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On Hadamard's Variational Formula

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INTRODUCTION

In his famous memoir [2] (see also pp. 303-312 of the book [3]) Hadamard obtains the following variational formula for Green's function G_1 and Neumann's function G_0 associated with the Laplace operator Δ in a bounded 3-dimensional domain Ω with boundary $\partial\Omega$:

$$\delta G_1(x, y) = \int_{\partial\Omega} \frac{\partial G_1(\cdot, x)}{\partial N} \frac{\partial G_1(\cdot, y)}{\partial N} \delta N \, \sigma, \tag{0}$$

$$\delta G_0(x, y) = -\int_{\partial\Omega} (\operatorname{grad} G_0(\cdot, x) \operatorname{grad} G_0(\cdot, y) + \frac{H}{S}\left(G_0(\cdot, x) + G_0(\cdot, y) - \frac{1}{S^2}\right) \delta N \, \sigma, \tag{1}$$

where δN denotes variation of Ω in the direction of the *exterior* normal of $\partial\Omega$, and S and H denote respectively the surface area and the mean curvature of $\partial\Omega$, σ being the standard surface area element. An analogous formula holds also for the fundamental solution G_2 associated with the biharmonic operator Δ^2 (with Dirichlet boundary conditions), namely

$$\delta G_2(x, y) = \int_{\partial\Omega} \frac{\partial^2 G_2(\cdot, x)}{\partial N^2} \frac{\partial^2 G_2(\cdot, y)}{\partial N^2} \delta N \, \sigma. \tag{2}$$

In fact the most part of [2] is devoted to a study of the biharmonic operator.

However the derivation given by Hadamard does not entirely meet up to present day standards of rigor.

In the book [1] by Bergman-Schiffer an entirely rigorous derivation of (1) and (0) based on integral equations is obtained in the case of 2 dimensions for the operator $-\Delta + c$, c a positive function. (In the case (0) the formula has to be modified somewhat: the terms involving S drop out, instead appears the term $cG_0(\cdot, x)\, G_0(\cdot, y)$.)

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In [5] the extension of (0) to general second order elliptic operators is given but the proof is not quite complete.

The purpose of the present note is to provide a rigorous, up-to-date proof of the analogous formula associated with a general energy integral of type

$$E(u, v) = \sum_{|\alpha| \leqslant m, |\beta| \leqslant m} a_{\alpha\beta} D_\alpha u D_\beta v.$$

Now Ω is a relatively compact domain of an n-dimensional manifold $\mathscr{M}$ and ω a chosen volume element on $\mathscr{M}$. We pick up a vector field N on $\mathscr{M}$ which is transversal on $\partial\Omega$. For a given function (or distribution) f on $\mathscr{M}$ we seek a function u vanishing up to order k on $\partial\Omega$, k being a fixed integer with $0 \leqslant k \leqslant m$, such that

$$E(u, v) = \int_\Omega fv\omega$$

for any test function v on $\mathscr{M}$ which too vanishes up to order k on $\partial\Omega$. (Thus there are $m + 1$ separate possibilities.) By integration by parts we see that this is the same as to impose

$$\begin{aligned} Au &= f \text{ on } \Omega \\ N^j u &= 0 \text{ on } \partial\Omega \ (0 \leqslant j < k) \\ B_j u &= 0 \text{ on } \partial\Omega \ (k \leqslant j < m) \end{aligned}$$

where A and B_j are certain partial differential operators on Ω and $\partial\Omega$ respectively, the latter also depending on the choice of N. We make the basic assumption that this is a correctly posed boundary value problem in suitable Sobolev spaces $H^s(\Omega)$. If G_k denotes the corresponding fundamental solution our generalization if Hadamard's formula reads:

$$\delta G_k(x, y) = \int_{\partial\Omega} \Big(-e(G_k(\cdot, y), G_k(x, \cdot)) + \frac{\partial^k}{\partial N^k} G_k(\cdot, y) B_{k-1}^+ G_k(x, \cdot) + B_{k-1} G_k(\cdot, y) \frac{\partial^k}{\partial N^k} G_k(x, \cdot)\Big) \delta N \sigma \tag{k}$$

where we have put

$$e(u, v) = \sum a_{\alpha\beta}(x) D_\alpha u D_\beta v$$

and where B_j^+ are the operators obtained in an analogous fashion as B_j when u and v are interchanged. We also briefly indicate a generalization to the non-linear case.

