On Farthest Points of Sets

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A set $K$ in a normed linear space is said to be $M$-compact if any maximizing sequence in $K$ is compact. A sequence $\{g_n\}$ in $K$ is called maximizing if for some $x \in X$, $\{\|x - g_n\|\}$ converges to the farthest distance between $x$ and $K$. In this paper we study $M$-compact sets, relate the continuity behavior of the associated farthest-point map with the Gateaux differentiability of the farthest-distance function, and prove that in a normed space admitting centers any nonempty $M$-compact set having the unique farthest-point property must be a singleton.

1. Introduction

First we present some notation and definitions. Let $X$ be a real normed linear space and let $X^*$ be its dual. Let $K$ be a nonempty, bounded subset of $X$. Let $F_K: X \to \mathbb{R}$ be the farthest-distance function defined by $F_K(x) = \sup \{\|x - y\|: y \in K\}$. The set-valued map $q: X \to K$ defined by $q(x) = \{y \in K: \|x - y\| = F_K(x)\}$ is called the farthest-point map supported by $K$. Every element $y \in q(x)$ is called farthest point of $K$ from $x$ and the set of all farthest points of $K$ is written as $\text{far}(K)$. If $q(x)$ is nonempty (respectively singleton) for each $x \in X$, then $K$ is said to have the farthest-point property (respectively unique farthest-point property). A sequence $\{g_n\}$ in $K$ is said to be maximizing if for some $x \in X$, $\|x - g_n\| \to F_K(x)$. $K$ is said to be $M$-compact (or $\text{w}M$ compact, see [4]) if every maximizing sequence in $K$ is compact.

The notation $U(X)$, $S(X)$ will denote the unit ball $\{x \in X: \|x\| \leq 1\}$ and the unit sphere $\{x \in X: \|x\| = 1\}$ of the space $X$. The closed ball with center $x$ and radius $r$ will be denoted by $B[x, r]$. We say that the space $X$ is compactly locally uniformly rotund (CLUR) if for any $x \in S(X)$, $g_n \in U(X)$ with $\|x + g_n\| \to 2$, the sequence $\{g_n\}$ is compact. $X$ is ($\text{w}M$) if for any $x_0, x_n \in S(X)$ and $f_0 \in S(X^*)$ with $f_0(x_0) = 1$ and $\|x_n + x_n\| \to 2$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such
that \( f_0(x_n) \to 1 \). \( X \) is (I) if every closed and bounded convex set in \( X \) can be represented as the intersection of a family of closed balls.

**Remark.** In a previous paper [8] we called CLUR spaces "spaces with property (M)." Later we found that these spaces were extensively used by Vlasov, who called them compactly locally uniformly rotund [10]. See, for example, the survey paper by Vlasov [11].

The question of existence of farthest points in bounded closed sets has been investigated by Asplund [1] and Edelstein [5]. In the next section we strengthen some of their results in the setting of CLUR and (\( wM \)) spaces. In Section 3 we construct \( M \)-compact sets in CLUR spaces and study some of their properties. In Section 4, a formula for the subdifferential of \( F_K \), where \( K \) is an \( M \)-compact set, is obtained. The continuity behavior of the farthest-point map and the Gateaux differentiability of the farthest-distance function are considered in Section 5. In the last section we prove that in a normed linear space which admits centers any nonempty \( M \)-compact set having unique farthest-point property must be a singleton.

## 2. Existence of Farthest Points

It has been shown by Asplund [1] that in a reflexive, locally uniformly convex Banach space \( X \), the set of points which admit farthest points in a given closed bounded set \( K \) is dense in \( X \). There is also the result of Edelstein [5] that in a uniformly convex Banach space \( X \) with the property (I), the relation \( \overline{co}(K) = \overline{co}(\text{far}(K)) \) is satisfied for every closed bounded subset \( K \) of \( X \). We have the following extensions of these results.

**Theorem 2.1.** Let \( K \) be either (i) a bounded, closed subset of a reflexive, (CLUR) space \( X \), or (ii) a bounded, weakly sequentially closed subset of a reflexive, (\( wM \)) space \( X \). Then the set \( D \) of all points in \( X \) that admit farthest points in \( K \) is dense in \( X \). Moreover, \( X \sim D \) is of the first category in \( X \).

**Proof.** With some modifications the proof given by Asplund [1] for his result gives the present result.

**Theorem 2.2.** Let \( K \) and \( X \) be as in Theorem 2.1. In addition let \( X \) have the property (I). Then \( \overline{co}(K) = \overline{co}(\text{far}(K)) \).

