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Control of Wave Motion on the Half-Line*

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1. INTRODUCTION

This paper is concerned with a problem on control theory of hyperbolic differential equations as proposed by Russell in a paper [1] where he solves a related problem. Russell considers the equation:

$$\frac{\partial^2 u}{\partial x^2} - r(x)u - \frac{\partial^2 u}{\partial t^2} = \gamma(x) f(t), \qquad 0 \leq x \leq l, \tag{1.1}$$

where γ belongs to $L^2[0, l]$, r is continuous on the interval $[0, l]$ and the function f is an admissible control on the interval $[0, T]$ where $T < \infty$, provided f is an element of $L^2[0, T]$.

He assumes that $u(x, t)$ obeys the boundary conditions

$$a_0 u(0, t) + b_0 \frac{\partial u}{\partial x}(0, t) = a_1 u(l, t) + b_1 \frac{\partial u}{\partial x}(l, t) \equiv 0, \tag{1.2}$$

where a_0, b_0, a_1, b_1 are constants with

$$0 \neq a_0^2 + b_0^2; \qquad a_1^2 + b_1^2 \neq 0.$$

The state space, I, consists of all pairs of functions $u_0(x)$, $v_0(x)$ with $d^2u_0(x)/dx^2$ and $dv_0(x)/dx$ in $L^2[0, l]$, the boundary condition corresponding to (1.2) is satisfies by $u(x)$, and the initial condition:

$$u(x, 0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x, 0) = v_0(x). \tag{1.3}$$

He defines a system as (1.1), (1.2) to be controllable in a fixed time $T > 0$ if, for each initial conditions in I, there exists an admissible control on $[0, T]$ such that the solutions of (1.1), (1.2), (1.3) further satisfy

$$u(x, T) = 0, \qquad \frac{\partial u}{\partial x}(x, T) = 0, \qquad x \in [0, l]. \tag{1.4}$$

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This control problem is reduced to a moment problem, and using results of the theory of nonharmonic series he arrives at the following facts:

(I) If $T < 2l$ the system (1.1), (1.2) is not controllable in time T.

(II) If $T > 2l$ the system (1.1), (1.2) is controllable in time T, and for each set of initial conditions in I the problem has infinitely many solutions $f \in L^2[0, T]$.

(III) If $T = 2l$ then

(i) When $b_0 = b_1 = 0$ the system (1.6) is controllable in time $T = 2l$ and for each set of initial conditions in I the solution set for the problem is of the form $f(t) + E$, where $f(t)$ is a certain specified solution of the problem and E is a fixed (for all initial conditions in I) one-dimensional subspace of $L_2[0, 2l]$.

(ii) When exactly one of the numbers b_0, b_1 is different from zero, the system (1.1) is controllable in time $T = 2l$ and for each initial condition in I the problem has a unique solution.

(iii) When neither of the numbers b_0, b_1 is equal to zero, the system (1.1) is not controllable in time $T = 2l$ but becomes controllable if we replace I by a certain $\hat{I} \subset I$ whose complement in I is one-dimensional.

At the end of his paper, Russell proposes the question: What happens for $T = \infty$? This motivates the problem which we consider here. The requirement $T = \infty$ must be thought of as the condition (1.4) replaced by:

$$\lim_{t\to\infty} u(x, t) = 0, \qquad \lim_{t\to\infty} \frac{\partial u}{\partial t}(x, t) = 0, \tag{1.5}$$

for every $x \in [0, l]$. The limit will be taken in the sense defined below (1.6).

If $l < \infty$ the answer is trivial. (For if we take some T, $2l < T < \infty$ and apply Russell's result II we find a control $f(t)$ for $t \in [0, T]$. Then extending $f(t)$ to be 0 for $t > T$ we answer the question affirmatively with the construction of a suitable control.)

Consequently we are going to consider the case $l = \infty$. Then we have to replace the conditions (1.2) by:

$$\begin{aligned} a_0 u(0, t) + b_0 \frac{\partial u}{\partial x}(0, t) &= 0, \qquad a_0^2 + b_0^2 \neq 0, \\ a_1 \lim_{x\to\infty} u(x, t) + b_1 \lim_{x\to\infty} \frac{\partial u}{\partial x}(x, t) &= 0, \qquad a_1^2 + b_1^2 \neq 0, \end{aligned} \tag{1.6}$$

and (1.5) by the L^2-limit as $t \to \infty$.

In order to solve this controllability problem, we again reduce it to a moment problem. But to solve it we cannot use nonharmonic series method because the interval is not finite in our case. Instead we use interpolation in the Hardy space H^2 of all analytic functions defined on the right half plane and such that $\int_{-\infty}^{\infty} | G(x + iy)| \, dy$ is bounded uniformly in $x > 0$.

This way we arrive at a condition on infinite products of the eigenvalues of an appropriate boundary value problem. Such a condition is satisfied for example if the eigenvalues are $\lambda_n = K(n + o(n^{S-1}))$; $n = 1, 2, \ldots$ with $1 < S < 2$, K a constant which is the case if $r(x) = x^p$ for $p > 2$. To prove the condition on the eigenvalues for this case we follow a method used by Fattorini in [2]. As a preliminary to our controllability theorem we prove in Section 2 the existence and uniqueness of a weak solution of the system (1.1), (1.3), (1.6) for $(x, t) \in \mathbb{R}_+^2$ under the conditions arising in the control problem.