All necessary information on elliptic partial differential operators can be found in the book [4] by Lions-Magenes. (That we work on a general manifold and not in $\mathbf{R}^n$ is of course immaterial.)

Sections 1-2 contain preliminary material. The variational formula itself is established in Section 3. Section 4 is devoted to the non-linear case.

1. Boundary Value Problems Associated with Energy Integrals

Let $\mathscr{M}$ be a paracompact orientable n-dimensional C^∞ manifold. If $U \subset \mathscr{M}$ is a local coordinate neighborhood with corresponding local coordinates $x^1,..., x^n$ we set $D_\alpha = \partial/\partial x_{\alpha_1} \cdots \partial/\partial x_{\alpha_k}$ where $\alpha = \alpha_1 \cdots \alpha_k$ is a multi-index of degree $k = |\alpha|$. We pick up once for all a volume element (n-form) ω on $\mathscr{M}$, i.e. locally holds

$$\omega = \rho(x)\, dx^1 \cdots dx^n$$

where ρ is a positive C^∞ function defined in U.

By an *energy form* on $\mathscr{M}$ of degree m we mean a bilinear differential expression ("Differenzialausdruck") which locally is of the form

$$e(u, v) = \sum_{|\alpha| \leqslant m, |\beta| \leqslant m} a_{\alpha\beta}(x)\, D_\alpha u\, D_\beta v$$

where $a_{\alpha\beta}$ are certain C^∞ functions defined in the local coordinate neighborhood U in question. Given an energy form on Ω we can for each relatively compact domain $\Omega \subset \mathscr{M}$ with C^∞ boundary $\partial\Omega$ form the *energy integral*

$$E(u, v) = \int_\Omega e(u, v)\omega. \tag{1}$$

Let us choose a vector field N on $\mathscr{M}$ which is transversal on $\partial\Omega$. As surface area element on $\partial\Omega$ we can then use the $(n - 1)$-form $\sigma = N \lrcorner\, \omega$. Then holds the following Green's formula:

$$E(u, v) = \int_\Omega Auv\omega + \int_{\partial\Omega} \sum_{j=0}^{m-1} B_j u N^j v\sigma. \tag{2}$$

Here A is a partial differential operator on Ω of degree $2m$ and the B_j are partial differential operators on $\partial\Omega$ of degree $2m - 1 - j$ ($j = 0, 1,..., m - 1$). They are uniquely determined. The B_j depend on N byt A can in fact be defined on the whole of $\mathscr{M}$ independent of Ω, Namely holds the formula:

$$Au = \sum D_\beta{}^+ a_{\alpha\beta}\, D_\alpha u$$

where $D_\beta{}^+$ is the adjoint of D_β (with respect to ω):

$$D_\beta{}^+ = \rho^{-1}(-D)_\beta\rho.$$

To prove (1) we perform a partition of unity. We then see that it suffices to consider the case when the $a_{\alpha\beta}$ vanish outside a local coordinate neighborhood U. We can also take U so small that N can be taken of the form $N = (1,..., 0)$. But then everything reduces to a simple integration by parts.

We can now formulate the following boundary value problem for *every* Ω, and every integer k with $0 \leqslant k \leqslant m$: If $f \in H^{s-m} = H^{s-m}(\mathscr{M})$ (Sobolev space; see [4]) is given find $u \in H^s(\Omega)$ vanishing up to order k on $\partial\Omega$ such that

$$E(u, v) = \int_\Omega fv\omega \tag{3}$$

for every C^∞ test function v which vanishes up to order k on $\partial\Omega$. In view of our above Green's formula this is the same as to require

$$\begin{aligned} Au &= f \text{ on } \Omega \\ N^j u &= 0 \text{ on } \partial\Omega \ (0 \leqslant j < k) \\ B_j u &= 0 \text{ on } \partial\Omega \ (k \leqslant j < m - 1). \end{aligned} \tag{4}$$

