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# Weighted Ostrowski type inequalities for functions with one point of nondifferentiability

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**Abstract.** We present a weighted generalization involving derivatives of arbitrary order of the recently obtained Ostrowski type inequality for functions with one point of nondifferentiability.

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## 1. Introduction

The well known **Ostrowski inequality** states:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leqslant \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\|f'\|_\infty. \tag{1.1}$$

It holds for every $x \in [a,b]$ whenever $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on $(a,b)$ with bounded derivative. Ostrowski proved it in 1938 in [15] and since

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then it has been generalized in a number of ways. Over the last decades some new inequalities of this type have been intensively considered and applied in Numerical analysis and Probability (see for instance [1,3,5–10,12] or monographs [2,11,13]).

M. Niezgoda in the recent paper [14] considered Ostrowski type inequalities for continuous functions with possibly one point of nondifferentiability. For $c_0 \in [a,b]$, let $\mathcal{D}(c_0)$ be the class of all continuous functions $f: [a,b] \to \mathbb{R}$ differentiable on the set $\langle a, c_0\rangle \cup \langle c_0, b\rangle$ and such that

$$M_l = \sup_{x\in\langle a,c_0\rangle} |f'(x)| < \infty \text{ and } M_r = \sup_{x\in\langle c_0,b\rangle} |f'(x)| < \infty.$$

In case $c_0 = a$ (resp. $c_0 = b$) we set $M_l = 0$ (resp. $M_r = 0$). M. Niezgoda in [14] established the following Ostrowski type inequality:

**Theorem 1.** *Let $f: I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a function differentiable in the interior $\mathring{I}$ of $I$, and let $[a,b] \subset \mathring{I}$. Suppose that $f' \in \mathcal{D}(c_0)$ for some $c_0 \in [a,b]$. Denote*

$$K_l = \sup_{x\in\langle a,\ c_0\rangle} |f''(x)| < \infty \text{ and } K_r = \sup_{x\in\langle c_0,\ b\rangle} |f''(x)| < \infty,$$

*where for $K_l = 0$ (resp. $K_r = 0$) if $c_0 = a$ (resp. $c_0 = b$). Then for $x \in [a, b]$ we have the following three inequalities*

$$\left|f(x) - \frac{1}{(b-a)}\int_a^b f(t)\ dt - \left(x - \frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a}\right|$$
$$\leqslant \begin{cases} \frac{b-a}{2(p+1)^{1/p}(q+1)^{1/q}}\left\{[K_l(c_0-a)]^p\frac{c_0-a}{b-a} + [K_r(b-c_0)]^p\frac{b-c_0}{b-a}\right\}^{1/p} & \text{if } 1<p<\infty, \\ \frac{1}{4}\left[K_l(c_0-a)^2 + K_r(b-c_0)^2\right] & \text{if } p=1, \\ \frac{1}{4}(b-a)\max\{K_l(c_0-a), K_r(b-c_0)\} & \text{if } p=\infty. \end{cases}$$

*where $1/p + 1/q = 1$.*

The aim of this paper is to give a weighted generalization of Theorem 1 involving derivatives of the function $f$ of arbitrary order. This will be done using the following extension of Montgomery identity via the Taylor's formula recently obtained in [4]:

let $f: I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geqslant 1$, $I \subset \mathbb{R}$ an open interval, $a,b \in I$, $a < b$. If $w:[a,b] \to [0,\infty\rangle$ is some normalized weighted function. Then the following identity holds

$$f(x) = \int_a^b w(t)f(t)dt - \sum_{i=0}^{n-2}\frac{f^{(i+1)}(x)}{(i+1)!}\int_a^b w(t)(t-x)^{i+1}ds$$
$$+ \int_a^b K_w^n(x,t)f^{(n)}(t)dt \tag{1.2}$$

where

$$K_w^n(x,t) = \begin{cases} \frac{1}{(n-1)!}\int_a^t w(u)(u-t)^{n-1}du, & a\leqslant t\leqslant x, \\ -\frac{1}{(n-1)!}\int_t^b w(u)(u-t)^{n-1}du, & x<t\leqslant b. \end{cases}$$

Since we assume $\sum_{i=0}^{-1} \cdot = 0$, for $n = 1$, (1.2) reduces to the **weighted Montgomery identity** obtained by J. Pečarić in [16] which states

$$f(x) = \int_a^b w(t)\, f(t)\, dt + \int_a^b P_w(x,t)\, f'(t) dt \tag{1.3}$$

where $P_w(x,t)$ is the weighted Peano kernel

$$P_w(x,t) = \begin{cases} \int_a^t w(u)du, & a \leqslant t \leqslant x, \\ \int_a^t w(u)du - 1, & x < t \leqslant b. \end{cases} \tag{1.4}$$