2. Well-Posed Nature of the Control Problem

We shall consider the controllability problem with control input $\gamma(x) f(t)$ distributed on $0 \leqslant x < \infty$ for each $t \geqslant 0$ as specified by the real hyperbolic partial differential equation:

$$\frac{\partial^2 u}{\partial x^2} - r(x)u - \frac{\partial^2 u}{\partial t^2} = \gamma(x) f(t). \tag{2.1}$$

We assume:

(i) $r(x)$ is continuous on $[0, \infty)$; $\lim_{x\to\infty} r(x) = \infty$, and

(ii) $\gamma(x) \in L^2[0, \infty)$ and the controller $f(t) \in L^2 \cap L^1[0, \infty)$.

The boundary conditions are:

$$a_0 u(0, t) + b_0 \frac{\partial u}{\partial x}(0, t) = 0 = \lim_{x\to\infty} \left[a_1 u(x, t) + b_1 \frac{\partial u}{\partial x}(x, t)\right] \tag{2.2i}$$

for constants $a_0^2 + b_0^2 \neq 0$, $a_1^2 + b_1^2 \neq 0$.

The initial conditions are:

$$u(x, 0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x, 0) = v_0(x), \tag{2.2ii}$$

for real, continuous functions u_0, v_0 in $L^2[0, \infty)$ we seek for each such controller $f(t)$ and initial data $u_0(x)$, $v_0(x)$ a response or solution $u(x, t)$ in $0 \leqslant x < \infty$, $0 \leqslant t < \infty$ satisfying (2.1) in the open quadrant $\mathbb{R}_+^2$, and also the given conditions (2.2).

Since the coefficient $\gamma(x) f(t)$ may not be continuous, the solution $u(x, t)$ is only required in the weak sense that $u(x, t) \in L^2_{\text{loc}}(\mathbb{R}_+^2)$ and

$$\int_0^\infty \int_0^\infty u(x, t)\, [\phi_{xx} - r(x)\phi - \phi_{tt}]\, dx\, dt - \int_0^\infty v_0(x)\, \phi(x, 0)\, dx + \int_0^\infty u_0(x)\, \phi_t(x, 0)\, dx = \int_0^\infty \int_0^\infty \gamma(x) f(t) \phi\, dx\, dt, \tag{2.3}$$

where $\phi(x, t)$ is an arbitrary real function $C_0^\infty(\mathbb{R}^2)$ with compact support in $\mathbb{R}_+ \times \mathbb{R}$.

Let us first note that any classical solution $u(x, t) \in C^2(\mathbb{R}_+^2)$ must satisfy (2.3) (this can be easily proved using a standard integration by parts argument) and hence it is necessarily a weak solution.

In order to construct a weak solution we proceed by the method of Fourier series and seek first a product solution $u = e^{\lambda^{1/2}t}\varphi(x)$ of the homogeneous equation

$$\frac{\partial^2 u}{\partial x^2} - r(x)u - \frac{\partial^2 u}{\partial t^2} = 0.$$

This yields the eigenvalue problem:

$$\varphi''(x) - r(x)\,\varphi(x) + \lambda\varphi(x) = 0 \qquad \text{on} \quad 0 \leqslant x < \infty, \tag{2.4}$$

with boundary data

$$a_0\varphi(0) + b_0\varphi'(0) = 0 = \lim_{x\to\infty} [a_1\varphi(x) + b_1\varphi'(x)].$$

Under our assumption (2.1i) there exists an increasing sequence of eigenvalues $\{\lambda_k\}$ with corresponding eigenfunctions φ_k, $k = 1, 2,\ldots$, that form an orthonormal basis for $L^2[0, \infty)$. (See [3, p. 26, Theorem 2.7(ii)].) Also $\lim_{k\to\infty} \lambda_k = +\infty$.

Theorem 1. Consider the partial differential equation (2.1) with boundary and initial data (2.2), as above. In terms of the eigenvalues λ_k and orthonormal eigenfunctions $\varphi_k(x)$, $k = 1, 2,\ldots$, for (2.4) consider the Fourier expansions in $L^2[0, \infty)$

$$\gamma(x) = \sum_{k=1}^{\infty} \gamma_k\varphi_k(x),$$

$$u_0(x) = \sum_{k=1}^{\infty} \mu_k\varphi_k(x),$$

$$v_0(x) = \sum_{k=1}^{\infty} \nu_k\varphi_k(x).$$

Then there exists a unique weak solution $u(x, t)$ in the sense of (2.3).

Proof. We seek a weak solution $u(x, t)$ as a Fourier series

$$u(x, t) = \sum_{k=1}^{\infty} \beta_k(t)\,\varphi_k(x).$$

The partial differential equation (2.1) indicates that $\beta_k(t)$ should be defined as the solution of

$$\beta_k''(t) + \lambda_k\beta_k(t) = -\gamma_k f(t) \qquad \text{on} \quad 0 \leqslant t < \infty \tag{2.5}$$

with

$$\beta_k(0) = \mu_k , \qquad \beta_k'(0) = \nu_k , \tag{2.6}$$

which, for $\lambda_k > 0$, is:

$$\beta_k(t) = \sin(\lambda_k)^{1/2}t \left[\frac{\nu_k}{(\lambda_k)^{1/2}} - \frac{\gamma_k}{(\lambda_k)^{1/2}} \int_0^t f(s) \cos(\lambda_k)^{1/2}s \, ds\right]$$

$$+ \cos(\lambda_k)^{1/2}t \left[\mu_k + \frac{\gamma_k}{(\lambda_k)^{1/2}} \int_0^t f(s) \sin(\lambda_k)^{1/2}s \, ds\right]. \tag{2.7}$$

Then for all $t \geqslant 0$ and each $k \geqslant k_0 = \min\{k \mid \lambda_k > 0\}$.