If $k = m$ this is Dirichlet's problem for E and if $k = 0$ Neumann's problem. We shall assume that A is elliptic and that the operators N^j and B_j in (4) cover A in the usual sense (cf. [4]). Then it is known that for s sufficiently large holds the Fredholm alternative. For simplicity we also assume that we are in the case of uniqueness and existence so that we have an entirely correctly posed problem. (This in particular rules out Neumann's problem for $-\Delta$; see Introduction.)

Our problem is to investigate how the solution u varies if Ω is varied.

2. Derivatives

Consider generally an element $u \in H^s(\Omega)$ which depends on Ω ("a function of domain"). We can then write $u = u_\Omega$. We wish to define derivatives of u.

To this end consider more specifically a deformation Ω_t of a given domain Ω, i.e. a family of domains which depend smoothly on a parameter t, with $\Omega_0 = \Omega$. (Below we will make this vague notion more precise.) We then have a family of functions (distributions) $u_t = u_{\Omega_t} \in H^s(\Omega_t)$. For an *interior* point $x \in \Omega$ we now can set

$$\dot{u}(x) = \lim_{t \to 0} \frac{u_t(x) - u(x)}{t} \tag{1}$$

if the limit exists. This formula makes sense because for t sufficiently small we have $x \in \Omega_t$. However this presupposes the continuity of u_t and in praxis will require the restriction $s > n/2$ (by Sobolev's lemma!). A better definition is therefore

$$\dot{u}(v) = \lim_{t \to 0} \frac{u_t(v) - u(v)}{t} \tag{2}$$

where v is any C^∞ test function whose support is a compact subset $K \subset \Omega$ (and where we have written

$$u(v) = \int_\Omega u(x)\, v(x) \omega,$$

with the integral interpreted in the distribution sense, if necessary). Again $K \subset \Omega_t$ for t sufficiently small so (2) too is a meaningful formula. The drawback of both (1) and (2) is that we have to keep away from the boundary.

To obtain a more workable notion of derivative we proceed as follows. We adopt the general point of view that, heuristically speaking, the set of all Ω's is a kind of "manifold" on which operates a group, namely the group G of all diffeomorphisms of $\mathscr{M}$ onto itself. There is also a natural well-defined candidate for the "Lie-algebra" of G, namely the space $\mathfrak{g}$ of all C^∞ tangent vector fields over $\mathscr{M}$. Moreover we can use the elements of $\mathfrak{g}$ to define "tangent vectors" at Ω (regarded as an element of our hypothetical "manifold"). More precisely let ϕ_t be a "curve" in G, i.e. a C^∞ family of diffeomorphism of $\mathscr{M}$, with $\phi_0 = 1$. Then we can set $\Omega_t = \phi_t(\Omega)$ which may be considered as a "curve" in our manifold. Leaving heuristics aside *we have now an entirely well-defined notion of deformation.* Putting

$$X = \frac{d\phi_t}{dt}. \tag{3}$$

we get a vector field on $\mathscr{M}$ which realizes the "tangent vector" corresponding to this deformation. Turning again back to the heuristical picture we can now consider the assignment $\Omega \mapsto H^s(\Omega)$, for s fixed, as a "vector bundle" over our "manifold" on which G operates (on the left): If $\phi \in G$ is a diffeomorphism of $\mathscr{M}$ and $u \in H^s(\Omega)$ then the effect of the action of ϕ on u is the element $\phi^{-1*}u \in H^s(\phi(\Omega))$. Consider a "curve" u_t in the "vector bundle", whose projection to the base is precisely the above "curve" $\Omega_t = \phi_t(\Omega)$, i.e. $u_t \in H^s(\Omega_t)$ as above. Then for each t holds $\phi^* u_t \in H^s(\Omega)$ and we are free to put (with $u = u_0$)

$$\theta_X u = \frac{d}{dt} \phi_t^* u_t \bigg|_{t=0} \quad \text{(Lie or flow derivative)} \tag{4}$$

where the derivation is taken in the sense of the topology of the space $H^s(\Omega)$. As (3) previously, (4) *can be regarded as an entirely rigorous definition* for which the heuristics only served as a motivation.