Here and hereafter the symbol $L^p_{[a,\ b]}$ $(p \geqslant 1)$ denotes the space of $p$-power integrable functions on the interval $[a,b]$ equipped with the norm

$$\|f\|_p = \left( \int_a^b |f(t)|^p dt \right)^{\frac{1}{p}}$$

and $L^\infty_{[a,b]}$denotes the space of essentially bounded functions on $[a,b]$ with the norm

$$\|f\|_\infty = ess \sup_{t \in [a,b]} |f(t)|.$$

## 2. Weighted generalization involving derivatives of the function f of arbitrary order

We denote

$$T^n_w(f;x) = f(x) - \int_a^b w(t) f(t) dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(x)}{(i+1)!} \int_a^b w(t)(t-x)^{i+1} dt.$$

In the next theorem we will obtain a bound for the $T^n_w(f;x)$ for functions $f$ such that $f^{(n-1)} \in \mathcal{D}(c_0)$.

**Theorem 2.** *Let $f: I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a function differentiable in the interior $\mathring{I}$ of $I$, and let $[a,b] \subset \mathring{I}$. Suppose that $f^{(n-1)} \in \mathcal{D}(c_0)$ for some $c_0 \in [a,b]$ and $w: [a,b] \to [0,\infty\rangle$ is a normalized weighted function. Denote*

$$M_l(n) = \sup_{x \in \langle a, c_0 \rangle} |f^{(n)}(x)| < \infty \text{ and } M_r(n) = \sup_{x \in \langle c_0, b \rangle} |f^{(n)}(x)| < \infty.$$

*Then for $x \in [a,b]$ we have*

$$|T^n_w(f;x)| \leqslant \begin{cases} \left\| K^n_w(x,\cdot) \right\|_q [M_l(n)^p (c_o - a) + M_r(n)^p (b - c_o)]^{1/p} \\ \left\| K^n_w(x,\cdot) \right\|_1 \max\{M_l(n), M_r(n)\} \end{cases} \tag{2.1}$$

*where $1 \leqslant p < \infty$ and $1/p + 1/q = 1$.*

**Proof.** Since $f^{(n-1)}$ is absolutely continuous, from (1.2) we have

$$T_w^n(f;x) = \int_a^b K_w^n(x,t) f^{(n)}(t)dt = \int_a^{c_0} K_w^n(x,t) f^{(n)}(t)dt + \int_{c_0}^b K_w^n(x,t) f^{(n)}(t)dt.$$

By taking the absolute value and applying triangle inequality and Hölder inequality we get

$$\begin{aligned} \left|T_w^n(f;x)\right| &\leqslant \left|\int_a^{c_0} K_w^n(x,t) f^{(n)}(t)dt\right| + \left|\int_{c_0}^b K_w^n(x,t) f^{(n)}(t)dt\right| \\ &\leqslant \left\|K_w^n(x,\cdot)\right\|_{q,[a,c_0]} \|f^{(n)}\|_{p,[a,c_0]} + \left\|K_w^n(x,\cdot)\right\|_{q,[c_0,b]} \|f^{(n)}\|_{p,[c_0,b]}. \end{aligned}$$

In case $1 \leqslant p < \infty$ we apply discrete Hölder inequality

$$\begin{aligned} &\left\|K_w^n(x,\cdot)\right\|_{q,[a,c_0]} \|f^{(n)}\|_{p,[a,c_0]} + \left\|K_w^n(x,\cdot)\right\|_{q,[c_0,b]} \|f^{(n)}\|_{p,[c_0,b]} \\ &\leqslant \left\|K_w^n(x,\cdot)\right\|_{q,[a,c_0]} M_l(n)(c_o-a)^{1/p} + \left\|K_w^n(x,\cdot)\right\|_{q,[c_0,b]} M_r(n)(b-c_o)^{1/p} \\ &\leqslant [M_l(n)^p(c_o-a) + M_r(n)^p(b-c_o)]^{1/p} \\ &\quad \cdot \left[\left(\left\|K_w^n(x,\cdot)\right\|_{q,[a,c_0]}\right)^q + \left(\left\|K_w^n(x,\cdot)\right\|_{q,[c_0,b]}\right)^q\right]^{1/q} \\ &= [M_l(n)^p(c_o-a) + M_r(n)^p(b-c_o)]^{1/p} \cdot \left\|K_w^n(x,\cdot)\right\|_q. \end{aligned}$$

Thus we have

$$\left|T_w^n(f;x)\right| \leqslant [M_l(n)^p(c_o-a) + M_r(n)^p(b-c_o)]^{1/p} \cdot \left\|K_w^n(x,\cdot)\right\|_q.$$

In case $p = \infty$ ($q = 1$) we have

$$\begin{aligned} &\left\|K_w^n(x,\cdot)\right\|_{1,[a,c_0]} \|f^{(n)}\|_{\infty,[a,c_0]} + \left\|K_w^n(x,\cdot)\right\|_{1,[c_0,b]} \|f^{(n)}\|_{\infty,[c_0,b]} \\ &\leqslant \left[\left\|K_w^n(x,\cdot)\right\|_{1,[a,c_0]} + \left\|K_w^n(x,\cdot)\right\|_{1,[c_0,b]}\right] \cdot \max\{M_l(n), M_r(n)\} \\ &= \left\|K_w^n(x,\cdot)\right\|_1 \max\{M_l(n), M_r(n)\} \end{aligned}$$

and

$$\left|T_w^n(f;x)\right| \leqslant \max\{M_l(n), M_r(n)\} \left\|K_w^n(x,\cdot)\right\|_1$$

which completes the proof. □

If we take a normalized weighted function $w(t) = \frac{1}{b-a}$, $t \in [a,b]$ in the last theorem we obtain the next corollary.

