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On some applications of the Briot–Bouquet differential subordination

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Abstract

Recently Srivastava et al. [J. Dziok, H.M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, Integral Transforms Spec. Funct. 14 (2003) 7–18; J. Dziok, H.M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, Appl. Math. Comput. 103 (1999) 1–13; Y.C. Kim, H.M. Srivastava, Fractional integral and other linear operators associated with the Gaussian hypergeometric function, Complex Var. Theory Appl. 34 (1997) 293–312] introduced and studied a class of analytic functions associated with the generalized hypergeometric function. In the present paper, by using the Briot–Bouquet differential subordination, new results in this class are obtained.

Keywords: Analytic functions; The generalized hypergeometric function; The Carlson–Shaffer operator; The Briot–Bouquet differential subordination

1. Introduction

Let $\mathcal{A}$ denote the class of functions which are *analytic* in $\mathcal{U} = \mathcal{U}(1)$, where

$$\mathcal{U}(r) = \{z\colon\ z \in \mathbf{C} \text{ and } |z| < r\}.$$

We denote by $\mathcal{A}_0$ the class of functions $f \in \mathcal{A}$ with the normalization $f(0) = f'(0) - 1 = 0$.

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We say that a function $f \in \mathcal{A}$ is *subordinate* to a function $F \in \mathcal{A}$ and write $f(z) \prec F(z)$, if and only if there exists a function $\omega \in \mathcal{A}$,

$$\omega(0) = 0, \qquad \left|\omega(z)\right| < 1 \quad (z \in \mathcal{U}),$$

such that

$$f(z) = F\big(\omega(z)\big) \quad (z \in \mathcal{U}).$$

Moreover, we say that f is subordinate to F in $\mathcal{U}(r)$, if $f(rz) \prec F(rz)$. We shall write

$$f(z) \prec_r F(z)$$

in this case. In particular, if F is univalent in $\mathcal{U}$, we have the following equivalence (cf. [10]):

$$f(z) \prec F(z) \quad \Longleftrightarrow \quad f(0) = F(0) \quad \text{and} \quad f(\mathcal{U}) \subset F(\mathcal{U}).$$

For analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

by $f * g$ we denote the *Hadamard product or convolution* of f and g, defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Let $q, s \in \mathbf{N} = \{1, 2, \ldots\}$, $q \leqslant s + 1$. For complex parameters $a_1, \ldots, a_q$ and $b_1, \ldots, b_s$ ($b_j \neq 0, -1, -2, \ldots;\ j = 1, \ldots, s$), we define the *generalized hypergeometric function* ${}_qF_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z)$ by

$${}_qF_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_q)_n}{(b_1)_n \cdots (b_s)_n} \frac{z^n}{n!} \quad (z \in \mathcal{U}),$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0), \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in \mathbf{N}). \end{cases}$$

Corresponding to a function $h(a_1, \ldots, a_q; b_1, \ldots, b_s; z)$ defined by

$$h(a_1, \ldots, a_q; b_1, \ldots, b_s; z) = z\, {}_qF_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z),$$

we consider a linear operator

$$H(a_1, \ldots, a_q; b_1, \ldots, b_s) : \mathcal{A}_0 \to \mathcal{A}_0,$$

defined by the convolution:

$$H(a_1, \ldots, a_q; b_1, \ldots, b_s) f(z) = h(a_1, \ldots, a_q; b_1, \ldots, b_s; z) * f(z).$$

In particular, for $s = 1$ and $q = 2$ and $a_2 = 1$, we have the Carlson–Shaffer operator

$$\mathcal{L}(a_1, b_1) f(z) = H_1(a_1, 1; b_1) f(z),$$

which was introduced by Carlson and Shaffer [1] (see also [8]).

After some calculations we obtain

$$aH(a + 1) f(z) = zH'(a) f(z) + (a - 1)H(a) f(z), \tag{1}$$

where, for convenience,

$$H(a_1)f(z) = H(a_1, \dots, a_q; b_1, \dots, b_s)f(z).$$

The linear operator $H(a_1, \dots, a_q; b_1, \dots, b_s)$ and some other linear operators and fractional calculus was investigated by many mathematicians (cf. [2,3,9,11,12]).