It is easy to relate the two kinds of derivatives. Assume that $\theta_X u$ exists (and is thus an element of $H^s(\Omega)$) and that $u \in H^{s+1}(\Omega)$ (not only $u \in H^s(\Omega)$). Then $\dot{u}$ too exists and we have

$$\theta_X u = \dot{u} + Xu. \tag{5}$$

For the proof of (5) it suffices to write down the identity

$$\phi^* u - u = \phi_t^*(u_t - u) + (\phi_t^* - 1)u.$$

Again (5) implies that in the said assumption holds $\dot{u} \in H^s(\Omega)$.

3. The Variational Formula

We return to the set-up of Section 1.

Let us make a deformation $\Omega_t = \phi_t(\Omega)$ corresponding to the vector field X (see Section 2, (3)). Then we have for each t a boundary value problem in Ω_t: To find $u_t \in H^s(\Omega_t)$ vanishing up to order k on $\partial\Omega_t$ such that (compare Section 1, (3))

$$E_t(u_t, v) = \int_{\Omega_t} fv\omega$$

for all test functions v vanishing up to order k on $\partial\Omega_t$. Here we have put (compare Section 1, (1))

$$E_t(u, v) = \int_{\Omega_t} e(u, v)\omega.$$

We claim that this is again for t *sufficiently small* a correctly posed problem.

The simplest way to do this is to transform it to a problem on Ω. Let v be a test function for the unperturbed problem (on Ω), i.e. we have $N^j v = 0$ on $\partial\Omega$ for $0 \leqslant j < k$. Then $\phi^{-1*}v$ is for each t a test function for the perturbed problem (on Ω_t). Thus we obtain

$$\int_{\Omega_t} e(u, \phi^{-1*}v)\omega = \int_{\Omega_t} f\phi^{-1*}v\omega$$

or, making a change of variable in the integrals,

$$\int_{\Omega} e_t(\phi_t^* u_t, v)\, \phi_t^*\omega = \int_{\Omega} \phi_t^* f v \phi_t^* \omega \tag{1}$$

where we have defined e_t by

$$e_t(u, v) = \phi_t^*(e(\phi_t^{-1*}u, \phi_t^{-1*}v)).$$

It is clear that e_t is again an energy integral

$$e_t(u, v) = \sum_{|\alpha|\leqslant m, |\beta|\leqslant m} a_{\alpha\beta t}(x)\, D_\alpha u\, D_\beta v$$

with coefficients $\alpha_{\alpha\beta t}$ depending smoothly (C^∞) on t. Also $\phi_t^*\omega$ is a new volume element and we have $\phi_t^*\omega = \lambda_t\omega$ where λ_t too depends smoothly on t. It follows now from simple Functional Analysis arguments that we here have a correctly posed problem for determining $\phi_t^* u_t$. A little more generally we can consider the case when f is replaced by f_t. We now assume that $\theta_X f$ exists and belongs to $H^{s-m}(\Omega)$. (In particular this is so in the preceding special case $f_t = f$ with $f \in H^{s-m+1}$ in which case $\theta_X f = Xf$, by Section 2, (5).) Then it follows easily that $\theta_X u$ exists and belongs to $H^s(\Omega)$. By Section 2, (5) then $\dot{u}$ too exists.

There remains only to find a more explicit expressions for $\dot{u}$.

Formal differentiation of (1) (with f replaced by f_t) yields, using once more Section 2, (5) and defining $\theta_X e$ and $\theta_X\omega$ in the obvious way,

$$\int_\Omega (e(\dot{u} + Xu, v) + \theta_X e(u, v))\omega + e(u, v)\,\theta_X\omega$$

$$= \int_\Omega (\dot{f} + Xf)\omega + f\theta_X\omega. \tag{2}$$

We now need concrete expressions for $\theta_X e$ and $\theta_X\omega$.