**Corollary 1.** *Suppose that all assumptions of Theorem* 2 *hold. Then for $x \in [a,b]$ we have*

$$|T^n(f;x)| \leqslant \begin{cases} \|K^n(x,\cdot)\|_q [M_l(n)^p(c_o-a) + M_r(n)^p(b-c_o)]^{1/p} \\ \|K^n(x,\cdot)\|_1 \max\{M_l(n), M_r(n)\} \end{cases} \tag{2.2}$$

*where $1 \leqslant p < \infty$ and $1/p + 1/q = 1$,*

$$T^n(f;x) = f(x) - \frac{1}{(b-a)} \int_a^b f(t)dt + \sum_{i=0}^{n-2} f^{(i+1)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!(b-a)}$$

*and*

$$K^n(x,t) = \begin{cases} \frac{-1}{n!(b-a)}(a-t)^n, & a \leqslant t \leqslant x, \\ \frac{-1}{n!(b-a)}(b-t)^n, & x < t \leqslant b. \end{cases}$$

**Corollary 2.** *Suppose that all assumptions of Theorem* 2 *hold. Then for* $x \in [a,b]$ *we have*

$$\left|f(x) - \frac{1}{(b-a)}\int_a^b f(t)dt\right| \leqslant \frac{1}{b-a}\left(\frac{(x-a)^{q+1} + (b-x)^{q+1}}{(q+1)}\right)^{\frac{1}{q}}[M_l(1)^p(c_o-a) + M_r(1)^p(b-c_o)]^{1/p},$$

*where* $1/p + 1/q = 1$*, while for* $p = 1$ *we have*

$$\left|f(x) - \frac{1}{(b-a)}\int_a^b f(t)dt\right| \leqslant \left(\frac{1}{2} + \frac{1}{b-a}\left|x - \frac{b+a}{2}\right|\right)[M_l(1)(c_o-a) + M_r(1)(b-c_o)],$$

*and for* $p = \infty$

$$\left|f(x) - \frac{1}{(b-a)}\int_a^b f(t)dt\right| \leqslant \frac{(x-a)^2 + (b-x)^2}{2(b-a)}\max\{M_l(1), M_r(1)\}.$$

**Proof.** We take a normalized weighted function $w(t) = \frac{1}{b-a}$, $t \in [a,b]$, and $n = 1$ in (2.2). Thus we have

$$T^1(f;x) = f(x) - \frac{1}{(b-a)}\int_a^b f(t)dt$$

$$K^1(x,t) = \begin{cases} \frac{1}{b-a}(t-a), & a \leqslant t \leqslant x, \\ \frac{1}{b-a}(t-b) & x < t \leqslant b. \end{cases}$$

and for $1 \leqslant q < \infty$

$$\|K^1(x,\cdot)\|_q = \frac{1}{b-a}\left(\int_a^x |t-a|^q dt + \int_x^b |b-t|^q dt\right)^{\frac{1}{q}} = \frac{1}{b-a}\left(\frac{(x-a)^{q+1} + (b-x)^{q+1}}{(q+1)}\right)^{\frac{1}{q}},$$

and

$$\|K^1(x,\cdot)\|_\infty = \frac{1}{b-a}\max\{x-a,b-x\} = \frac{1}{2}+\frac{1}{b-a}\left|x-\frac{b+a}{2}\right|.$$

Taking all this in (2.2) the proof follows. □

**Corollary 3.** *Suppose that all assumptions of Theorem 2 hold. Then for $x \in [a,b]$ we have*

$$\left|\frac{1}{(b-a)}\int_a^b f(t)dt - f(x) + f'(x)\frac{(b-x)^2-(a-x)^2}{2(b-a)}\right|$$
$$\leqslant \frac{1}{2(b-a)}\left(\frac{(x-a)^{2q+1}+(b-x)^{2q+1}}{(2q+1)}\right)^{\frac{1}{q}}[M_l(2)^p(c_o-a)+M_r(2)^p(b-c_o)]^{1/p},$$

