On some applications of the Briot–Bouquet differential subordination

J. Dziok

Institute of Mathematics, University of Rzeszów, ul. Rejtana 16A, PL-35-310 Rzeszów, Poland

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Abstract

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1. Introduction

Let \( A \) denote the class of functions which are analytic in \( U = \mathbb{U}(1) \), where

\[ U(r) = \{z : z \in \mathbb{C} \text{ and } |z| < r\}. \]

We denote by \( A_0 \) the class of functions \( f \in A \) with the normalization \( f(0) = f'(0) - 1 = 0 \).

E-mail address: jdziok@univ.rzeszow.pl.

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We say that a function \( f \in \mathcal{A} \) is subordinate to a function \( F \in \mathcal{A} \) and write \( f(z) \prec F(z) \), if and only if there exists a function \( \omega \in \mathcal{A} \),

\[
\omega(0) = 0, \quad |\omega(z)| < 1 \quad (z \in \mathcal{U}),
\]
such that

\[
f(z) = F(\omega(z)) \quad (z \in \mathcal{U}).
\]

Moreover, we say that \( f \) is subordinate to \( F \) in \( \mathcal{U}(r) \), if \( f(rz) \prec F(rz) \). We shall write \( f(z) \prec r F(z) \) in this case. In particular, if \( F \) is univalent in \( \mathcal{U} \), we have the following equivalence (cf. [10]):

\[
f(z) \prec F(z) \iff f(0) = F(0) \quad \text{and} \quad f(\mathcal{U}) \subseteq F(\mathcal{U}).
\]

For analytic functions \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \), by \( f \ast g \) we denote the Hadamard product or convolution of \( f \) and \( g \), defined by

\[
(f \ast g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.
\]

Let \( q, s \in \mathbb{N} = \{1, 2, \ldots\}, q \leq s + 1 \). For complex parameters \( a_1, \ldots, a_q \) and \( b_1, \ldots, b_s \) \( (b_j \neq 0, -1, -2, \ldots; j = 1, \ldots, s) \), we define the generalized hypergeometric function \( qF_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z) \) by

\[
qF_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_q)_n z^n}{(b_1)_n \cdots (b_s)_n n!} \quad (z \in \mathcal{U}),
\]

where \( (\lambda)_n \) is the Pochhammer symbol defined, in terms of the Gamma function \( \Gamma \), by

\[
(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0), \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in \mathbb{N}). \end{cases}
\]

Corresponding to a function \( h(a_1, \ldots, a_q; b_1, \ldots, b_s; z) \) defined by

\[
h(a_1, \ldots, a_q; b_1, \ldots, b_s; z) = z_q F_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z),
\]

we consider a linear operator

\[
H(a_1, \ldots, a_q; b_1, \ldots, b_s): \mathcal{A}_0 \to \mathcal{A}_0,
\]

defined by the convolution:

\[
H(a_1, \ldots, a_q; b_1, \ldots, b_s) f(z) = h(a_1, \ldots, a_q; b_1, \ldots, b_s; z) \ast f(z).
\]

In particular, for \( s = 1 \) and \( q = 2 \) and \( a_2 = 1 \), we have the Carlson–Shaffer operator

\[
L(a_1, b_1) f(z) = H_1(a_1, 1; b_1) f(z),
\]

which was introduced by Carlson and Shaffer [1] (see also [8]).

After some calculations we obtain

\[
a H(a + 1) f(z) = z H'(a) f(z) + (a - 1) H(a) f(z),
\]

(1)
where, for convenience,
\[ H(a_1) f(z) = H(a_1, \ldots, a_q; b_1, \ldots, b_s) f(z). \]

The linear operator \( H(a_1, \ldots, a_q; b_1, \ldots, b_s) \) and some other linear operators and fractional calculus was investigated by many mathematicians (cf. [2,3,9,11,12]).