**Proof.** Once we have the proof of Theorem 2.1 it is quite easy to construct a proof of this theorem along the lines of one given by Edelstein [5].

## 3. \( M \)-Compact Sets

Recall that a set is \( M \)-compact if every maximizing sequence in it is compact. Clearly every compact set is \( M \)-compact. We shall see below that there are
$M$-compact sets which are not even precompact. The set $K$ consisting of the open unit square together with its corners in the two-dimensional Euclidean space $\mathbb{R}^2$ is $M$-compact but is not closed.

**Proposition 3.1.** Every $M$-compact set has the farthest-point property.

**Proof.** This follows from the definition of $M$-compact sets.

**Proposition 3.2.** The closure of an $M$-compact set is $M$-compact.

**Proof.** Let $K$ be an $M$-compact set in a normed linear space $X$, and let $\{g_n\} \subset K$, $x \in X$, and $\|x - g_n\| \rightarrow F_K(x)$. Clearly $F_K(x) = F_K(x)$ and for each $n$, there exists $g_n' \in K$ such that $\|g_n - g_n'\| < 1/n$. Clearly, then, the sequence $\{g_n\}$ and $\{g_n'\}$ has the same set of cluster points and the result follows.

**Proposition 3.3.** The farthest-point map supported by an $M$-compact set is upper semicontinuous.

**Proof.** See Blatter [4].

The converse of this proposition is not true. Indeed the unit ball $U(X)$ of an infinite-dimensional (CLUR) space supports an upper semicontinuous farthest-point map but it is not $M$-compact.

The following property of the unit ball of a (CLUR) space will be needed in constructing a large class of $M$-compact sets in these spaces.

**Lemma 3.4.** Let $X$ be a (CLUR) space. Let $x$ be a nonzero element of $X$ and $\{g_n\} \subset U(X)$ be such that $\|x + g_n\| \rightarrow 1 + \|x\|$. Then the sequence $\{g_n\}$ is compact.

**Proof.** See Panda and Kapoor [8].

**Proposition 3.5.** Let $X$ be a reflexive (CLUR) Banach space, and let $K \subset X$ be a compact set which intersects the complement of the unit ball of $X$. Then both $U(X) \cup K$ and $\text{co}(U(X) \cup K)$ are $M$-compact.

**Proof.** This is easy to prove with the help of Lemma 3.4.

**Proposition 3.6.** Let $X$ be a (CLUR) normed linear space and let $K$ be an $M$-compact set consisting of more than one element. Then the set $K_a = K + aU(X)$, $a > 0$ is also $M$-compact.

**Proof.** Let $x \in X$ and $y \in K_a$. Then $y$ can be written in the form $y = u + av$, where $u \in K$ and $v \in U(X)$. It is easy to see that $F_{K_a}(x) = F_K(x) + a$. Let
\{g_n\} \subset K_a$ be a maximizing sequence for an $x \in X$. Expressing $g_n$ in the form $g_n = u_n + \alpha v_n$, we see that

$$F_K(x) + a = \lim_{n \to \infty} \| x - g_n \| \leq \lim \inf_{n \to \infty} \| x - u_n \| + a$$

$$\leq \lim \sup_{n \to \infty} \| x - u_n \| + a \leq F_K(x) + a.$$ 

This shows that $\| x - u_n \| \to F_K(x)$ and hence there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} \to u_0 \in K$. Therefore,

$$\lim_{i \to \infty} \frac{1}{a} (x - u_0) - v_{n_i} = \lim_{i \to \infty} \frac{1}{a} (x - u_{n_i}) - u_{n_i} = 1 + \frac{1}{a} \| x - u_0 \|$$

and, since $F_K(x) = \| x - u_0 \| \neq 0$, applying Lemma 3.4 we see that the sequence $\{v_{n_i}\}$ is compact in $U(X)$. Consequently, $\{g_n\}$ is compact in $K_a$ and the result follows.

4. THE SUBDIFFERENTIAL OF THE FUNCTION $F_K$

Let $f$ be a proper convex function on $X$. Then the subgradient of $f$ at $x$ is an $x^* \in X^*$ such that

$$f(y) \geq f(x) + (y - x, x^*) \quad \forall y \in X.$$ 

The set of all subgradient of $f$ at $x$ is denoted by $\partial f(x)$ and is called the subdifferential of $f$ at $x$.