*where $1/p + 1/q = 1$, while for $p = 1$ we have*

$$\left|\frac{1}{(b-a)}\int_a^b f(t)dt - f(x) + f'(x)\frac{(b-x)^2-(a-x)^2}{2(b-a)}\right|$$
$$\leqslant \frac{1}{2(b-a)}\max\left\{(x-a)^2,(b-x)^2\right\}[M_l(2)(c_o-a)+M_r(2)(b-c_o)]$$

*and for $p = \infty$*

$$\left|\frac{1}{(b-a)}\int_a^b f(t)dt - f(x) + f'(x)\frac{(b-x)^2-(a-x)^2}{2(b-a)}\right|$$
$$\leqslant \frac{(x-a)^3+(b-x)^3}{6(b-a)}\max\{M_l(2),M_r(2)\}.$$

**Proof.** We take a normalized weighted function $w(t) = \frac{1}{b-a}$, $t \in [a,b]$, and $n = 2$ in (2.2). Thus we have

$$T^2(f;x) = f(x) - \frac{1}{(b-a)}\int_a^b f(t)dt + f'(x)\frac{(b-x)^2-(a-x)^2}{2(b-a)},$$

$$K^2(x,t) = \begin{cases} \frac{-1}{2(b-a)}(a-t)^2, & a \leqslant t \leqslant x, \\ \frac{-1}{2(b-a)}(b-t)^2, & x < t \leqslant b. \end{cases}$$

and for $1 \leqslant q < \infty$

$$\|K^2(x,\cdot)\|_q = \frac{1}{2(b-a)}\left(\int_a^x |t-a|^{2q}dt + \int_x^b |b-t|^{2q}dt\right)^{\frac{1}{q}}$$
$$= \frac{1}{2(b-a)}\left(\frac{(x-a)^{2q+1}+(b-x)^{2q+1}}{(2q+1)}\right)^{\frac{1}{q}}$$

and

$$\|K^2(x,\cdot)\|_\infty = \frac{1}{2(b-a)} \max\left\{(x-a)^2, (b-x)^2\right\}.$$

Taking all this in (2.2) the proof follows. □

Our next goal is to obtain a bound for

$$T_w^n(f;x) - \frac{1}{b-a}\int_a^b K_w^n(x,t)dt \int_a^b f^{(n)}(t)dt$$

for functions $f$ such that $f^{(n)} \in \mathcal{D}(c_0)$. We will show that this is a weighted generalization of Theorem 1 involving derivatives of the function $f$ of arbitrary order.

For that purpose we shall need **Sonin's identity**(see [17])

$$\int_a^b w(t)f(t)g(t)dt - \left(\int_a^b w(t)f(t)dt\right)\left(\int_a^b w(t)g(t)dt\right)$$
$$= \int_a^b w(t)(g(t)-\lambda)\left(f(t) - \int_a^b w(t)f(t)dt\right)dt$$

where $f, g : [a,b] \to \mathbb{R}$ are two Lebesgue integrable functions, $w$:$[a,b] \to [0,\infty\rangle$ is some normalized weighted function and $\lambda$ is an arbitrary real number.

**Theorem 3.** *Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a function differentiable in the interior $\mathring{I}$ of $I$, and let $[a,b] \subset \mathring{I}$. Suppose that $f^{(n)} \in \mathcal{D}(c_0)$ for some $c_0 \in [a,b]$ and $w$:$[a,b] \to [0,\infty\rangle$ is a normalized weighted function. Then for $x \in [a,b]$ and $1 \leqslant p < \infty$ we have*

$$\left|T_w^n(f;x) - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}\int_a^b K_w^n(x,t)dt\right|$$
$$\leqslant \frac{1}{(p+1)^{1/p}}\left[M_l(n+1)^p(c_0-a)^{p+1} + M_r(n+1)^p(b-c_0)^{p+1}\right]^{1/p}\|K_w^n(x,\cdot)-\mu\|_q, \tag{2.3}$$

*where $1/p + 1/q = 1$, while for $p = \infty$ we have*

$$\left|T_w^n(f;x) - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}\int_a^b K_w^n(x,t)dt\right|$$
$$\leqslant \max\{M_l(n+1)(c_0-a), M_r(n+1)(b-c_0)\}\|K_w^n(x,\cdot)-\mu\|_1 \tag{2.4}$$

*where $\mu = \frac{1}{b-a}\int_a^b K_w^n(x,t)dt$.*

**Proof.** By using the identity (1.2) we have

$$T_w^n(f;x) - \frac{1}{b-a}\int_a^b K_w^n(x,t)dt \int_a^b f^{(n)}(t)dt$$
$$= \int_a^b K_w^n(x,t)f^{(n)}(t)dt - \frac{1}{b-a}\int_a^b K_w^n(x,t)dt\int_a^b f^{(n)}(t)dt$$

and further by applying the Sonin's identity for the functions $K_w^n(x,t)$ and $f^{(n)}(t)$ with $\lambda = f^{(n)}(c_0)$ and $w(t) = \frac{1}{b-a}$, $t \in [a,b]$, we obtain