Now suppose that the parameters \( a_1, \ldots, a_q \) and \( b_1, \ldots, b_s \) are positive real numbers. Also let
\[ 0 \leq B \leq 1 \quad \text{and} \quad -B \leq A < B. \]

We denote by
\[ V(a_1; A, B) = V(a_1, \ldots, a_q; b_1, \ldots, b_s; A, B) \]
the class of functions \( f \in A_0 \) which satisfy the following condition:
\[ a_1 \frac{H(a_1 + 1) f(z)}{H(a_1) f(z)} + 1 - a_1 \leq \frac{1 + Az}{1 + Bz}. \]  
(2)

The class \( V(a_1, \ldots, a_q; b_1, \ldots, b_s; A, B) \) for functions with negative coefficients was introduced and studied by Dziok and Srivastava [5] (see also [4,6]). The class \( V(a, l; c; 2\alpha - 1, 1) \) was investigated by Kim and Srivastava [8].

Let \( h \) and \( q \) be analytic functions in \( U \) with \( h(0) = q(0) = 1 \) and let \( h \) be univalent. The first-order differential subordination
\[ q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} < h(z) \]  
(3)
is called the Briot–Bouquet differential subordination. This particular differential subordination has a surprising number of important applications in the theory of analytic functions (for details see [10]).

In the paper we present one more application of the Briot–Bouquet differential subordination.

2. Main results

Eenigenburg et al. [7] proved, that for convex function \( h \), with \( \text{Re}(\beta h(z) + \gamma) \geq 0 \), the Briot–Bouquet differential subordination (3) implies \( p(z) < h(z) \). Thus we have the following lemma.

**Lemma 1.** If \( q \) is an analytic function in \( U(r) \), \( q(0) = 1 \) and
\[ q(z) + \frac{zq'(z)}{q(z) + \gamma} < r \frac{1 + Az}{1 + Bz} \left( \gamma + \frac{1 + A}{1 + B} \geq 0 \right), \]
then
\[ q(z) < r \frac{1 + Az}{1 + Bz}. \]

Making use of the above lemma, we get the following theorem.

**Theorem 1.** If \( a \geq \frac{B - A}{1 + B} \), then
\[ V(a + m; A, B) \subset V(a; A, B) \quad (m \in \mathbb{N}). \]
Proof. It is clear that it is sufficient to prove the theorem for \( m = 1 \). Let a function \( f \) belong to the class \( V(a + 1; A, B) \) or equivalently

\[
(a + 1) \frac{H(a + 2)f(z)}{H(a + 1)f(z)} - a < \frac{1 + Az}{1 + Bz}.
\]

(4)

It is sufficient to verify condition (2). If we put

\[
R = \sup \{ r : H(a)f(z) \neq 0, \ z \in \mathcal{U}(r) \},
\]

then the function

\[
q(z) = a \frac{H(a + 1)f(z)}{H(a)f(z)} + 1 - a
\]

(5)

is analytic in \( \mathcal{U}(R) \) and \( q(0) = 1 \). Taking the logarithmic derivative of (5) we get

\[
\frac{z[H(a + 1)f(z)]'}{H(a + 1)f(z)} - \frac{z[H(a)f(z)]'}{H(a)f(z)} = \frac{zq'(z)}{q(z) + a - 1} \quad (z \in \mathcal{U}(R)).
\]

Applying (1) and (5) we obtain

\[
(a + 1) \frac{H(a + 2)f(z)}{H(a + 1)f(z)} - a = q(z) + \frac{zq'(z)}{q(z) + a - 1} \quad (z \in \mathcal{U}(R)).
\]

(6)

Thus by (4) we have

\[
q(z) + \frac{zq'(z)}{q(z) + \gamma} < R \frac{1 + Az}{1 + Bz}.
\]

(7)

By (5) it suffices to verify that \( R = 1 \). From (7), (5) and (1) we conclude that \( H(a)f(z) \) is starlike in \( \mathcal{U}(R) \) and consequently it is univalent in \( \mathcal{U}(R) \). Thus we see that \( H(a)f(z) \) cannot vanish on \( |z| = R \) if \( R < 1 \). Hence \( R = 1 \) and this proves Theorem 1. \( \square \)

Using Lemma 1 we show the following sufficient conditions for functions to belong to the class \( V(a; A, B) \).