First we notice that $\theta_X e$ is given by

$$\theta_X e(u, v) = X(e(u, v)) - e(Xu, v) - e(u, Xv). \tag{3}$$

To handle $\theta_X\omega$ it is convenient to introduce the divergence div X by the formula

$$\theta_X\omega = \operatorname{div} X\omega. \tag{4}$$

It clearly obeys the product rule

$$\operatorname{div}(fX) = f \operatorname{div} X + Xf \tag{5}$$

where f denotes a scalar function. We have the following integral formula.

LEMMA. *For any scalar function f holds*

$$\int_\Omega Xf\omega + \int_\Omega f \operatorname{div} X\omega = \int_{\partial\Omega} fX_N\sigma \tag{6}$$

where X_N is the "normal" component of X in the obvious decomposition of X into a sum of a tangential and a "normal" component ($X = X_N N + Y$, Y tangential).

Proof. We first recall E. Cartan's general formula:

$$\theta_X = d_{iX} + i_X d$$

where θ_X stands for the Lie derivative of forms and i_X for interior multiplication. Using this along with (4) and (5) we get

$$d(f(X \lrcorner\, \omega)) = Xf\omega + f \operatorname{div} X\omega.$$

Now $X \lrcorner\, \omega = X_N \lrcorner\, \omega = X_N\sigma$. Thus (6) follows upon application of Stokes' theorem.

If we use (3) and (6) in conjunction with (2) we see that many terms cancel: We get on one hand

$$\int_\Omega \theta_X e(u, v)\omega = \int_{\partial\Omega} e(u, v)\, X_N\sigma - \int_\Omega e(u, v) \operatorname{div} X\omega$$
$$- \int_\Omega e(Xu, v)\omega - \int_\Omega e(u, Xv)\omega,$$

on the other hand

$$\int_\Omega Xfv\omega = \int_{\partial\Omega} fvX_N\sigma - \int_\Omega fXv\omega - \int_\Omega \operatorname{div} Xfv\omega.$$

This is thus the outcome:

$$\int_\Omega e(\dot{u}, v)\omega + \int_{\partial\Omega} e(u, v)\, X_N\sigma$$
$$= \int_\Omega e(u, Xv)\omega - \int_\Omega fXv\omega + \int_\Omega \dot{f}v\omega + \int_{\partial\Omega} fvX_N\sigma. \tag{7}$$

If we also invoke our Green's formula, Section 1, (2), recalling that $Au = f$, we see that the two first terms of the right hand side of (7) give

$$\sum_{j=0}^{k-1} \int_{\partial\Omega} B_j u N^j Xv\sigma.$$

Next recall also that $N^j v = 0$ if $0 \leqslant j < k$ on $\partial\Omega$. Writing again $X = X_N N + Y$, Y tangential, it is quite easy to see that this implies

$$N^j Xv = \begin{cases} 0 & \text{if } \ 0 \leqslant j < k - 1 \\ X_N N^k v & \text{if } \ j = k - 1 \end{cases}$$

Thus we get in fact

$$\int_\Omega e(\dot{u}, v)\omega + \int_{\partial\Omega} e(u, v)\, X_N\sigma$$
$$= \int_{\partial\Omega} B_{k-1}uN^k v X_N\sigma + \int_\Omega \dot{f}v\omega + \int_\Omega fvX_N\sigma \tag{8}$$

which might be considered as a formula for recapturing $\dot{u}$. If v vanishes in a neighborhood of $\partial\Omega$ (8) simplifies to

$$\int_\Omega e(\dot{u}, v)\omega = \int_\Omega \dot{f}v\omega$$

which gives $A\dot{u} = \dot{f}$. (This latter result can of course be seen more directly too.) (8) does provide the additional boundary conditions needed for specifying $\dot{u}$.