In the following theorem a formula for the subdifferential of the function $F_K$, where $K$ is an M-compact set, at a point $x$ in $X$ is obtained. Such a formula for the subdifferential of a function $f$ which is equal to the supremum of a family of convex functions $f_\alpha$, $\alpha$ ranging over a compact set $\Omega$, has been given by Valadier [9]. A formula for $\partial F_K$, where $K$ is a compact set, has been observed by Holmes [6, p. 182].

**Theorem 4.1.** Let $X$ be a normed linear space, $K$ an M-compact set, and $x_0 \in X$. Let $f_\alpha(x) = \| x - y \|$. Then

$$\partial F_K(x_0) = \text{weak*}-\text{closed convex hull of} \{\partial f_\alpha(x_0) : y \in q(x_0)\}$$

$$= \text{weak*}-\text{closed convex hull of} \{\psi \in S(X^*) : \psi(x_0 - y) = F_K(x_0) \text{ for some } y \in K\}.$$ 

**Proof.** The first part of the proof is exactly the same as that in [6, p. 182].
We need only show that for any fixed $z \in X$, there is a point $y_0 \in q(x_0)$ such that
\[ F'_K(x_0; z) \leq f'_{y_0}(x_0; z), \]
where the prime denotes the right directional derivative.

Let $g_n \in q(x_0 + (1/n) z)$, then $\{g_n\}$ is a maximizing sequence in $K$ for $x_0$ and, by the $M$-compactness of $K$, contains a subsequence converging to some element $y_0$ of $q(x_0)$. Without any loss of generality, we may assume that $g_n \to y_0$. Then by the property of a convex function
\[ F'(x_0; z) \leq \frac{F_K(x_0 + (1/n) z) - F_K(x_0)}{1/n}, \]
for $n \geq 1$.\hfill (4.1)

If $\phi_n$ is a subgradient of the norm at $x_0 + (1/n) z - g_n$, then
\[ \|x_0 - g_n\| \geq \|x_0 + \frac{1}{n} z - g_n\| - \frac{1}{n} \phi_n(z), \]
where
\[ \phi_n = 1 \quad \text{and} \quad \phi_n(x_0 + \frac{1}{n} z - g_n) = \|x_0 + \frac{1}{n} z - g_n\|. \hfill (4.2) \]

Let $X_0$ be the linear span of $x_0 - y_0$ and $z$. Since $\phi_n(z) \leq \|z\|$, it can be assumed that $\phi_n(z)$ is convergent. Moreover,
\[ \lim_{n \to \infty} \phi_n(x_0 + (1/n) z - g_n) = \lim_{n \to \infty} \phi_n(x_0 - y_0) = \|x_0 - y_0\|. \]
Thus we can define a bounded linear functional $\phi_0$ on $X_0$ by the relation
\[ \phi_0(x) = \lim_{n \to \infty} \phi_n(x) \quad \text{for all} \quad x \in X_0. \]
It is clear that $\|\phi_0\|_{x_0} = 1$ and $\phi_0(x_0 - y_0) = \|x_0 - y_0\|$. Extend $\phi_0$, by the Hahn–Banach theorem, to the whole of $X$ with preservation of the norm. Clearly, then, $\phi_0$ is a subgradient of the norm at $x_0 - y_0$. Therefore,
\[ \|x_0 - y_0 + \frac{1}{n} z\| \geq \|x_0 - y_0\| + \frac{1}{n} \phi_0(z), \quad \text{for all} \quad n \geq 1. \hfill (4.3) \]

But from (4.1) and (4.2), we have after taking the limit
\[ F'_K(x_0; z) \leq \phi_0(z), \]
and so from (4.3)
\[ F''_K(x_0; z) \leq \lim_{n \to \infty} \frac{\|x_0 - y_0 + (1/n) z\| - \|x_0 - y_0\|}{1/n} = f'_{y_0}(x_0; z) \]
and the theorem is established.
5. CONTINUITY OF THE FARthest-POINt MAP

We shall now consider the continuity of the farthest-point map.

**Theorem 5.1.** Let $X$ be a reflexive Banach space satisfying the (CLUR) condition, and let $K$ be any nonempty, bounded, closed subset of $X$. Then there exists a subset $G$ dense in $X$ such that

(a) if $x \in G$, then every maximizing sequence in $K$ for $x$ is compact in $K$, and

(b) the farthest-point map $q$ restricted to $G$ is upper semicontinuous.