$$\begin{aligned}
&\int_a^b K_w^n(x,t)f^{(n)}(t)dt - \frac{1}{b-a}\int_a^b K_w^n(x,t)dt\int_a^b f^{(n)}(t)dt \\
&= \int_a^b (f^{(n)}(t) - f^{(n)}(c_0))\left(K_w^n(x,t) - \frac{1}{b-a}\int_a^b K_w^n(x,t)dt\right)dt \\
&= \int_a^{c_0} (f^{(n)}(t) - f^{(n)}(c_0))\big(K_w^n(x,t) - \mu\big)dt \\
&\quad + \int_{c_0}^b (f^{(n)}(t) - f^{(n)}(c_0))\big(K_w^n(x,t) - \mu\big)dt.
\end{aligned}$$

It is obvious that $\int_a^b f^{(n)}(t)d = f^{(n-1)}(b) - f^{(n-1)}(a)$. By taking the absolute value and then applying the triangle inequality and Hölder's inequality we get

$$\begin{aligned}
&\left|T_w^n(f;x) - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}\int_a^b K_w^n(x,t)dt\right| \\
&\leqslant \left|\int_a^{c_0} (f^{(n)}(t) - f^{(n)}(c_0))\big(K_w^n(x,t) - \mu\big)dt\right| \\
&+ \left|\int_{c_0}^b (f^{(n)}(t) - f^{(n)}(c_0))\big(K_w^n(x,t) - \mu\big)dt\right| \\
&\leqslant \|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{p,[a,c_0]}\big\|K_w^n(x,\cdot) - \mu\big\|_{q,[a,c_0]} \\
&+ \|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{p,[c_0,b]}\big\|K_w^n(x,\cdot) - \mu\big\|_{q,[c_0,b]}.
\end{aligned}$$

In case $1 \leqslant p < \infty$ we apply discrete Hölder inequality

$$\begin{aligned}
&\left|T_w^n(f;x) - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}\int_a^b K_w^n(x,t)dt\right| \\
&\leqslant \left[\|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{p,[a,c_0]}^p + \|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{p,[c_0,b]}^p\right]^{\frac{1}{p}} \\
&\cdot \left[\big\|K_w^n(x,\cdot) - \mu\big\|_{q,[a,c_0]}^q + \big\|K_w^n(x,\cdot) - \mu\big\|_{q,[c_0,b]}^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Since the mean value theorem guarantees the existence of $\xi \in \langle t, c_0 \rangle$ such that $f^{(n)}(t) - f^{(n)}(c_0) = f^{(n+1)}(\xi)(t - c_0)$ we have

$$f^{(n)}(t) - f^{(n)}(c_0) = M_l(n+1)(t - c_0)$$

and thus

$$\big\|f^{(n)}(\cdot) - f^{(n)}(c_0)\big\|_{p,[a,c_0]}^p \leqslant M_l(n+1)^p \int_a^{c_0} (c_0 - t)^p dt = M_l(n+1)^p \frac{(c_0 - a)^{p+1}}{p+1}.$$

The same reasoning leads to

$$\|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{p,[c_0,b]}^p \leqslant M_r(n+1)^p \int_{c_0}^b (t - c_0)^p dt = M_r(n+1)^p \frac{(b - c_0)^{p+1}}{p+1}.$$

Finally for $1 < q < \infty$ we have

$$\left[\left\|K_w^n(x,\cdot)-\mu\right\|_{q,[a,c_0]}^q+\left\|K_w^n(x,\cdot)-\mu\right\|_{q,[c_0,b]}^q\right]^{\frac{1}{q}}=\left\|K_w^n(x,\cdot)-\mu\right\|_{q,[a,b]}$$

and

$$\begin{aligned}&\left\|K_w^n(x,\cdot)-\mu\right\|_{\infty,[a,c_0]}\leqslant\left\|K_w^n(x,\cdot)-\mu\right\|_{\infty,[a,b]}\\&\left\|K_w^n(x,\cdot)-\mu\right\|_{\infty,[c_0,b]}\leqslant\left\|K_w^n(x,\cdot)-\mu\right\|_{\infty,[a,b]},\end{aligned}$$

which implies the first inequality.