$$|\beta_k(t)| \leqslant \frac{2\gamma_k}{(\lambda_k)^{1/2}} \|f\|_1 + \frac{|\nu_k|}{(\lambda_k)^{1/2}} + \mu_k \leqslant \frac{2\|f\|_1\gamma_k}{M} + \frac{|\nu_k|}{M} + \mu_k ,$$

where $M = \min_{k>k_0} \lambda_k$. Since $\beta_k(t)$ forms a sequence in l^2 we conclude that

$$u(x, t) = \sum \beta_k(t) \, \varphi_k(x)$$

belongs to $L^2_{1oc}(\mathbb{R}_+{}^2)$. It is easily seen that $u(x, t)$ satisfies (2.3). In order to prove that $u(x, t)$ is unique let $\Psi_N(x) \in C_0^\infty(R_+)$ with $|\Psi_N(x)| \leqslant 1$ for every x, $\Psi_N(x) = 1$ on $[0, N]$, and $\operatorname{supp} \Psi_N \subset [0, N + 1]$, let $\phi(t) \in C_0^\infty(-\infty, T]$.

For a function ζ let $\zeta_\epsilon = \zeta * g_\epsilon$, where $g_\epsilon(x) = (1/\epsilon) \exp(-x^2/\epsilon)$ denote the mollifier function.

Set $\eta_{\epsilon,k,N}(x, t) = \phi(t)[\varphi_k(x) \Psi_N(x)]_\epsilon$.

If $u(x, t)$ is a solution of (2.3)

$$\int_0^T \int_{\mathbb{R}^+} u(x, t) \frac{\partial^2 \eta_{\epsilon,k,N}}{\partial t^2} (x, t) \, dx \, dt$$

$$= \int_0^T \int_{\mathbb{R}^+} u(x, t) \frac{\partial^2 \eta_{\epsilon,k,N}}{\partial x^2} (x, t) \, dx \, dt - \int_0^T \int_{\mathbb{R}^+} u(x, t) \, r(x) \eta_{\epsilon,k,N} (x, t) \, dx \, dt$$

$$- \int_{\mathbb{R}^+} v_0(x) \, \eta_{\epsilon,k,N}(x, 0) \, dx + \int_{\mathbb{R}^+} u_0(x) \frac{\partial \eta_{\epsilon,k,N}}{\partial t} (x, 0) \, dx$$

$$+ \int_0^T \int_{\mathbb{R}^+} \gamma_k f(t) \eta_{\epsilon,k,N} (x, t) \, dx \, dt.$$

Therefore,

$$\lim_{N\to\infty} \lim_{\epsilon\to\infty} \int_0^T \int_{\mathbb{R}^+} u(x, t) \frac{\partial^2 \eta_{\epsilon,k,N}}{\partial t^2} (x, t) \, dx \, dt$$

$$= \int_0^T \int_{\mathbb{R}^+} u(x, t) \, \phi''(t) \, \varphi_k(x) \, dx \, dt = \int_0^T \alpha_k(t) \, \phi''(t) \, dt,$$

where $\alpha_k(t) = \int_{\mathbb{R}^+} u(x, t) \, \varphi_k(x) \, dx$.

On the other hand,

$$\lim_{N\to\infty}\lim_{\epsilon\to\infty}\int_0^T\int_{\mathbb{R}^+} u(x,t)\left[\frac{\partial^2\eta_{\epsilon,k,N}}{\partial x^2}(x,t) - r(x)\,\eta_{\epsilon,k,N}(x,t)\right] dx\,dt$$

$$= \lim_{N\to\infty}\lim_{\epsilon\to\infty}\int_0^T\int_{\mathbb{R}^+} u(x,t)\,\phi(t)\,[-\lambda_k\varphi_k(x)\,\Psi_N(x)]_\epsilon\,dx\,dt$$

$$= \int_0^T\int_{\mathbb{R}^+} u(x,t)\,\phi(t)\,[-\lambda_k\varphi_k(x)]\,dx\,dt.$$

Furthermore,

$$\lim_{N\to\infty}\lim_{\epsilon\to 0}\int_{\mathbb{R}^+} v_0(x)\,\eta_{\epsilon,k,N}(x,0)\,dx = \int_{\mathbb{R}^+} v_0(x)\,\phi(0)\,\varphi_k(x)\,dx = \phi(0)\nu_k,$$

$$\lim_{N\to\infty}\lim_{\epsilon\to 0}\int_{\mathbb{R}^+} u_0(x)\,\frac{\partial\eta_{\epsilon,k,N}}{\partial t}(x,0)\,dx = \mu_k\phi'(0),$$

$$\lim_{N\to\infty}\lim_{\epsilon\to 0}\int_0^T\int_{\mathbb{R}^+} \gamma_k f(t)\,\eta_{\epsilon,k,N}(x,t)\,dx\,dt = \int_0^T \gamma_k f(t)\,\phi(t)\,dt.$$

Then taking limits in (2.7) for $N\to\infty$; $\epsilon\to 0$ we have:

$$\int_0^T \alpha_k(t)\,\phi''(t)\,dt = -\int_0^T \phi(t)\,\lambda_k\alpha_k(t)\,dt + \phi(0)\,\nu_k - \mu_k\phi'(0)$$
$$+ \nu_k\phi(0) - \int_0^T f(t)\,\phi(t)\gamma_k\,dt. \tag{2.9}$$

Hence α_k is a solution in the weak sense of (2.4), (2.5). Also α_k is unique in such a class. If there should be another solution $\alpha_k^0(t)$ the function $\tilde{\alpha}_k(t) = \alpha_k(t) - \alpha_k^0(t)$ satisfies

$$\int_0^T \tilde{\alpha}_k(t)\,[\phi''(t) + \lambda_k\phi(t)]\,dt = 0. \tag{2.10}$$

On the other hand, for each ζ belonging to C_0^∞ we have a solution ϕ of $\phi''(t) + \lambda_k\phi(t) = \zeta(t)$ with ϕ in C_0^∞ then

$$\int_0^T \tilde{\alpha}_k(t)\,\zeta(t)\,dt = 0$$

for every ζ belonging to C_0^∞. Therefore $\tilde{\alpha}_k(t) = 0$. Since the problem also has a regular classical solution, the regularity of the weak solution α_k follows. Then $u(x,t) = \sum_k \alpha_k(t)\,\varphi_k(x)$ with $\alpha_k(t)$ satisfying (2.4) and (2.5).