Now suppose that the parameters $a_1, \dots, a_q$ and $b_1, \dots, b_s$ are positive real numbers. Also let

$$0 \leqslant B \leqslant 1 \quad \text{and} \quad -B \leqslant A < B.$$

We denote by

$$V(a_1; A, B) = V(a_1, \dots, a_q; b_1, \dots, b_s; A, B)$$

the class of functions $f \in \mathcal{A}_0$ which satisfy the following condition:

$$a_1 \frac{H(a_1+1)f(z)}{H(a_1)f(z)} + 1 - a_1 \prec \frac{1+Az}{1+Bz}. \tag{2}$$

The class $V(a_1, \dots, a_q; b_1, \dots, b_s; A, B)$ for functions with negative coefficients was introduced and studied by Dziok and Srivastava [5] (see also [4,6]). The class $V(a, 1; c; 2\alpha - 1, 1)$ was investigated by Kim and Srivastava [8].

Let h and q be analytic functions in U with $h(0) = q(0) = 1$ and let h be univalent. The first-order differential subordination

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec h(z) \tag{3}$$

is called the Briot–Bouquet differential subordination. This particular differential subordination has a surprising number of important applications in the theory of analytic functions (for details see [10]).

In the paper we present one more application of the Briot–Bouquet differential subordination.

2. Main results

Eenigenburg et al. [7] proved, that for convex function h, with $\mathrm{Re}(\beta h(z) + \gamma) \geqslant 0$, the Briot–Bouquet differential subordination (3) implies $p(z) \prec h(z)$. Thus we have the following lemma.

Lemma 1. *If q is an analytic function in $\mathcal{U}(r)$, $q(0) = 1$ and*

$$q(z) + \frac{zq'(z)}{q(z) + \gamma} \prec_r \frac{1+Az}{1+Bz} \quad \left(\gamma + \frac{1+A}{1+B} \geqslant 0\right),$$

then

$$q(z) \prec_r \frac{1+Az}{1+Bz}.$$

Making use of the above lemma, we get the following theorem.

Theorem 1. *If $a \geqslant \frac{B-A}{1+B}$, then*

$$V(a+m; A, B) \subset V(a; A, B) \quad (m \in \mathbf{N}).$$

Proof. It is clear that it is sufficient to prove the theorem for $m = 1$. Let a function f belong to the class $V(a+1; A, B)$ or equivalently

$$(a+1)\frac{H(a+2)f(z)}{H(a+1)f(z)} - a \prec \frac{1+Az}{1+Bz}. \tag{4}$$

It is sufficient to verify condition (2). If we put

$$R = \sup\{r\colon\ H(a)f(z) \neq 0,\ z \in \mathcal{U}(r)\},$$

then the function

$$q(z) = a\frac{H(a+1)f(z)}{H(a)f(z)} + 1 - a \tag{5}$$

is analytic in $\mathcal{U}(R)$ and $q(0) = 1$. Taking the logarithmic derivative of (5) we get

$$\frac{z[H(a+1)f(z)]'}{H(a+1)f(z)} - \frac{z[H(a)f(z)]'}{H(a)f(z)} = \frac{zq'(z)}{q(z)+a-1} \quad \big(z \in \mathcal{U}(R)\big).$$

Applying (1) and (5) we obtain

$$(a+1)\frac{H(a+2)f(z)}{H(a+1)f(z)} - a = q(z) + \frac{zq'(z)}{q(z)+a-1} \quad \big(z \in \mathcal{U}(R)\big). \tag{6}$$

Thus by (4) we have

$$q(z) + \frac{zq'(z)}{q(z)+\gamma} \prec_R \frac{1+Az}{1+Bz}.$$

Lemma 1 now yields

$$q(z) \prec_R \frac{1+Az}{1+Bz}. \tag{7}$$

By (5) it suffices to verify that $R = 1$. From (7), (5) and (1) we conclude that $H(a)f(z)$ is starlike in $\mathcal{U}(R)$ and consequently it is univalent in $\mathcal{U}(R)$. Thus we see that $H(a)f(z)$ cannot vanish on $|z| = R$ if $R < 1$. Hence $R = 1$ and this proves Theorem 1. □

Using Lemma 1 we show the following sufficient conditions for functions to belong to the class $V(a; A, B)$.

Theorem 2. *Let $m \in \mathbf{N}$, $B - A \leqslant (1+B)a$, $2B^2a \leqslant (2B+1)(B-A)$. If a function $f \in \mathcal{A}_0$ satisfies the following inequality*:

$$\left|\frac{H(a+m+1)f(z)}{H(a+m)f(z)} - 1\right| < \frac{2B-A}{(a+m)(1+B)} + \frac{B-A-aB}{(a+m)(B-A+a-aB)} \quad (z \in \mathcal{U}), \tag{8}$$

then f belongs to the class $V(a; A, B)$.