The final result can be put in yet another form. Interchanging the rôles of u and v have the following dual form of our Green's formula (Section 1, (2))

$$E(u, v) = \int_\Omega uA^+v\omega + \int_{\partial\Omega} \sum_{j=0}^{m-1} N^j u B_j^+ v\sigma \tag{9}$$

where A^+ and B_j^+ are certain partial differential operators having analogous meaning as A and B_j. In particular A^+ is the formal adjoint of A. Let now v satisfy the dual boundary conditions to u, i.e. in addition to $N_j v = 0$ for $0 \leqslant j < k$ we likewise impose $B_j^+ v = 0$ for $k \leqslant j < m - 1$. Then (8) gives

$$\int_\Omega e(\dot{u}, v)\omega = \int_\Omega \dot{u}A^+v\omega + \int_{\partial\Omega} \sum_{j=0}^{k-1} N^j\dot{u}B_j^+v\sigma.$$

Recall now that $N^j u_t = 0$ for $0 \leqslant j < k$ on $\partial\Omega_t$. Equivalently $\phi_t^* N^j \phi_t^{-1*}\phi_t^* u_t = 0$ on $\partial\Omega$. Differentiation of the latter relation yields

$$[X, N^j]u + N^j\dot{u} = 0$$

which again leads to

$$N^j\dot{u} = \begin{cases} 0 & \text{for } j < k - 1 \\ -X_N N^k u & \text{for } j = k. \end{cases}$$

Thus we end up with the formula

$$\boxed{\begin{aligned} \int_\Omega \dot{u}A^+v\omega &= \int_\Omega \dot{f}v\omega + \int_{\partial\Omega} fvX_N\sigma \\ &\quad + \int_{\partial\Omega} (-e(u, v) + N^k u B_{k-1}^+ v + B_{k-1}uN^k v)\, X_N\sigma \end{aligned}} \tag{10}$$

which we regard as a generalization of Hadamard's formula. Namely if we apply (10) formally to the situation when v is the fundamental solution of the dual problem with singularity at $x \in \Omega$, i.e. $v = G_k{}^+(\cdot, x)$, we get

$$\dot{u}(x) = \int_{\partial\Omega} (-e(u, G_k{}^+(\cdot, x)) + N^k u B_{k-1}^+ G_k{}^+(\cdot, x) + B_{k-1} u N^k G_k{}^+(\cdot, x))\, X_N \sigma. \tag{10'}$$

Again taking $u = G_k(\cdot, y)$ with $y \in \Omega$, and noticing that $G_k{}^+(x, y) = G_k(y, x)$ (10′) yields

$$\dot{G}_k(x, y) = \int_{\partial\Omega} (-e(G_k(\cdot, y), G_k(\cdot, x)) + N^k G_k(\cdot, y)\, B_{k-1}^+ G_k(x, \cdot) + B_{k-1} G_k(\cdot, y)\, N^k G_k(x, \cdot))\, X_N \sigma. \tag{10''}$$

This is essentially Introduction, (k).

We consider in particular the two extremal cases, viz. $k = m$ and $k = 0$.

If $k = m$ (Dirichlet's problem) then the three terms in the surface integral of (10) are all of the form $aN^k u N^k v$. Thus (10) simplifies to

$$\int_\Omega \dot{u} A^+ v \omega = \cdots - \int_{\partial\Omega} a N^k u N^k v X_N \sigma$$

where the dots now stand for the f terms (unaltered!). To (10″) corresponds now

$$\dot{G}_m(x, y) = \int_{\partial\Omega} a N^k G_m(y, \cdot)\, N^k G_m(\cdot, x)\, X_N \sigma$$

which is indeed very similar in form to Introduction, (1) and (2).

If $k = 0$ (Neumann's problem) then, since we have to make the interpretation $B_{-1} = 0$, the last two terms in (10). Thus (10) takes the form

$$\int_\Omega \dot{u} A^+ v = \cdots - \int_{\partial\Omega} e(u, v)\, X_N \sigma$$

and similarly (10″) gives

$$\dot{G}_0(x, y) = - \int_{\partial\Omega} e(G_0(y, \cdot), G_0(\cdot, x))\, X_N \sigma$$

which should be compared to Introduction, (0).