**Proof.** By Theorem 2.1, there exists a subset $D$ dense in $X$ such that every point $x \in D$ admits at least one farthest point in $K$. If $y \in q(x)$, then every point of the half-ray $\{\alpha x + (1 - \alpha) y : \alpha > 1\}$ admits $y$ as a farthest point in $K$. The set $G$ of the union of all half-rays of the form $\{\alpha x + (1 - \alpha) y : \alpha > 1\}$, $x \in D$, and $y \in q(x)$ is clearly dense in $X$. As $K \subset B[x, F_K(x)]$, by Lemma 3.4, it is clear that any maximizing sequence $\{x_n\} \subset K$ for an element $\alpha x + (1 - \alpha) y$, $\alpha > 1$ is compact in $B[x, F_K(x)]$. As $K$ is closed, the sequence $\{x_n\}$ is also compact in $K$ and this establishes (a). The proof of (b) follows from Proposition 3.3.

**Theorem 5.2.** Let $X$ be a locally uniformly convex Banach space, and let $K$ be a subset of $X$ having unique the farthest-point property. Then the farthest-point map supported by $K$ is continuous on a dense subset of $X$.

**Proof.** Clearly, the set $G = \bigcup_{x \in X} \{\alpha x + (1 - \alpha) q(x) : \alpha > 1\}$ is dense in $X$ and for all $\alpha \geq 1$, $q(\alpha x + (1 - \alpha) q(x)) = q(x)$. If $x_n \to \alpha x + (1 - \alpha) q(x)$, then $\{q(x_n)\}$ is a maximizing sequence for $\alpha x + (1 - \alpha) q(x)$ and, by the local uniform convexity of the norm (see Lemma 3.4), $q(x_n) \to q(x)$. This proves the result.

**Theorem 5.3.** Let $X$ be a smooth normed linear space, and let $K$ be a subset of $X$ having unique farthest-point property. Then at a point $x$ of continuity of the farthest-point map, the function $F_K$ is Gateaux differentiable and the derivative

$$F'_K(x; z) = G(x - q(x); z),$$

for all $z \in X$, where $G(x; y)$ is the Gateaux derivative of the norm at $x$ in the direction $y$.

**Proof.** The derivative $G(x; y)$ being, in particular, a subgradient of the norm, we get

$$\|x - q(x + tz)\| \geq \|x + tz - q(x + tz)\| + G(x + tz - q(x + tz); -tz);$$

that is,

$$F_K(x + tz) \leq F_K(x) + G(x + tz - q(x + tz); tz). \quad (5.1)$$
Similarly, by interchanging $x$ and $x + tz$, we get

$$F_K(x + tz) \geq F_K(x) + G(x - q(x); tz). \quad (5.2)$$

The $G$-derivative of the norm being norm-to-weak*-continuous [3, Cor. 3, p. 461], letting $t \to 0$, we obtain from (5.1) and (5.2) the required result.

If the space $X$ is not necessarily smooth, then the function $F_K$ is still differentiable in the direction of a certain vector $z$ (see [4]). This is shown in the following theorem.

**Theorem 5.4.** Let $X$ be a normed linear space, and let $K$ be a subset of $X$ having the unique farthest-point property. Then at a point $x$ of continuity of the farthest-point map

$$\lim_{t \to 0} \frac{F_K(x + t(x - q(x)) - F_K(x)}{t} = F_K(x).$$

**Proof.** This result can also be easily proved by following the proof of the preceding theorem.

**Theorem 5.5.** Let $X$ be a locally uniformly convex Banach space, and let $K \subset X$ have the unique farthest-point property. If $x \in X$ is a point of continuity of the farthest-point map $q: X \to K$, then every maximizing sequence in $K$ for $x$ is convergent.

**Proof.** Let \( \{g_n\} \subset K \) and \( \|x - g_n\| \to F_K(x) \). By the Hahn–Banach theorem, there exists a \( \psi_n \in S(X^*) \) such that \( \psi_n(x - g_n) = \|x - g_n\| \). Let \( t_n^2 = F_K(x) - \|x - g_n\| \) and \( t_n \leq 0 \). We can assume that \( t_n < 0 \), if necessary, by passing onto a subsequence. Then

$$F_K(x + t_n(x - q(x)) \geq \|x + t_n(x - q(x)) - g_n\|$$

$$\geq \|x - g_n\| + t_n\psi_n(x - q(x))$$

$$= F_K(x) - t_n^2 + t_n\psi_n(x - q(x));$$

that is,

$$\frac{F_K(x + t_n(x - q(x)) - F_K(x)}{t_n} \leq -t_n + \psi_n(x - q(x))$$

$$\leq -t_n + \|x - q(x)\|.$$

Applying the previous theorem, we get \( \lim_{n \to \infty} \psi_n(x - q(x)) = F_K(x) \). Set