Proof. By Theorem 1 it is sufficient to consider the case $m = 1$. Let a function f belong to the class $\mathcal{A}_0$. Putting

$$q(z) = \frac{1+Aw(z)}{1+Bw(z)} \quad \big(z \in \mathcal{U}(R)\big) \tag{9}$$

in (6), we obtain

$$(a+1)\frac{H(a+2)f(z)}{H(a+1)f(z)}-a=\frac{1+Aw(z)}{1+Bw(z)}+\frac{(aB+A-B)zw'(z)}{a+(aB+A-B)w(z)}-\frac{Bzw'(z)}{1+Bw(z)}.$$

Consequently, we have

$$F(z)=w(z)\left\{\frac{zw'(z)}{w(z)}\left(\frac{aB+A-B}{a+(aB+A-B)w(z)}-\frac{B}{1+Bw(z)}\right)-\frac{B-A}{1+Bw(z)}\right\}, \tag{10}$$

where

$$F(z)=(a+1)\frac{H(a+2)f(z)}{H(a+1)f(z)}-a-1.$$

By (2), (5) and (9) it is sufficient to verify that w is analytic in U and

$$\left|w(z)\right|<1 \quad (z\in\mathcal{U}).$$

Now, suppose that there exists a point $z_0\in\mathcal{U}(R)$, such that

$$\left|w(z_0)\right|=1, \quad \left|w(z)\right|<1 \quad \bigl(|z|<|z_0|\bigr).$$

Then, applying Lemma 1, we can write

$$z_0w'(z_0)=kw(z_0), \quad w(z_0)=e^{i\theta} \quad (k\geqslant 1).$$

Combining these with (10), we obtain

$$\begin{aligned}\left|F(z_0)\right|&=\left|k\left(\frac{B-A-aB}{a+(aB+A-B)e^{i\theta}}+\frac{B}{1+Be^{i\theta}}\right)+\frac{B-A}{1+Be^{i\theta}}\right|\\&\geqslant k\operatorname{Re}\left(\frac{B-A-aB}{a+(aB+A-B)e^{i\theta}}+\frac{B}{1+Be^{i\theta}}\right)+\frac{B-A}{1+B}\\&\geqslant k\left(\frac{B-A-aB}{a+B-A-aB}+\frac{B}{1+B}\right)+\frac{B-A}{1+B}\\&\geqslant\frac{2B-A}{1+B}+\frac{B-A-aB}{a+B-A-aB}.\end{aligned}$$

Since this result contradicts (8) we conclude that w is the analytic function in $\mathcal{U}(R)$ and $|w(z)|<1$ $(z\in\mathcal{U}(R))$. Applying the same methods as in the proof of Theorem 1 we obtain $R=1$, which completes the proof of Theorem 2. □

Putting $A=2\alpha-1$ and $B=1$ in Theorems 1 and 2 we obtain the following two corollaries.

Corollary 1. *Let* $0\leqslant\alpha<1$, $a\geqslant 1-\alpha$, $m\in\mathbf{N}$. *If a function* $f\in\mathcal{A}_0$ *satisfies the following inequality*:

$$\operatorname{Re}\left\{(a+m)\frac{H(a+m+1)f(z)}{H(a+m)f(z)}+1-a-m\right\}>\alpha \quad (z\in\mathcal{U}),$$

then

$$\operatorname{Re}\left\{a\frac{H(a+1)f(z)}{H(a)f(z)}+1-a\right\}>\alpha \quad (z\in\mathcal{U}).$$

Corollary 2. *Let $m \in \mathbf{N}$, $0 \leqslant \alpha < 1$, $1-\alpha \leqslant a \leqslant 3(1-\alpha)$. If a function $f \in \mathcal{A}_0$ satisfies the following inequality*:

$$\left|\frac{H(a+m+1)f(z)}{H(a+m)f(z)} - 1\right| < \frac{2(1-\alpha)^2 + 3(1-\alpha) - a}{2(a+m)(1-\alpha)} \quad (z \in \mathcal{U}),$$

then

$$\mathrm{Re}\left\{a\frac{H(a+1)f(z)}{H(a)f(z)} + 1 - a\right\} > \alpha \quad (z \in \mathcal{U}).$$

Putting $s=1$, $q=2$, $b_1=b$ and $a_2=1$, in Theorems 1 and 2 we obtain the following two corollaries.