In case $p = \infty$

$$\begin{aligned}&\left|T_w^n(f;x)-\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\int_a^b K_w^n(x,t)dt\right|\\&\leqslant\|f^{(n)}(\cdot)-f^{(n)}(c_0)\|_{\infty,[a,c_0]}\left\|K_w^n(x,\cdot)-\mu\right\|_{1,[a,c_0]}\\&+\|f^{(n)}(\cdot)-f^{(n)}(c_0)\|_{\infty,[c_0,b]}\left\|K_w^n(x,\cdot)-\mu\right\|_{1,[c_0,b]}\\&\leqslant M_l(n+1)(c_0-a)\left\|K_w^n(x,\cdot)-\mu\right\|_{1,[a,c_0]}\\&+M_r(n+1)(b-c_0)\left\|K_w^n(x,\cdot)-\mu\right\|_{1,[c_0,b]}\\&\leqslant\max\{M_l(n+1)(c_0-a),M_r(n+1)(b-c_0)\}\left\|K_w^n(x,\cdot)-\mu\right\|_{1,[a,b]}\end{aligned}$$

and the second inequality is proved too. □

**Corollary 4.** *Let $f: I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a function differentiable in the interior $\mathring{I}$ of I, and let $[a,b] \subset \mathring{I}$. Suppose that $f' \in \mathcal{D}(c_0)$ for some $c_0 \in [a,b]$. Then if $x \in [a,b]$ and $1 < p < \infty$ we have*

$$\begin{aligned}&\left|f(x)-\frac{1}{(b-a)}\int_a^b f(t)dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right|\\&\leqslant\frac{(b-a)^{1/q}}{2(p+1)^{1/p}(q+1)^{1/q}}\left[M_l(2)^p(c_0-a)^{p+1}+M_r(2)^p(b-c_0)^{p+1}\right]^{1/p},\end{aligned}\tag{2.5}$$

*while for $p = 1$ we have*

$$\begin{aligned}&\left|f(x)-\frac{1}{(b-a)}\int_a^b f(t)dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right|\\&\leqslant\frac{1}{4}\left[M_l(2)(c_0-a)^2+M_r(2)(b-c_0)^2\right],\end{aligned}\tag{2.6}$$

*and for $p = \infty$ we have*

$$\begin{aligned}&\left|f(x)-\frac{1}{(b-a)}\int_a^b f(t)dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right|\\&\leqslant\frac{1}{4}(b-a)\max\{M_l(2)(c_0-a),M_r(2)(b-c_0)\}.\end{aligned}\tag{2.7}$$

**Proof.** If we take a normalized weighted function $w(t) = \frac{1}{b-a}$, $t \in [a,b]$ and $n = 1$ in (2.3) we have

$$\int_a^b K^1(x,t)dt = \left(\int_a^x \frac{t-a}{b-a}dt - \int_x^b \frac{b-t}{b-a}dt\right) = \frac{(x-a)^2 - (b-x)^2}{2(b-a)}$$
$$= \left(x - \frac{a+b}{2}\right),$$

$$\mu = \frac{1}{b-a}\int_a^b K^1(x,t)dt = \frac{1}{b-a}\left(x - \frac{a+b}{2}\right),$$

$$T^1(f;x) - \frac{f(b)-f(a)}{b-a}\int_a^b K^1(x,t)dt$$
$$= f(x) - \frac{1}{(b-a)}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)$$

and for $1 \leqslant q < \infty$

$$\|K^1(x,\cdot) - \mu\|_q = \left\|K^1(x,.) - \frac{1}{b-a}\left(x - \frac{a+b}{2}\right)\right\|_q$$
$$= \frac{1}{(b-a)}\left(\int_a^x \left|(t-a) - \left(x - \frac{a+b}{2}\right)\right|^q dt + \int_x^b \left|(t-b) - \left(x - \frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}.$$

First we observe $I_1 = \int_a^x \left|(t-a) - \left(x - \frac{a+b}{2}\right)\right|^q dt$. In case $x \in \left[a, \frac{a+b}{2}\right]$ we have

$$I_1 = \int_a^x \left(t - x + \frac{b-a}{2}\right)^q dt = \frac{\left(\frac{b-a}{2}\right)^{q+1} - \left(\frac{a+b}{2} - x\right)^{q+1}}{q+1},$$

while in case $x \in \left\langle \frac{a+b}{2}, b\right]$

$$I_1 = \int_a^{x-(b-a)/2} \left(-t + x - \frac{b-a}{2}\right)^q dt + \int_{x-(b-a)/2}^x \left(t - x + \frac{b-a}{2}\right)^q dt$$
$$= \frac{\left(x - \frac{a+b}{2}\right)^{q+1} + \left(\frac{b-a}{2}\right)^{q+1}}{q+1}.$$

Now, observing $I_2 = \int_x^b \left|(t-b) - \left(x - \frac{a+b}{2}\right)\right|^q dt$ in case $x \in \left[a, \frac{a+b}{2}\right]$ we have

$$I_2 = \int_x^{x+(b-a)/2} \left(-t + x + \frac{b-a}{2}\right)^q dt + \int_{x+(b-a)/2}^b \left(t - x - \frac{b-a}{2}\right)^q dt$$
$$= \frac{\left(\frac{b-a}{2}\right)^{q+1} + \left(\frac{a+b}{2} - x\right)^{q+1}}{q+1},$$

while in case $x \in \left\langle \frac{a+b}{2}, b\right]$

$$I_2 = \int_x^b \left(-t + x + \frac{b-a}{2}\right)^q dt = \frac{-\left(x - \frac{a+b}{2}\right)^{q+1} + \left(\frac{b-a}{2}\right)^{q+1}}{q+1}.$$