**Theorem 2.** Let \( m \in \mathbb{N}, \ B - A \leq (1 + B)a, \ 2B^2a \leq (2B + 1)(B - A) \). If a function \( f \in \mathcal{A}_0 \) satisfies the following inequality:

\[
\left| \frac{H(a + m + 1)f(z)}{H(a + m)f(z)} - 1 \right| < \frac{2B - A}{(a + m)(1 + B)} + \frac{B - A - aB}{(a + m)(B - A + a - aB)} \quad (z \in \mathcal{U}),
\]

(8)

then \( f \) belongs to the class \( V(a; A, B) \).

**Proof.** By Theorem 1 it is sufficient to consider the case \( m = 1 \). Let a function \( f \) belong to the class \( \mathcal{A}_0 \). Putting

\[
q(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \quad (z \in \mathcal{U}(R))
\]

(9)

in (6), we obtain
\[(a + 1) \frac{H(a + 2)f(z)}{H(a + 1)f(z)} - a = \frac{1 + Aw(z)}{1 + Bw(z)} + \frac{(aB + A - B)w'(z)}{a + (aB + A - B)w(z)} - \frac{Bzw'(z)}{1 + Bw(z)}.\]

Consequently, we have

\[F(z) = w(z) \left\{ zw'(z) \left( \frac{aB + A - B}{a + (aB + A - B)w(z)} - \frac{B}{1 + Bw(z)} \right) - \frac{B - A}{1 + Bw(z)} \right\}, \quad (10)\]

where

\[F(z) = (a + 1) \frac{H(a + 2)f(z)}{H(a + 1)f(z)} - a - 1.\]

By (2), (5) and (9) it is sufficient to verify that \(w\) is analytic in \(U\) and

\[|w(z)| < 1 \quad (z \in U).\]

Now, suppose that there exists a point \(z_0 \in \mathcal{U}(R)\), such that

\[|w(z_0)| = 1, \quad |w(z)| < 1 \quad (|z| < |z_0|).\]

Then, applying Lemma 1, we can write

\[z_0 w'(z_0) = kw(z_0), \quad w(z_0) = e^{i\theta} \quad (k \geq 1).\]

Combining these with (10), we obtain

\[|F(z_0)| = \left| k \left( \frac{B - A - aB}{a + (aB + A - B)e^{i\theta}} + \frac{B}{1 + Be^{i\theta}} \right) + \frac{B - A}{1 + Be^{i\theta}} \right| \geq k \text{Re} \left( \frac{B - A - aB}{a + (aB + A - B)e^{i\theta}} + \frac{B}{1 + Be^{i\theta}} \right) + \frac{B - A}{1 + B} \geq k \left( \frac{B - A - aB}{a + B - A - aB} + \frac{B}{1 + B} \right) + \frac{B - A}{1 + B} \geq 2B - A + \frac{B - A - aB}{a + B - A - aB}.\]

Since this result contradicts (8) we conclude that \(w\) is the analytic function in \(\mathcal{U}(R)\) and \(|w(z)| < 1 \quad (z \in \mathcal{U}(R))\). Applying the same methods as in the proof of Theorem 1 we obtain \(R = 1\), which completes the proof of Theorem 2.

Putting \(A = 2\alpha - 1\) and \(B = 1\) in Theorems 1 and 2 we obtain the following two corollaries.

