Note on a Proof of the Extended Kirby–Paris Theorem on Labeled Finite Trees

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Buchholz [2] extended a certain game of unlabeled finite trees of Kirby–Paris [6] to the case of labeled finite trees whose nodes have labels from $\omega + 1 = \{0, 1, 2, \ldots, \omega\}$, and proved that this game stops in finite time. He used an infinitary notion of 'well-founded infinite trees' to prove this property on the finite-tree game. In this note, we avoid the use of any infinitary notion and reduce the infinitary technique to a finitary technique, by utilizing Takeuti's system of ordinal diagrams [7]. Also we generalize Buchholz's game by introducing higher ordinal numbers as the labels of the trees, and show the termination property of this generalized game.

Kirby and Paris [5] formulated a game of 'Hercules and a hydra' and showed that (1) Hercules always wins the game in a finite number of steps, but (2) that this fact cannot be proved in Peano arithmetic (PA). Buchholz [2] generalized the Kirby–Paris game and extended the independence result. This is, in the author's opinion, one of the best examples of the application of logic to finite combinatorics. For the case of the Kirby–Paris game, it is straightforward to show the fact that Hercules always wins in a finite number of steps, because one can easily correspond each hydra to a unique ordinal number less than $\varepsilon_0$ and show that at each step of the game (i.e. at each of Hercules' chops), the corresponding ordinal number decreases. Then well-orderedness of ordinal numbers up to $\varepsilon_0$ implies the fact that the game stops, i.e. Hercules chops off every head of the hydra in a finite time. However, the proof [2] of the extended version requires a infinitary method, i.e. every hydra is in correspondence to an infinite tree, and the well-foundedness of the infinite tree is used to show that the game stops in a finite number of steps.

In this note, we avoid the use of any infinitary objects in the proof, i.e. reduce the infinitary technique to a finitary technique by the direct correspondence of each hydra to an ordinal term. For this purpose we utilize Takeuti’s system of ordinal diagrams in [9].

1. Buchholz’s Extended Theorem on Hydra and the Theory of Ordinal Diagrams

We generalize slightly the notion of hydra in [2]. A hydra is a labeled finite forest $T$ whose roots have the label 0, and whose other nodes have labels from $\omega + 1 = \{0, 1, 2, \ldots, \omega\}$. If Hercules chops off a head (i.e. a top node) $\sigma$ of a given hydra $T$, the hydra will choose an arbitrary number $n \in \mathbb{N}$ and transfers itself into a new hydra $T(\sigma, n)$ as follows.

Let $\tau$ denote the node of $T$ which is immediately below $\sigma$, and let $T^-\tau$ denote that part of $T$ which remains after $\sigma$ has been chopped off. The definition of $T(\sigma, n)$ depends on the label of $\sigma$:

**Case 1.** label ($\sigma$) = 0:
If $\sigma$ is one of roots of $T$, we set $T(\sigma, n) = T^-\tau$ (Figure 1). Otherwise $T(\sigma, n)$ results from $T^-\tau$ by sprouting $n$ replicas of $T^-\tau$ from the node immediately below $\tau$. Here $T^-\tau$ denotes the subtree of $T^-\tau$ determined by $\tau$ (Figure 2).

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Case 2. label ($\sigma$) = $u + 1$:
Let $\varepsilon$ be the first node below $\sigma$ with label $v \leq u$. Let $S$ be that tree which results from the subtree $T_\varepsilon$ by changing the label of $\varepsilon$ to $v$ and the label of $\sigma$ to 0. $T(\sigma, n)$ is obtained from $T$ by replacing $\sigma$ by $S$. In this case, $T(\sigma, n)$ does not depend on $n$ (Figure 3).

Case 3. label ($\sigma$) = $\omega$:
Let $S$ be the component tree of forest $T$ in which $\sigma$ occurs. $S'$ is obtained from $S$ by changing the label of $\sigma$ (i.e. $\omega$) by $n + 1$. Then $T(\sigma, n)$ is obtained from $T$ by deleting $S$ and adding $n + 1$ replicas of tree $S'$ as new components of the forest (Figure 4).