Finally, in both cases we have $I_1 + I_2 = \frac{2}{q+1}\left(\frac{b-a}{2}\right)^{q+1}$ and (2.5) is proved. Since all this is also valid for $q = 1$ ($p = \infty$) we have

$$\|K^1(x,\cdot) - \mu\|_1 = \frac{1}{4}(b-a)$$

and (2.4) implies (2.7). Last case for $q = \infty$ ($p = 1$)

$$\|K^1(x,\cdot) - \mu\|_\infty = \sup_{t\in[a,b]} |K^1(x,t) - \mu|$$

$$= \frac{1}{(b-a)} \max\left\{ \sup_{t\in[a,x]} \left|(t-a) - \left(x - \frac{a+b}{2}\right)\right|, \sup_{t\in[x,b]} \left|(t-b) - \left(x - \frac{a+b}{2}\right)\right| \right\}$$

$$= \frac{1}{(b-a)} \max\left\{ \left(x - \frac{a+b}{2}\right), \frac{b-a}{2} \right\} = \frac{1}{2}$$

follows from (2.5). □

**Remark 1.** Inequalities from the last Corollary are identical to those from the Theorem 1 since

$$\frac{(b-a)^{1/q}}{2(p+1)^{1/p}(q+1)^{1/q}} \left[M_l(2)^p(c_0-a)^{p+1} + M_r(2)^p(b-c_0)^{p+1}\right]^{1/p}$$

$$= \frac{(b-a)^{1-1/p}}{2(p+1)^{1/p}(q+1)^{1/q}} \left[M_l(2)^p(c_0-a)^{p+1} + M_r(2)^p(b-c_0)^{p+1}\right]^{1/p}$$

$$= \frac{b-a}{2(p+1)^{1/p}(q+1)^{1/q}} \left\{ [M_l(2)(c_0-a)]^p \frac{c_0-a}{b-a} + [M_r(2)(b-c_0)]^p \frac{b-c_0}{b-a} \right\}^{1/p}.$$

The last goal is to obtain a bound for

$$T_w^n(f;x) - f^{(n)}(c_0) \int_a^b K_w^n(x,t)dt$$

for functions $f$ such that $f^{(n)} \in \mathcal{D}(c_0)$.

**Theorem 4.** *Suppose that all assumptions of the Theorem 3 hold. Then for $x \in [a,b]$ we have*

$$\left| T_w^n(f;x) - f^{(n)}(c_0) \int_a^b K_w^n(x,t)dt \right|$$
$$\leqslant \begin{cases} \frac{1}{(p+1)^{1/p}} \left[M_l(n+1)^p(c_0-a)^{p+1} + M_r(n+1)^p(b-c_0)^{p+1}\right]^{1/p} \left\|K_w^n(x,\cdot)\right\|_q \\ \max\{M_l(n+1)(c_0-a), M_r(n+1)(b-c_0)\} \left\|K_w^n(x,\cdot)\right\|_1 \end{cases} \tag{2.8}$$

**Proof.** By applying the identity (1.2) we have

$$T_w^n(f;x) - f^{(n)}(c_0)\int_a^b K_w^n(x,t)dt = \int_a^b K_w^n(x,t)(f^{(n)}(t) - f^{(n)}(c_0))dt$$

and further by taking the absolute value, applying the triangle inequality and Hölder's inequality we get

$$\begin{aligned}
&\left|T_w^n(f;x) - f^{(n)}(c_0)\int_a^b K_w^n(x,t)dt\right| \\
&\leqslant \left|\int_a^{c_0} K_w^n(x,t)\big(f^{(n)}(t) - f^{(n)}(c_0)\big)dt\right| + \left|\int_{c_0}^b K_w^n(x,t)(f^{(n)}(t) - f^{(n)}(c_0))dt\right| \\
&\leqslant \|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{p,[a,c_0]}\left\|K_w^n(x,\cdot)\right\|_{q,[a,c_0]} \\
&\quad + \left\|f^{(n)}(\cdot) - f^{(n)}(c_0)\right\|_{p,[c_0,b]}\left\|K_w^n(x,\cdot)\right\|_{q,[c_0,b]}.
\end{aligned}$$

In case $1 \leqslant p < \infty$ we apply discrete Hölder's inequality

$$\begin{aligned}
&\left|T_w^n(f;x) - f^{(n)}(c_0)\int_a^b K_w^n(x,t)dt\right| \\
&\leqslant \left[\|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{p,[a,c_0]}^p + \left\|f^{(n)}(\cdot) - f^{(n)}(c_0)\right\|_{p,[c_0,b]}^p\right]^{\frac{1}{p}} \\
&\quad \cdot \left[\left\|K_w^n(x,\cdot)\right\|_{q,[a,c_0]}^q + \left\|K_w^n(x,\cdot)\right\|_{q,[c_0,b]}^q\right]^{\frac{1}{q}}.
\end{aligned}$$

