda}| \right\}.$$

Lemma 5.2. *For any* $g \in \mathrm{SL}(n, \mathbb{C})$,

$$\mathbb{Q}_0^+\big(wt_{\tilde{\lambda}}(g)\big) \cap N_{\tilde{\lambda}} = \mathbb{N}\big(wt_{\tilde{\lambda}}(g)\big).$$

Proof. Suppose that $v \in \mathbb{Q}_0^+(wt_\lambda(g)) \cap N_{\tilde{\lambda}}$. Then $|v| = d|\tilde{\lambda}|$ for some $d \in \mathbb{N}$. Hence v belongs to the convex hull of the dth dilate of $wt_{\tilde{\lambda}}(g)$, so v is in the convex hull of $wt_{d\tilde{\lambda}}(g)$, since $wt_{d\tilde{\lambda}}(g)$ is the d-fold Minkowski sum of $wt_{\tilde{\lambda}}(g)$. But $wt_{d\tilde{\lambda}}(g) - d\tilde{\lambda}$ is root-saturated, and since $v - d\tilde{\lambda} \in Q(R)$ we have that $v - d\tilde{\lambda} \in wt_{d\tilde{\lambda}}(g) - d\tilde{\lambda}$. Equivalently, $v \in wt_{d\tilde{\lambda}}(g)$. Since $wt_{d\tilde{\lambda}}(g)$ is the d-fold Minkowski sum of $wt_{\tilde{\lambda}}(g)$, we have that $v \in \mathbb{N}(wt_{\tilde{\lambda}}(g))$. □

Corollary 5.3. *The semigroup algebra* $\mathbb{C}[\mathbb{N}(wt_{\tilde{\lambda}}(g))]$ *is normal.*

Now suppose that λ is dominant and P_λ is the associated parabolic subgroup. Choose a basis $(s_1, s_2, \ldots, s_N)$ of $V_\lambda = \Gamma(\mathrm{SL}(n, \mathbb{C})/P_\lambda, L_\lambda)$ such that each basis vector is a generalized eigenvector for the democratic T-action. (Recall the democratic action is the restriction of the natural action of $\mathrm{SL}(n, \mathbb{C})$ on V_λ to T.) Let $\iota_\lambda : \mathrm{SL}(n, \mathbb{C})/P_\lambda \to \mathbb{P}(V_\lambda)$ be the projective embedding determined by this choice of basis. (Note that one typically embeds G/P_λ into $\mathbb{P}(V_\lambda^*)$ as there is no need for a choice of basis, but it is more convenient for us to embed into $\mathbb{P}(V_\lambda)$.)

The following theorem has been proven by R. Dabrowski for certain *generic* T-orbits in G/P for G an arbitrary semisimple complex Lie group, see [Dab]. Herein lies the first proof for *arbitrary* T-orbits in the case $G = \mathrm{SL}(n, \mathbb{C})$.

Theorem 5.4. *The Zariski closure of any T-orbit in* $\mathrm{SL}(n, \mathbb{C})/P_\lambda \hookrightarrow \mathbb{P}(V_\lambda)$ *is a projectively normal toric variety.*

Proof. Let $x \in \mathrm{SL}(n, \mathbb{C})/P_\lambda \subset \mathbb{P}(V_\lambda)$. Let $cl(T \cdot x)$ denote the Zariski closure of the orbit $T \cdot x$. Let $\mathrm{Aff}(cl(T \cdot x)) \subset V_\lambda$ denote the associated affine cone; it is easy to see that $\mathrm{Aff}(cl(T \cdot x)) = cl(H \cdot v_x)$ where v_x is any nonzero vector on the line x, since the scalar matrices in H fill out all nonzero multiples of points in $T \cdot v_x$.

Let $g \in \mathrm{SL}(n, C)$ be such that $gP_\lambda = x$. Now $wt_{\tilde{\lambda}}(g) = A(v_x)$. Hence by Proposition 5.1, the affine toric variety $\mathrm{Aff}(cl(T \cdot x))$ is normal if and only if the semigroup algebra $\mathbb{C}[\mathbb{N}(wt_{\tilde{\lambda}}(g))]$ is normal, which we have already shown. This means that the projective toric variety $cl(T \cdot x)$ is projectively normal. □

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