Corollary 3. *If $a \geqslant \frac{B-A}{1+B}$ and*

$$(a+m)\frac{\mathcal{L}(a+m+1,b)f(z)}{\mathcal{L}(a+m,b)f(z)} + 1 - a - m \prec \frac{1+Az}{1+Bz},$$

then

$$a\frac{\mathcal{L}(a+1,b)f(z)}{\mathcal{L}(a,b)f(z)} + 1 - a \prec \frac{1+Az}{1+Bz}.$$

Remark 1. Putting $m=a=B=1$ and $A=2\beta-1$ in Corollary 3 we have the result of Kim and Srivastava [8], obtained by using another methods.

Corollary 4. *Let $m \in \mathbf{N}$, $B-A \leqslant (1+B)a$, $2B^2a \leqslant (2B+1)(B-A)$. If a function $f \in \mathcal{A}_0$ satisfies the following inequality*:

$$\left|\frac{\mathcal{L}(a+m+1,b)f(z)}{\mathcal{L}(a+m,b)f(z)} - 1\right| < \frac{2B-A}{(a+m)(1+B)} + \frac{B-A-aB}{(a+m)(B-A+a-aB)} \quad (z \in \mathcal{U}),$$

then

$$a\frac{\mathcal{L}(a+1,b)f(z)}{\mathcal{L}(a,b)f(z)} + 1 - a \prec \frac{1+Az}{1+Bz}.$$

Putting $a=b=m=1$ in Corollary 4 we obtain the sufficient condition for starlikeness.

Corollary 5. *Let $B-A \geqslant 2AB$. If a function $f \in \mathcal{A}_0$ satisfies the following inequality*:

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{2B-A}{1+B} - \frac{A}{1-A} \quad (z \in \mathcal{U}),$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz},$$

i.e., the function f is starlike in $\mathcal{U}$.

Putting $a=2$ and $b=m=1$ in Corollary 4 we obtain the sufficient condition for convexity.

Corollary 6. *Let $B - B^2 - 2AB - A \geqslant 0$. If a function $f \in \mathcal{A}_0$ satisfies the following inequality*:

$$\left|\frac{z^3 f'''(z) + 4z^2 f''(z) + 2zf''(z)}{z^2 f''(z) + 2f'(z)}\right| < \frac{2B - A}{1 + B} - \frac{B + A}{2 - (B + A)} \quad (z \in \mathcal{U}),$$

then

$$\frac{zf''(z)}{f'(z)} + 1 \prec \frac{1 + Az}{1 + Bz},$$

i.e., the function f is convex in $\mathcal{U}$.

Remark 2. Putting $B = 1$ and $A = 2\alpha - 1$ in Corollaries 5 and 6 we obtain the sufficient conditions for starlikeness of order α and convexity of order α, respectively.

References

[1] B.C. Carlson, D.B. Shaffer, Starlike and prestarlike hypergeometric functions, SIAM J. Math. Anal. 15 (1984) 737–745.
[2] J. Dziok, Applications of the Jack lemma, Acta Math. Hungar. 105 (2004) 93–102.
[3] J. Dziok, Some relations including various linear operators, Demonstratio Math., in press.
[4] J. Dziok, H.M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, Integral Transforms Spec. Funct. 14 (2003) 7–18.
[5] J. Dziok, H.M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, Appl. Math. Comput. 103 (1999) 1–13.
[6] J. Dziok, H.M. Srivastava, Some subclasses of analytic functions with fixed argument of coefficients associated with the generalized hypergeometric function, Adv. Stud. Contemp. Math. 5 (2) (2002) 115–125.
[7] P.J. Eenigenburg, S.S. Miller, P.T. Mocanu, O.M. Reade, Second order differential inequalities in the complex plane, J. Math. Anal. Appl. 65 (1978) 289–305.
[8] Y.C. Kim, H.M. Srivastava, Fractional integral and other linear operators associated with the Gaussian hypergeometric function, Complex Var. Theory Appl. 34 (1997) 293–312.
[9] J.-L. Liu, H.M. Srivastava, Certain properties of the Dziok–Srivastava operator, Appl. Math. Comput. 159 (2004) 485–493.
[10] S.S. Miller, P.T. Mocanu, Differential Subordinations: Theory and Applications, Monogr. Textbooks Pure Appl. Math., vol. 225, Dekker, New York, 2000.
[11] H.M. Srivastava, S. Owa, Some characterization and distortion theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators, and certain subclasses of analytic functions, Nagoya Math. J. 106 (1987) 1–28.
[12] H.M. Srivastava, S. Owa, Univalent Functions, Fractional Calculus, and Their Applications, Halsted Press, New York, 1989.