Remarks. (1) Set $k_0 = \min\{k; \lambda_k > 0\}$. Since for $k > k_0$

$$|\beta_k(t)| \leqslant \frac{2\|f\|_1 |\gamma_k|}{M} + \frac{|\nu_k|}{M} + \frac{|\mu_k|}{M},$$

we have

$$\| u(x, t)\|_{2,x}^2 \leqslant 2 \frac{\|f\|_1^2 \|\gamma\|_2^2}{M^2} + \frac{\|v_0\|_2^2}{M} + \|u_0\|^2 + \sum_{k=1}^{k_0} |\beta_k(t)|^2.$$

Thus the mapping

$$T: \mathbb{R}^+ \to L^2(\mathbb{R}^+),$$
$$T: t \to u(0, t)$$

is an element of the space $L^\infty(\mathbb{R}_+, L^2(\mathbb{R}_+))$. The reader can easily verify that in fact T maps $\mathbb{R}_+$ into $C(\mathbb{R}^+, L^2(\mathbb{R}_+))$, since the β_k's are continuous.

(2) If $\{\mu_k(\lambda_k)^{1/2}\gamma_k\}$ is an element of l^2 as it is required in the control problem, it follows that

$$|\beta_k'(t)| \leqslant 2|\gamma_k| \|f\|_1 + |\nu_k| + (\lambda_k)^{1/2} |\mu_k|$$

for $k = k_0 + 1, k_0 + 2, \ldots$.

Also

$$\tilde{T}: \mathbb{R}_+ \to L^2(\mathbb{R}_+)$$
$$\tilde{T}: t \to u_t(\cdot, t)$$

is bounded and continuous.

3. Interpolation Problem and Main Result

From Theorem 1 we have that the solution of (1.1), (1.3), (1.6) can be written as:

$$u(x, t) = \sum_k \beta_k(t)\, \varphi_k(x).$$

Since

$$\| u(x, t)\|_{2,x} = \sum_0^\infty |\beta_k(t)|^2; \qquad \left\| \frac{\partial u}{\partial t}(x, t)\right\|_{2,x} = \sum |\beta_k'(t)|^2$$

in order to fulfill conditions (1.5) it must be

$$\lim_{t\to\infty} \beta_k(t) = 0, \qquad \lim_{t\to\infty} \beta_k'(t) = 0. \tag{3.1}$$

It follows from remarks (1) and (2) to Theorem 1 that they are quivalent.

Case Where All the Eigenvalues λ_k $k = 1, 2, \ldots$ Are Positive

In this case the solutions β_k of (2.4) are as on (2.6), thus conditions (3.1) for this case are fulfilled if the following conditions arise for $k = 1, 2, \ldots$.

$$\int_0^\infty f(s)\cos(\lambda_k)^{1/2}s\, ds = -\nu_k/\gamma_k = a_k\,,$$

$$\int_0^\infty f(s)\sin(\lambda_k)^{1/2}s\, ds = \mu_k(\lambda_k)^{1/2}\gamma_k = b_k\,.$$

To continue our analysis we must make use of interpolation on H^2 and therefore we need to impose the condition that the sequences $\{a_k\}$, $\{b_k\}$ be elements of l^2. Then the problem is reduced to: Find $f \in L^2[0, \infty)$ such that for a given sequence $\{C_k\} \in l^2$ it is true that:

$$C_k = \int_0^\infty f(s)e^{i(\lambda_k)^{1/2}s}\, ds, \tag{3.2}$$

where $C_k = a_k + ib_k$.

The assumption that $\{a_k\}$ and $\{b_k\}$ belong to l^2 is satisfied, for example, if $[u_0(x)\, r(x)]$, $[v_0(x)\, r(x)]$, and $(d^2v_0/dx^2)(x)$ belong to $L^2[0, \infty)$ and $\liminf |(\lambda_k)^{1/2}\gamma_k| > 0$ $\gamma_k \neq 0$, since in this case:

$$\begin{aligned}\mu_k &= \int_0^\infty u_0(x)\,\varphi_k(x)\, dx = -\frac{1}{\lambda_k}\int_0^\infty \left[\frac{d^2u_0}{dx^2}(x)\,\varphi_k(x) + r(x)\,u_0(x)\,\varphi_k(x)\right] dx \\ &= -\frac{1}{\lambda_k}\tilde{\mu}_k\end{aligned}$$

with $\tilde{\mu}_k \in l^2$.

Also in the same way $\nu_k = (1/\lambda_k)\,\tilde{\nu}_k$, $\tilde{\nu}_k \in l^2$.

We consider the moment problem (3.2). In order to solve it we will use facts on the Hardy space H^2 of the half-plane. We recall that because of the Paley–Wiener theorem a complex valued function G in the right half-plane belongs to the class H^2 if and only if G has the form:

$$G(w) = \int_0^\infty g(t)e^{-wt}\, dt$$

for some function $g \in L^2[0, \infty)$.

Thus if we can find a function $G \in H^2$ such that

$$G(1 - i(\lambda_k)^{1/2}) = C_k \tag{3.3}$$

the problem will be solved because in this case:

$$G(1 - i(\lambda_k)^{1/2}) = \int_0^\infty g(s)e^{-s}e^{i(\lambda_k)^{1/2}s}\, ds = C_k \tag{3.4}$$

and the solution to the control problem will be: $f(s) = g(s)\, e^{-s}$, consequently $f \in (L^2 \cap L^1)[0, \infty)$.