**Corollary 1.** Let \(0 \leq \alpha < 1, \quad a \geq 1 - \alpha, \quad m \in \mathbb{N}\). If a function \(f \in A_0\) satisfies the following inequality:

\[\text{Re} \left\{ (a + m) \frac{H(a + m + 1)f(z)}{H(a + m)f(z)} + 1 - a - m \right\} > \alpha \quad (z \in \mathcal{U}),\]

then

\[\text{Re} \left\{ a \frac{H(a + 1)f(z)}{H(a)f(z)} + 1 - a \right\} > \alpha \quad (z \in \mathcal{U}).\]
Corollary 2. Let \( m \in \mathbb{N}, 0 \leq \alpha < 1, 1 - \alpha \leq a \leq 3(1 - \alpha) \). If a function \( f \in \mathcal{A}_0 \) satisfies the following inequality:
\[
\left| \frac{H(a + m + 1)f(z)}{H(a + m)f(z)} - 1 \right| < \frac{2(1 - \alpha)^2 + 3(1 - \alpha) - a}{2(a + m)(1 - \alpha)} \quad (z \in \mathcal{U}),
\]
then
\[
\Re \left\{ a \frac{H(a + 1)f(z)}{H(a)f(z)} + 1 - a \right\} > \alpha \quad (z \in \mathcal{U}).
\]

Putting \( s = 1, q = 2, b_1 = b \) and \( a_2 = 1 \), in Theorems 1 and 2 we obtain the following two corollaries.

Corollary 3. If \( a \geq \frac{B - A}{1 + B} \) and
\[
(a + m) \frac{\mathcal{L}(a + m + 1, b)f(z)}{\mathcal{L}(a + m, b)f(z)} + 1 - a - m < \frac{1 + Az}{1 + Bz},
\]
then
\[
a \frac{\mathcal{L}(a + 1, b)f(z)}{\mathcal{L}(a, b)f(z)} + 1 - a < \frac{1 + Az}{1 + Bz}.
\]

Remark 1. Putting \( m = a = B = 1 \) and \( A = 2\beta - 1 \) in Corollary 3 we have the result of Kim and Srivastava [8], obtained by using another methods.

Corollary 4. Let \( m \in \mathbb{N}, B - A \leq (1 + B)a, 2B^2a \leq (2B + 1)(B - A) \). If a function \( f \in \mathcal{A}_0 \) satisfies the following inequality:
\[
\left| \frac{\mathcal{L}(a + m + 1, b)f(z)}{\mathcal{L}(a + m, b)f(z)} - 1 \right| < \frac{2B - A}{(a + m)(1 + B)} + \frac{B - A - aB}{(a + m)(B - A + a - aB)} \quad (z \in \mathcal{U}),
\]
then
\[
a \frac{\mathcal{L}(a + 1, b)f(z)}{\mathcal{L}(a, b)f(z)} + 1 - a < \frac{1 + Az}{1 + Bz}.
\]

Putting \( a = b = m = 1 \) in Corollary 4 we obtain the sufficient condition for starlikeness.

Corollary 5. Let \( B - A \geq 2AB \). If a function \( f \in \mathcal{A}_0 \) satisfies the following inequality:
\[
\left| \frac{zf''(z)}{f'(z)} \right| < \frac{2B - A}{1 + B} - \frac{A}{1 - A} \quad (z \in \mathcal{U}),
\]
then
\[
\frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 + Bz},
\]
i.e., the function \( f \) is starlike in \( \mathcal{U} \).

Putting \( a = 2 \) and \( b = m = 1 \) in Corollary 4 we obtain the sufficient condition for convexity.
Corollary 6. Let $B - B^2 - 2AB - A \geq 0$. If a function $f \in A_0$ satisfies the following inequality:

$$
\left| \frac{z^3 f'''(z) + 4z^2 f''(z) + 2zf''(z)}{z^2 f''(z) + 2f'(z)} \right| < \frac{2B - A}{1 + B} - \frac{B + A}{2 - (B + A)} \quad (z \in \mathbb{U}),
$$

then

$$
\frac{zf''(z)}{f'(z)} + 1 \prec \frac{1 + Az}{1 + Bz},
$$

i.e., the function $f$ is convex in $\mathbb{U}$.

Remark 2. Putting $B = 1$ and $A = 2\alpha - 1$ in Corollaries 5 and 6 we obtain the sufficient conditions for starlikeness of order $\alpha$ and convexity of order $\alpha$, respectively.

References