**Notation.** If $\sigma$ is the rightmost head of $T$ (as in Figures 1–4) we write $T[n]$ instead of $T(\sigma, n)$.

**Theorem (Buchholz [2]).** By always chopping off the rightmost head, Hercules is able to kill every hydra in a finite number of steps, i.e. for each hydra $T$ and any sequence $(n_i)_{i \in \mathbb{N}}$ of natural numbers there exists $k \in \mathbb{N}$ such that $T[n_0][n_1] \ldots [n_k] = 0$. 
We identify each finite forest whose nodes have labels from set $I$, with an ordinal diagram in the system $0(I, \{0\})$ of [9], as follows:

1. If forest $T$ is composed of only one node (i.e. a root), and the label of the node is $i$, then $T$ is identified with $(i, 0)$.
2. If forest $T$ is composed of trees $\{T_1, T_2, \ldots, T_n\}$, and each $T_i$ is identified with ordinal diagram $\alpha_i$, then $T$ is identified with $\alpha_1 \neq \alpha_2 \neq \ldots \neq \alpha_n$.
3. If a labeled tree $T$ has the form $\prod X_i$, where $T_i$ is the subforest of $T$ and $i$ is the label of the root, and if $T_i$ is identified with $\alpha$, then $T$ is identified with $(i, \alpha)$.

E.g.

![Diagram](image)

is expressed by $(0, (5, (3, 0)) \neq (0, (4, 0))) \neq (4, 0))$. Then hydas, as special cases of labeled finite trees, are considered ordinal diagrams of system $0(\omega + 1, \{0\})$.

**Lemma.** For any hydra $\alpha \in 0(\omega + 1, \{0\})$, and for any $n \in \mathbb{N}$, $\alpha[n] < \alpha$ for all $i \in \omega + 1 \cup \{\infty\}$ in the sense of the system $0(\omega + 1, \{0\})$ of ordinal diagrams.

Then the well-orderedness of $\prec_0$ of $0(\omega + 1, \{0\})$ (cf. [8]) implies the above Theorem. Since the accessibility of $\alpha$ or the transfinite induction on $\alpha$ is provable in the system $(\Pi_1 - CA) + BI$ for each $\alpha \in 0(\omega + 1, \{0\})$, we have Theorem (cf. 2.3 of [3]). For each hydra $\alpha \in 0(\omega + 1, \{0\})$, $(\Pi_1 - CA) + BI + \forall(n) \in \mathbb{N} \exists k \ [n_0] \ldots [n_k] = 0$.

Next we generalize the Theorem by extending the notion of hydra. A hydra is defined to be a labeled forest $\in 0(\varepsilon_0, \{0\})$ whose roots have the label 0. For the definition of $\alpha(\sigma, n)$, Case 1 is defined in the same way. For the Case 2, instead of $u + 1$, we take $\xi + 1$ for any $\xi < \varepsilon_0$. We generalize Case 3 as follows.

**Case 3** label ($\sigma$) is a limit ordinal $\xi < \varepsilon_0$.

Let $S$ be the component tree of forest $T$ in which $\sigma$ occurs. $S'$ is obtained from $S$ by changing the label of $\sigma$ (i.e., $\xi$) by $\xi[n] + 1$. Here $\{\xi[n]\}_{n}$ is the fundamental sequence of $\xi$. If the normal form of $\xi$ is of the form $\beta + \omega^\omega + 1$ then $\xi[n] = \beta + \omega^\omega \cdot n$; if the normal form of $\xi$ is of the form $\beta + \omega^\omega$, where $\gamma$ is a limit ordinal, then $\alpha[n] = \beta + \omega^\omega$. Then $T(\sigma, n)$ is obtained from $T$ by deleting $S$ and adding $n + 1$ replicas of tree $S'$ as new components of the forest (Figure 5).

Since the accessibility of $\alpha \in 0(\varepsilon_0, \{0\})$ is provable in the system $A_2 - CA$ and the above Lemma can be generalized to the case of $0(\varepsilon_0, \{0\})$, we have

**Theorem.** For each hydra $\alpha \in 0(\varepsilon_0, \{0\})$, $A_2 - CA + \forall(n) \in \mathbb{N} \exists k \alpha[n] \ldots [n_k] = 0$.

Let $\xi$ be $\varepsilon_0$. Since for each $\alpha \in 0(\xi + 1, \{0\})$ accessibility of $\alpha$ is provable in the system $I_\xi$ (cf. [1]), we have,

**Theorem (cf. 2.2 of [2]).** For each hydra $\alpha \in 0(\xi + 1, \{0\})$,

$$I_\xi + \forall(n) \in \mathbb{N} \exists k \alpha[n] \ldots [n_k] = 0.$$
Since the accessibility of $|0(\xi + 1, \{0\})| = (\xi + 2, 0)$ is provable in $\text{ID}_{\xi + 1}$, we have

**Theorem.**

\[ \text{ID}_{\xi + 1} \vdash \forall \alpha \in 0(\xi + 1, \{0\}) \forall (n_i)_{i \in N} \exists k \alpha[n_0] \ldots [n_k] = 0. \]

**Remark.** On the other hand, the unprovability result of Buchholz [2] can be generalized as follows: The above statement is not provable in $\text{ID}_\xi$, and the above statement with $\varepsilon_0$ instead of $\xi + 1$ is not provable in $\Delta^1_2 - \text{AC}$. (The details of the proofs will appear elsewhere.)