It is known (see [4, p. 202, Lemma 4]), that if the sequence $\{z_k\}$, $z_k \in D$ is an interpolating sequence (D is the open unit disk in the complex plane), i.e., it satisfies the condition:

$$\prod_{j \neq k} \frac{| z_k - z_j |}{| 1 - \bar{z}_k z_j |} \geq \delta > 0, \qquad k = 1, 2, 3, \ldots, \tag{3.5}$$

then for any square-summable sequence $\{\alpha_k\}$ there is a function g in $H^2(D)$ such that:

(i) $\| g \|_2^2 \leq \frac{2}{\delta^4} (1 - 2 \log \delta) \sum_k | \alpha_k |^2,$

(ii) $g(z_k)(1 - | z_k |^2)^{1/2} = \alpha_k$, $k = 1, 2, 3, \ldots$.

By a conformal mapping $z = (w - 1)/(w + 1)$ of the right plane in D, the sequence $\{w_k\}$

$$w_k = 1 - i(\lambda_k)^{1/2}, \qquad k = 1, 2, 3, \ldots, \tag{3.6}$$

is transformed in

$$z_k = \frac{-i(\lambda_k)^{1/2}}{2 - i(\lambda_k)^{1/2}}, \qquad k = 1, 2, 3, \ldots, \tag{3.7}$$

and for this sequence the condition (3.5) is:

$$\prod_{k \neq j} \frac{|(\lambda_k)^{1/2} - (\lambda_j)^{1/2}|}{(4 + ((\lambda_k)^{1/2} - (\lambda_j)^{1/2})^2)^{1/2}} \geq \delta > 0. \tag{3.8}$$

If the condition (3.8) is satisfied we denote α_k:

$$\alpha_k = \frac{C_k(1 - | z_k |)^{1/2}}{1 - z_k}$$

with the z_k from (3.7). It follows that $| \alpha_k | = | C_k |$ and then the sequence $\{\alpha_k\}$ is square-summable.

From the above result on interpolation for $\{z_k\}$ and $\{\alpha_k\}$ we have a function $g \in H^2(D)$ such that

$$g(z_k)(1 - | z_k |)^{1/2} = \alpha_k .$$

Let

$$h(z) = g(z)(1 - z).$$

Then the function G analytic on the right half plane $G(w) = h((w-1)/(w+1))$ belongs to H^2 of the right half-plane and

$$G(1 - i(\lambda_k)^{1/2}) = h(z_k) = C_k .$$

Case Where There Are Negative Eigenvalues

Since $\lambda_k \uparrow \infty$ there are only a finite number of negative eigenvalues. Let $k_0 = \max\{k; \lambda_k < 0\}$.

For $k = 1, 2,..., k_0$ the variation of parameters formula yields:

$$\beta_k(t) = \frac{e^{|\lambda_k|^{1/2}t}}{2}\left[\frac{\nu_k}{|\lambda_k|^{1/2}} + \mu_k - \frac{\nu_k}{\gamma_k}\int_0^t f(s)e^{-|\lambda_k|^{1/2}s}\,ds\right]$$

$$+ \frac{e^{-|\lambda_k|^{1/2}t}}{2}\left[\mu_k - \frac{\nu_k}{|\lambda_k|^{1/2}} + \frac{\gamma_k}{|\lambda_k|^{1/2}}\int_0^t f(s)e^{|\lambda_k|^{1/2}s}\,ds\right].$$

If f is a function such that:

$$\lim_{k\to\infty} e^{|\lambda_k|^{1/2}t}\left[\frac{\nu_k}{|\lambda_k|^{1/2}} + \mu_k - \frac{\gamma_k}{|\lambda_k|^{1/2}}\int_0^t f(s)e^{-|\lambda_k|^{1/2}s}\,ds\right] = 0, \qquad (3.9\text{i})$$

$$\lim_{k\to\infty} e^{-|\lambda_k|^{1/2}t}\left[\mu_k - \frac{\nu_k}{|\lambda_k|^{1/2}} + \frac{\gamma_k}{|\lambda_k|^{1/2}}\int_0^t f(s)e^{|\lambda_k|^{1/2}s}\,ds\right] = 0 \qquad (3.9\text{ii})$$

condition (3.1) will follow.

If the function G in addition to condition (3.4) satisfies the condition:

$$G(1 + |\lambda_k|^{1/2}) = \frac{\nu_k}{\gamma_k} + \frac{\mu_k}{\gamma_k}|\lambda_k|^{1/2} = a_k + b_k . \qquad (3.10)$$

That is:

$$\int_0^\infty g(s)e^{-s}e^{|\lambda_k|^{1/2}s}\,ds = \int_0^\infty f(s)e^{|\lambda_k|^{1/2}s}\,ds = \frac{\nu_k}{\gamma_k} + \frac{\mu_k}{\gamma_k}|\lambda_k|^{1/2} \qquad (3.10)$$

then conditions (3.9) are satisfied since:

$$\left| e^{|\lambda_k|^{1/2}t}\left[\frac{\nu_k}{|\lambda_k|^{1/2}} + \mu_k - \frac{\gamma_k}{|\lambda_k|^{1/2}}\int_0^t f(s)e^{-|\lambda_k|^{1/2}s}\,ds\right]\right|$$

$$= e^{|\lambda_k|^{1/2}t}\left|\frac{\gamma_k}{|\lambda_k|^{1/2}}\int_t^\infty f(s)e^{-|\lambda_k|^{1/2}s}\,ds\right| \leqslant \frac{|\gamma_k|}{|\lambda_k|^{1/2}}\int_t^\infty |f(s)|\,ds$$

but f belongs to $L^1[0, \infty)$ and therefore the last integral tends to zero when t tends to infinity.