$$z_n = (x - g_n)/\|x - g_n\| \quad \text{and} \quad z = (x - q(x))/F_K(x).$$
Then
\[ 2 = \lim_{n \to \infty} \psi_n(z_n + z) \leq \lim \inf_{n \to \infty} \| z_n + z \| \leq \lim \sup_{n \to \infty} \| z_n + z \| \leq 2, \]
and hence \( \| z_n + z \| \to 2 \). By the local uniform convexity of the norm, \( z_n \to z \) and consequently, \( g_n \to q(x) \). Thus the theorem is proved.

6. Uniqueness of Farthest Points

Suppose that \( K \) is a nonempty bounded subset of a normed linear space \( X \) such that every point \( x \) in \( X \) admits a farthest point in \( K \). A natural question which then arises is: When are the farthest points unique? In that case must \( K \) be a singleton? This problem has been considered by Asplund [2], Blatter [4], Klee [7], and others; but except when \( X \) is finite dimensional [2] no solution is known for general infinite-dimensional normed linear spaces. A partial answer has been provided by Blatter [4], using the axiom of choice. In this section we shall use the idea of a Chebyshev center and obtain a result for spaces admitting centers. In this case we shall not assume the completeness of \( X \).

A center (or Chebyshev center) of a bounded, nonempty set \( K \) in a normed linear space \( X \) is an element \( x_0 \) in \( X \) for which \( F_K(x_0) = \inf\{F_K(x): x \in X\} \). The number \( F_K(x_0) \) is called the Chebyshev radius of \( K \) and is denoted by \( r(K) \). Clearly, \( r(K) \) is the radius of the smallest ball in \( X \) (if one exists) which contains the set \( K \). The collection of the centers of all such balls is denoted by \( E(K) \). We shall say that a normed linear space \( X \) admits centers if for every bounded, nonempty set \( K \) of \( X \) the set \( E(K) \) is nonempty. It is known [6] that all conjugate Banach spaces, the space \( L^1(\mu) \) of absolutely integrable functions, and the space \( C_0(\Omega) \) of real-valued, bounded continuous functions, where \( \Omega \) is paracompact, admit centers.

**Theorem 6.1.** Let \( X \) be a normed linear space admitting centers, and let \( K \) be a nonempty subset of \( X \) having the unique farthest-point property. Suppose that for every \( x \) in \( K + r(K) \cup (X) \), the farthest-point map \( q: X \to K \) restricted to the line segment \([x, q(x)]\) is continuous at \( x \), then \( K \) must consist of a single point.

**Proof.** Suppose that \( K \) is not a singleton. We may assume that the origin \( 0 \in E(K) \). Then there exists an element \( \theta \in K \) in the interior of the ball \( B[0, r(K)] \). So \( \theta \in \text{int} B[x, r(K)] \subseteq K + r(K) \cup (X) \). Denote \( x_0 = q(\theta) \) and \( g_n = q((1/n) x_0) \). Suppose that \( \phi_n \) is a subgradient of the norm at \( g_n - (1/n) x_0 \). Then
\[ \| g_n \| \geq \| g_n - \frac{1}{n} x_0 \| + \frac{1}{n} \psi_n(x_0). \] (6.1)
However, $B[\theta, r(K)]$ being the minimal ball containing $K$, we obtain

$$0 < \left\| g_n - \frac{1}{n} x_0 \right\| - \left\| g_n \right\| \leq - \frac{1}{n} \phi_n(x_0);$$

that is,

$$\phi_n(x_0) < 0. \quad (6.2)$$

Since $g_n \to x_0$, we get

$$\lim_{n \to \infty} \phi_n(x_0) = \lim_{n \to \infty} \phi_n \left( g_n - \frac{1}{n} x_0 \right) = \lim_{n \to \infty} \left\| g_n - \frac{1}{n} x_0 \right\| = \left\| x_0 \right\|,$$

which contradicts (6.2). Thus the theorem is proved.

**Corollary 6.2.** Let $X$ be as above, and let $K \subset X$ be a nonempty $M$-compact set having the unique farthest-point property. Then $K$ must be a singleton.

**Proof.** Follows from Theorem 6.1 and Proposition 3.3.

*Note added in proof.* Some of the results in this paper extend similar results in the authors' paper with the same title which has appeared in *Rev. Roumaine Math. Pures Appl.* 21 (1976), 1369–1377.

**References**