2. **Proof of The Lemma**

We use the notation $\alpha \lessdot \beta$ for "\(\alpha < \beta\) for all \(i \in \omega + 1 \cup \{\infty\}\)", notation $\alpha \lessdot_i \beta$ for "\(\alpha < i, \beta\) for all \(i \in \omega + 1 \cup \{\infty\}\) such that \(j \leq i \in \omega + 1 \cup \{\infty\}\)". If $\beta$ expresses a left most and upper most occurrence of a subforest of $\alpha$, $\alpha$ is written as $\alpha(\beta)$. $\alpha(\gamma/\beta)$ is the result which is obtained from $\alpha(\beta)$ by replacing the indicated occurrence of $\beta$ with $\gamma$.

The proof of the Lemma is carried out according to cases of the definition of $\alpha[n]$.

**Case 1:** Assume $T_i$ is expressed by $(i, \beta \neq (0, 0))$, where $\beta$ is of the form $\beta_1 \neq \ldots \neq \beta_k$. Then $(i, \beta) \neq (i, \beta) \neq \ldots \neq (i, \beta) \lessdot (i, \beta \neq (0, 0))$. Then by induction on the complexity of $\alpha$, we can see $\alpha[n] = \alpha((i, \beta) \neq (i, \beta) \neq \ldots \neq (i, \beta)/\beta \neq (0, 0)) \lessdot ((i, \beta \neq (0, 0)))$.

**Case 2:** Assume that $T_i$ is expressed by $(j, \delta_{m+1} \neq (k_m, \delta_m \neq (k_{m-1}, \ldots, (k_1, \delta_1 \neq (i+1, 0)) \ldots)))$. We denote $(k_m, \delta_m \neq (k_{m-1}, \ldots, (k_1, \delta_1 \neq (i+1, 0)) \ldots))$ by $\mu$.

**Sublemma 1.** $(i, \mu((0, 0)/(i + 1, 0))) \lessdot_{i+1} (i + 1, 0)$.

We express $(i, \mu((0, 0)/(i + 1, 0)))$ as $\nu$.

**Sublemma 2.** $(k_m, \delta_m \neq (\ldots (k_1, \delta_1 \neq \nu) \ldots)) \lessdot_{i+1} (k_m, \delta_m \neq (\ldots (k_1, \delta_1 \neq (i+1, 0)) \ldots)))$.

**Proof of Sublemma 2.** By induction on $m$. For the case $m = 0$, it follows from Sublemma 1.

We consider the case $m = l + 1$. We express the ordinal diagram of the left hand side as $\chi$ and that of the right hand side as $\lambda$. Then by IH, follows $\chi < \lambda$. For any $k (i < k < \infty)$, for any $k$-section $\delta$ of $\chi$, by IH we can find a $k$-section $\delta'$ of $\lambda$ such that $\delta \lessdot_{i+1} \delta'$, in particular $\delta < \lambda$ (see [9] for the definition of $k$-section). Therefore $\delta < \lambda$. Hence $\chi \lessdot_{i+1} \lambda$.

**Sublemma 3.** $\mu((0, 0)/(i + 1, 0)) \lessdot \mu((i + 1, 0))$.

**Proof.** Obvious.
SUBLEMMA 4. $\chi \leq \lambda$.

From Sublemma 2, $\chi \leq \lambda$. From Sublemma 3, $\mu((0, 0)/(i + 1, 0)) < \mu((i + 1, 0))$. Therefore for every $i$-section $\sigma$ of $\chi$, $\sigma < \lambda$. Hence $\chi < \lambda$. On the other hand, for every $h(0 \leq h < i)$, for every $h$-section $\sigma$ of $\chi$, we can find the same $\sigma$ such that $\sigma$ is an $h$-section of $\lambda$. Hence we have $\chi < h \lambda$ for all $h < i$. Therefore $\chi < \lambda$.

From Sublemma 4, Lemma for Case 2 follows.

Case 3: Assume that the component tree $S$ is expressed by $\beta \equiv \beta((\omega, 0))$. Then $\beta((n + 1, 0)/(\omega, 0)) \ll \beta(\omega, 0))$. From this, follows $\alpha[\eta] \ll \alpha$. End of the Proof.

The corresponding Lemma for the case of the extended hydra $\alpha \in 0(e_0, \{0\})$ can be proved in the similar way. The details of this and other generalizations of the above game will appear elsewhere.

REFERENCES


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