For (3.9ii) let us denote by $I(t)$:

$$I(t) = e^{-|\lambda_k|^{1/2}t}\int_0^t f(s)e^{|\lambda_k|^{1/2}s}\,ds,$$

$$|I(t)| \leqslant \|f\|_1 \qquad \text{for every} \quad t \in [0, \infty).$$

Let t_0 be such that $\int_{t_0}^{\infty} |f(s)|\,ds < \epsilon/2$, and $T > t_0$ such that:

$$e^{-|\lambda_k|^{1/2}(T-t_0)} \|f\|_1 < \frac{\epsilon}{2}.$$

Therefore

$$|I(T)| = \left| e^{-|\lambda_k|^{1/2}(T-t_0)} I(t_0) + e^{-|\lambda_k|^{1/2}T} \int_{t_0}^{T} f(s) e^{|\lambda_k|^{1/2}s}\,ds \right|$$

$$\leqslant e^{-|\lambda_k|^{1/2}(T-t_0)} \|f\|_1 + \int_{t_0}^{T} |f(s)|\,ds \leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $\lim_{t\to\infty} \beta_k(t) = 0$, and since

$$\beta_k'(t) = \frac{|\lambda_k|^{1/2} e^{|\lambda_k|^{1/2}t}}{2} \left[\frac{\nu_k}{|\lambda_k|^{1/2}} + \mu_k - \frac{\nu_k}{|\lambda_k|^{1/2}} \int_0^t f(s) e^{-|\lambda_k|^{1/2}s}\,ds\right]$$

$$- \frac{|\lambda_k|^{1/2} e^{-|\lambda_k|^{1/2}t}}{2} \left[\mu_k - \frac{\nu_k}{|\lambda_k|^{1/2}} + \frac{\gamma_k}{|\lambda_k|^{1/2}} \int_0^t f(s) e^{|\lambda_k|^{1/2}s}\,ds\right]$$

it follows that $\lim_{t\to\infty} \beta_k'(t) = 0$.

Now we seek a condition on the sequence $\{\lambda_k\}$ (analogous to (3.8)) so that $\{z_k\}$ is an interpolating sequence suitable for the verification of (3.4) and (3.10).

By the conformal mapping $z = (w-1)/(w+1)$ the points $w_k = 1 + |\lambda_k|^{1/2}$, $k = 1, 2, \ldots, k_0$, are transformed in

$$z_k = \frac{|\lambda_k|^{1/2}}{2 + |\lambda_k|^{1/2}}.$$

Thus the condition (3.8) is replaced by the two conditions

$$\prod_{k\neq j} \left(\frac{|\lambda_j|^{1/2} - |\lambda_k|^{1/2}}{2 + |\lambda_k|^{1/2} + |\lambda_j|^{1/2}} \times \prod_{k=k_0+1}^{\infty} \frac{|\lambda_j|^{1/2} + \lambda_k(|\lambda_j|^{1/2} + 1)^2}{((2 + |\lambda_j|^{1/2})^2 + \lambda_k(1 + |\lambda_j|^{1/2})^2)^{1/2}}\right)$$

$$\geqslant \delta > 0, \qquad j = 1, 2, \ldots, k_0, \tag{3.11i}$$

and

$$\prod_{j=1}^{k_0} \frac{(|\lambda_j| + \lambda_k(|\lambda_j|^{1/2} + 1)^2)^{1/2}}{((2 + |\lambda_j|^{1/2})^2 + \lambda_k(1 + |\lambda_j|^{1/2})^2)^{1/2}} \times \prod_{\substack{j\neq k\\ j=k_0+1}}^{\infty} \frac{|(\lambda_k)^{1/2} - (\lambda_j)^{1/2}|}{(4 + ((\lambda_k)^{1/2} - (\lambda_j)^{1/2})^2)^{1/2}}$$

$$\geqslant \delta > 0, \; k = k_0 + 1, k_0 + 2, \ldots \tag{3.11ii}$$

Case Where $\lambda_{k_0} = 0$ is an Eigenvalue

The solution to (2.4), (2.5) is in this case:

$$\beta_{k_0}(t) = \mu_{k_0} + \nu_{k_0} t - \gamma_{k_0} \int_0^t (t-s) f(s)\,ds.$$

In order to verify (3.1) for $k = k_0$ we seek an f that in addition to conditions (3.4) and (3.10′) also satisfies:

$$\int_0^\infty sf(s)\,ds = -\mu_{k_0}/\gamma_{k_0}, \tag{3.12i}$$

$$\lim_{t\to\infty} t\left(\int_0^t f(s)\,ds - \nu_{k_0}/\gamma_{k_0}\right) = 0. \tag{3.12ii}$$

The conformal mapping $z = (w-1)/(w+1)$, maps $w_{k_0} = 1$ into $z_{k_0} = 0$. If the sequence $\{z_k\}$ (the transformal of $\{1 + i(\lambda_k)^{1/2}\}$, $k = 1, 2, \ldots, k_0, k_0 + 1 \ldots$) is an interpolating sequence we may require that the function G verify (3.4), (3.10) and

$$G(1) = \int_0^\infty g(s)e^{-s}\,ds = \int_0^\infty f(s)\,ds = \nu_{k_0}/\gamma_{k_0}.$$

Therefore, since

$$\left| t\int_t^\infty f(s)\,ds \right| = \left| t\int_t^\infty g(s)e^{-s}\,ds \right| \leqslant t\left(\int_t^\infty |g(s)|^2\,ds\right)(e^{-2t}/2).$$

Then

$$\lim_{t\to\infty} t\left(\int_0^t f(s)\,ds - \frac{\nu_{k_0}}{\gamma_{k_0}}\right) = \lim_{t\to\infty}\left(t\int_t^\infty f(s)\,ds\right) = 0$$

and (3.11ii) follows.