A final remark on formula (10) is in order. Namely (10) contains explicitly the transversal vector field N which does not enter explicitely in the original formulation of the problem. It is however quite easy to see directly that thanks to the boundary conditions imposed on u *and* v all the *five* factors containing N—remember that B_j and B_j^+ too depend on N—really are independent of N. (In the same way as the Hessian of a function at a critical point is independent of the local coordinates!)

4. On the Non-Linear Case

Now we drop the basic assumption (see Section 1) that our energy form $e(u, v)$ is linear in u, i.e. we assume from now on that

$$e(u, v) = \sum_{|\beta| \leqslant m} a_\beta(x, D^m u) D_\beta v$$

where $D^m u$ stands for the collection of all derivatives $D_\alpha u$ of degree $\leqslant m$. The right hand side f can now be absorbed into E so our boundary problem is simply defined by the relation

$$E(u, v) = 0 \tag{1}$$

to hold for the same test functions v as in Section 1, (3). A and B_j are then also *non-linear* partial differential operators. Consider a deformation Ω_t of Ω. We then get the equation (cf. Sec. 3, (1))

$$\int_\Omega e_t(\phi_t^* u_t, v) \phi_t^* \omega = 0$$

with the analogous definition of e_t. Differentiation yields

$$\int_\Omega (e_u'(u + Xu, v) + \theta_X e(u, v))\omega + e(u, v) \theta_X \omega = 0$$

where we have introduced the *fiber derivative* e_u' of e at u,

$$e_u'(w, v) = \lim_{t \to 0} \frac{e(u + tw, v) - e(u, v)}{t}.$$

(Notice that e_u' is linear in *both* variables!) For the Lie derivative $\theta_X e$ of e holds now the following formula analogous to Sec. 3, (3):

$$\theta_X e(u, v) = X(e(u, v)) - e_u'(Xu, v) - e(u, Xv),$$

to which it reduces in the linear case. Using Section 3, Lemma we get corresponding to Section 3, (7)

$$\int_\Omega e_u'(\dot{u}, v)\omega + \int_{\partial\Omega} e(u, v) X_N \sigma = \int_\Omega e(u, Xv)\omega \tag{2}$$

(It is even a formally simpler expression than the latter precisely because there are no f terms present!) Finally introducing the fiber derivatives A'_u and B'_{ju} corresponding to A and B_j we get (compare Section 3, (10))

$$\int_\Omega \dot{u} A_u^{+\prime} v \omega = \int_{\partial\Omega} (-e(u, v) + N^k u B_{k-1,u}^{+\prime} v + B_{k+1} u N^k v) \, X_N \sigma. \tag{3}$$

Notice in particular that $\dot{u}$ satisfies the variational or Jacobi equation $A'_u \dot{u} = 0$. Let G_{ku}^+ be the corresponding fundamental solution of $A_u^{+\prime}$. Then (3) gives

$$\dot{u}(x) = \int_{\partial\Omega} (-e(u, G_{ku}^+(\cdot, x)) + N^k u B_{k-1}^+ G_{ku}^+(\cdot, x) + B_{k-1} u N^k G_{ku}^+(\cdot, x)) \, X_N \sigma \tag{3'}$$

which is as close as we can come to Hadamard's formula. (There is no fundamental solution in the non-linear case!) If $k = m$ (Dirichlet's problem) or $k = 0$ (Neumann's problem) we get

$$\dot{u}(x) = \int_{\partial X} a N^k u N^k G_{mu}^+(\cdot, x) \, X_N \sigma$$

respectively

$$\dot{u}(x) = -\int_{\partial\Omega} e(u, G_{0u}^+(\cdot, x)) \, X_N \sigma.$$

Note (*added Sept. 1979*). We recently noticed that the same problem, in a slightly different setting, already has been treated by Fujiwara and Ozawa (*Proc. Japan. Acad. Sci. Ser. A* **54** (1978), 215–220). See also the very interesting note by Ozawa (*Proc. Japan. Acad. Sci. Ser. A* **54** (1978), 303–340) dealing with the parabolic case.

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