The same reasoning as in the proof of the Theorem 3 leads to

$$\begin{aligned}
&\|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{p,[a,c_0]}^p + \|f^{(n)}(\cdot) - f^{(n)}(c_0)\|_{p,[c_0,b]}^p \\
&\leqslant M_l(n+1)^p\frac{(c_0-a)^{p+1}}{p+1} + M_r(n+1)^p\frac{(b-c_0)^{p+1}}{p+1},
\end{aligned}$$

which implies the first inequality. In case $p = \infty$

$$\begin{aligned}
&\left|T_w^n(f;x) - f^{(n)}(c_0)\int_a^b K_w^n(x,t)dt\right| \\
&\leqslant \left\|f^{(n)}(\cdot) - f^{(n)}(c_0)\right\|_{\infty,[a,c_0]}\left\|K_w^n(x,\cdot)\right\|_{1,[a,c_0]} \\
&\quad + \left\|f^{(n)}(\cdot) - f^{(n)}(c_0)\right\|_{\infty,[c_0,b]}\left\|K_w^n(x,\cdot)\right\|_{1,[c_0,b]} \\
&\leqslant M_l(n+1)(c_0-a)\left\|K_w^n(x,\cdot)\right\|_{1,[a,c_0]} \\
&\quad + M_r(n+1)(b-c_0)\left\|K_w^n(x,\cdot)\right\|_{1,[c_0,b]} \\
&\leqslant \max\{M_l(n+1)(c_0-a), M_r(n+1)(b-c_0)\}\left\|K_w^n(x,\cdot)\right\|_{1,[a,b]}
\end{aligned}$$

and the second inequality is proved too. □

**Corollary 5.** *Suppose that all assumptions of the Theorem* 4 *hold. Then if* $x \in [a,b]$ *and* $1 < p < \infty$ *we have*

$$\begin{aligned}&\left|f(x)-\frac{1}{(b-a)}\int_a^b f(t)dt-f'(c_0)\left(x-\frac{a+b}{2}\right)\right|\\&\leqslant\frac{\left((x-a)^{q+1}+(b-x)^{q+1}\right)^{1/q}}{(b-a)(p+1)^{1/p}(q+1)^{1/q}}\left[M_l(2)^p(c_0-a)^{p+1}+M_r(2)^p(b-c_0)^{p+1}\right]^{1/p},\end{aligned}\tag{2.9}$$

*while for p = 1 we have*

$$\begin{aligned}&\left|f(x)-\frac{1}{(b-a)}\int_a^b f(t)dt-f'(c_0)\left(x-\frac{a+b}{2}\right)\right|\\&\leqslant\left(\frac{1}{2}+\frac{1}{b-a}\left|\frac{b+a}{2}-x\right|\right)\left[M_l(2)(c_0-a)^2+M_r(2)(b-c_0)^2\right]\end{aligned}\tag{2.10}$$

*and for p = ∞ we have*

$$\begin{aligned}&\left|f(x)-\frac{1}{(b-a)}\int_a^b f(t)dt-f'(c_0)\left(x-\frac{a+b}{2}\right)\right|\\&\leqslant\frac{(x-a)^2+(b-x)^2}{2(b-a)}\max\{M_l(2)(c_0-a),M_r(2)(b-c_0)\}.\end{aligned}\tag{2.11}$$

**Proof.** If we take a normalized weighted function $w(t)=\frac{1}{b-a}$, $t\in[a,b]$ and $n=1$ in (2.8), as in the proof of the Corollary 4 we have

$$T^1(f;x)=f(x)-\frac{1}{(b-a)}\int_a^b f(t)dt,$$
$$\int_a^b K^1(x,t)dt=\left(x-\frac{a+b}{2}\right)$$

and

$$\begin{aligned}\|K^1(x,\cdot)\|_q&=\frac{1}{(b-a)}\left(\int_a^x|(t-a)|^q dt+\int_x^b|(t-b)|^q dt\right)^{\frac{1}{q}}\\&=\frac{1}{(b-a)}\left(\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right)^{\frac{1}{q}},\end{aligned}$$

which proves (2.9), while

$$\|K^1(x,\cdot)\|_1=\frac{(x-a)^2+(b-x)^2}{2(b-a)}$$

proves (2.11) and finally

$$\begin{aligned}\|K^1(x,\cdot)\|_\infty&=\max\left\{\sup_{t\in[a,x]}\left|\frac{t-a}{b-a}\right|,\sup_{t\in[x,b]}\left|\frac{t-b}{b-a}\right|\right\}\\&=\max\left\{\frac{x-a}{b-a},\frac{b-x}{b-a}\right\}=\left(\frac{1}{2}+\frac{1}{b-a}\left|\frac{b+a}{2}-x\right|\right)\end{aligned}$$

proves (2.10). □

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