On the other hand, for the interpolating sequence $\{z_k\}$ we can find a function $Q \in H_2$ such that:

$$Q(w_k) = 0 \quad \text{for} \quad k \neq k_0, \qquad Q(w_{k_0}) = Q(1) = 1.$$

Set

$$H(w) = \left(-\frac{\mu_{k_0}}{\gamma_{k_0}} - G'(1)\right)(w-1)\frac{1}{(w+1)}.$$

Since $H \in H^\infty$ of the right half-plane, the function $Q \cdot H$ belongs to H^2.
Set F:

$$F(w) = G(w) + Q(w)\,H(w).$$

Clearly

$$F(w_k) = G(w_k), \qquad k = 1, 2, \ldots,$$

and

$$F'(1) = -\mu_{k_0}/\gamma_{k_0}.$$

F has a representation:

$$F(w) = \int_0^\infty g_1(s)e^{-ws}\, ds$$

with $g_1 \in L^2[0, \infty)$.

$$F(1) = \int_0^\infty g_1(s)e^{-s}\, ds = \nu_{k_0}/\gamma_{k_0},$$

$$F'(1) = -\int_0^\infty sg_1(s)e^{-s}\, ds = -\mu_{k_0}/\gamma_{k_0}.$$

Therefore for $f(s) = g_1(s)\, e^{-s}$ conditions (3.12) are verified, moreover conditions (3.4) and (3.10) are also verified.

In this case the conditions which turns out from (3.5) are:

$$\prod_{k=1}^{k_0-1} \frac{|\lambda_k|^{1/2}}{2+|\lambda_k|^{1/2}} \times \prod_{k=k_0+1}^{\infty} \frac{(\lambda_k)^{1/2}}{(4+\lambda_k)^{1/2}} \geqslant \delta > 0, \tag{3.13i}$$

$$\prod_{\substack{k\neq j\\ k\leqslant k_0-1}} \left(\frac{|\lambda_j|^{1/2} - |\lambda_k|^{1/2}}{2+|\lambda_j|^{1/2}+|\lambda_k|^{1/2}}\right) \times \left(\frac{|\lambda_j|^{1/2}}{2+|\lambda_j|^{1/2}}\right)$$
$$\times \prod_{k=k_0+1}^{\infty} \frac{(|\lambda_j|^{1/2} + \lambda_k(|\lambda_j|^{1/2}+1)^2)^{1/2}}{((2+|\lambda_j|^{1/2})^2 + \lambda_k(1+|\lambda_k|^{1/2})^2)^{1/2}} \geqslant \delta, \quad j = 1,\ldots, k_0 - 1, \tag{3.13ii}$$

$$\prod_{j=1}^{k_0-1} \left(\frac{|\lambda_j| + \lambda_k(|\lambda_j|^{1/2}+1)^2}{((2+|\lambda_j|^{1/2})^2 + \lambda_k(1+|\lambda_j|^{1/2})^2)^{1/2}}\right) \times \frac{(\lambda_k)^{1/2}}{(4+\lambda_k)^{1/2}}$$
$$\times \prod_{\substack{j\neq k\\ j=k_0+1}}^{\infty} \frac{|(\lambda_k)^{1/2} - (\lambda_j)^{1/2}|}{(4+((\lambda_k)^{1/2}-(\lambda_j)^{1/2})^2)^{1/2}} \geqslant \delta, \quad k = k_0+1, k_0+2,\ldots. \tag{3.13iii}$$

Hence collecting results we have proved the following theorem.

THEOREM 2. *If the conditions of Theorem* 1 *are satisfied, and*

(i) *The sequences* $\{\mu_k(\lambda_k)^{1/2}/\gamma_k\}$ *and* $\{\nu_k/\gamma_k\}$ *are elements of* l^2,

(ii) *On each of the following cases*:

(I) *If all the eigenvalues* λ_k, $k = 1, 2\ldots$ *are positive the condition* (3.8) *is satisfied*;

(II) *If some eigenvalues are negative the conditions* (3.11) *are satisfied*;

(III) *If some eigenvalues are negative, and also* $\lambda_{k_0} = 0$ *is an eigenvalue*; *the conditions* (3.13) *are satisfied.*

Then the system (1.1), (1.3), (1.6) is controllable for $T = \infty$. (Note: When $k_0 = 0$ conditions (3.13) become:

$$\prod_{k=1}^{\infty} \frac{(\lambda_k)^{1/2}}{(4+\lambda_k)^{1/2}} \geqslant \delta > 0,$$

$$\frac{(\lambda_j)^{1/2}}{(4+\lambda_j)^{1/2}} \times \prod_{\substack{k=1\\k\neq j}}^{\infty} \frac{|(\lambda_k)^{1/2} - (\lambda_j)^{1/2}|}{(4+((\lambda_k)^{1/2} - (\lambda_j)^{1/2})^2)^{1/2}} \geqslant \delta > 0, \qquad j = 1, 2, \ldots.)$$

Remark. When $r(x) = x^b$, $b > 2$ the corresponding eigenvalues λ_k verify $\lambda_k = M(k^s + o(k^{s-1}))$ $1 < s < 2$. In this case, if the condition (i) of the theorem is satisfied, the system (1.1), (1.3) and (1.6) is controllable for $T = \infty$, since:

LEMMA. *If* $\lambda_k = M[k^s + o(k^{s-1})]$, $1 < s < 2$, *the condition* (4.5) *is satisfied.*

Proof. If $\lambda_k = M(k^s + o(k^{s-1}))$ then $\lambda_k^{1/2} = M^{1/2}k^{s/2}(1 + o(1/k))$. When $k \to \infty$ we call $w_k = \frac{1}{2}\lambda_k^{1/2}$, then

$$\frac{((\lambda_k)^{1/2} - (\lambda_j)^{1/2})^2}{4 + ((\lambda_k)^{1/2} - (\lambda_j)^{1/2})^2} = 1 - \frac{1}{1 + (w_k - w_j)^2},$$

let P:

$$P = \prod_{j\neq k} \frac{((\lambda_j)^{1/2} - (\lambda_k)^{1/2})^2}{4 + ((\lambda_k)^{1/2} - (\lambda_j)^{1/2})^2} = \prod_{j\neq k} \left[1 - \frac{1}{1 + (w_k - w_j)^2}\right].$$

Set $N_n(u)$ the function defined by:

$$\begin{aligned} N_n(u) &= 0 && \text{if } 0 \leqslant u < w_1, \\ &= k && \text{if } w_k \leqslant u < w_{k+1}, \quad k \neq n, \\ &= n - 1 && \text{if } w_{n-1} \leqslant u < w_{n+1}. \end{aligned}$$

$$\begin{aligned} \log P &= \sum_{j\neq k} \log\left(1 - \frac{1}{1 + (w_k - w_j)^2}\right) \\ &= \int_{w}^{w_{n-1}} \log\left(1 - \frac{1}{1 + (w_n - u)^2}\right) dN_n(u) \\ &\quad + \int_{w_{n+1}}^{\infty} \log\left(1 - \frac{1}{1 + (w_n - u)^2}\right) dN_n(u). \end{aligned}$$

Integrating by parts and using the following facts

$$N_n(w_1) = 0,$$

$$\lim_{u\to\infty} N_n(u) \log\left(1 - \frac{1}{1 + (w_n - u)^2}\right) = 0,$$

$$N_n(w_{n-1}) = N_n(w_{n+1}) = n - 1$$

we have:

$$\log P = (n-1)\left[\log\left(1-\frac{1}{1+(w_n-w_{n-1})^2}\right)\right.$$

$$\left.-\log\left(1-\frac{1}{1+(w_n-w_{n-1})^2}\right)\right]$$

$$+\int_{w_1}^{w_{n-1}}\frac{2N_n(u)}{[1+(w_n-u)^2](w_n-u)}\,du$$

$$+\int_{w_{n+1}}^{\infty}\frac{2N_n(u)}{[1+(w_n-u)^2](w_n-u)}\,du$$

$$>\int_{w_1}^{w_{n-1}}\frac{2N_n(u)}{[1+(w_n-u)^2](w_n-u)}\,du$$

$$+\int_{w_{n+1}}^{\infty}\frac{2N_n(u)}{[1+(w_n-u)^2](w_n-u)}\,du.$$

The asymptotic relationship (4.1) implies:

$$-2+o(1)<N_n(u)-u^{2/5}<o(1).$$

Then:

$$\log P>\int_{w_1}^{w_{n-1}}\frac{2n^{2/s}}{[1+(w_n-u)^2](w_n-u)}\,du$$

$$+\int_{w_{n+1}}^{\infty}\frac{2u^{2/s}}{[1+(w_n-u)^2](w_n-u)}\,du$$

$$-\int_{w_1}^{w_{n-1}}\frac{4+o(1)}{[1+(w_n-u)^2](w_n-u)}\,du$$

$$-\int_{w_{n+1}}^{\infty}\frac{4+o(1)}{[1+(w_n-u)^2](w_n-u)}\,du=I_1+I_2-I_3-I_4\,.$$

$$\lim_{n\to\infty}\log\frac{1+(w_n-u)^2}{(w_n-u)^2}=0;\qquad \log\frac{(w_n-w_1)^2}{1+(w_n-w_1)^2}=o\left(\frac{1}{n}\right),$$

$$\log\frac{w_{n+1}-w_n}{w_n-w_{n-1}}=o(1),\qquad \log\frac{1+(w_n-w_{n-1})^2}{1+(w_n-w_{n+1})^2}=o(1);$$

we have $I_3+I_4=o(1)$.

Also:

$$I_1+I_2=-2\int_{w_n-w_{n-1}}^{w_n-w_1}\frac{(w_n+v)^{2/s}-(w_n-v)^{2/s}}{(1+v^2)v}\,dv$$

$$-2\int_{w_{n+1}-w_n}^{w_n-w_{n-1}}\frac{(w_n+v)^{2/s}}{(1+v^2)v}\,dv-\int_{w_n-w_1}^{\infty}\frac{(w_n+v)^{2/s}}{(1+v^2)v}\,dv$$

$$=-s_1-s_2-s_3;$$

from $(w_n + v)^{2/s} - (w_n - v)^{2/s} < w(v)^{2/s}$ it follows that:

$$S_1 < \pi \Big/ \left(\cos\left(\frac{2}{s} - 1\right) \cdot \frac{\pi}{2}\right) = K(s),$$

$$S_3 < \int_{w_2 - w_1}^{\infty} \frac{(2v + w_1)^{2/s}}{(1 + v^2)v}\, dv = M$$

and as $((w_n + v)^{1/s})/((1 + v^2)v)$ is an increasing function

$$S_2 < (n + 1)\, \frac{w_n - w_{n-1}}{w_{n+1} - w_n} - 1 = \frac{2 - s}{4} + o(1) = N + o(1).$$

Then

$$P > e^{-K-N-N+o(1)} \geqslant \delta > 0.$$

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