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Representation of a gauge field via intrinsic "BRST" operator

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ABSTRACT

We show that there exists a representation of a matrix-valued gauge field via intrinsic "BRST" operator assigned to matrix-valued generators of a gauge algebra. In this way, we reproduce the standard formulation of the ordinary Yang–Mills theory. In the case of a generating quasigroup/groupoid, we give a natural counterpart to the Yang–Mills action. The latter counterpart does also apply as to the most general case of an involution for matrix-valued gauge generators.

1. Introduction and summary

All modern models describing the fundamental forces in the Nature are based on the concept of gauge fields [1–8]. It is a well-known fact that the BRST symmetry [9,10] is the most powerful method to represent the invariance properties of a gauge field system [11,12]. Usually, in simple examples, in Hamiltonian formalism, gauge generators have the form of secondary constraints similar to the "Gauss law" represented as covariant divergence of canonical momenta. These generators are in involution that represents a gauge algebra on the phase space of the system [13,14]. By introducing ghost canonical pairs, one is able to define the respective nilpotent BRST operator containing the first class constraints in its lowest terms, linear in ghost coordinates. In the respective Lagrangian formalism, the gauge generators are represented in terms of Lagrangian field variables, as the coefficients linear in the original antifields, entering the minimal master action. In this way, usually, space-like components of relativistic fields are identified with Hamiltonian coordinates, while time – like components are identified with Lagrange multipliers to secondary first-class constraints. In the simplest example, the Yang–Mills theory, Lagrangian matrix-valued gauge field is a linear combination of matrix-valued generators of adjoint representation of a generating Lie group. Thus, if one has defined the respective intrinsic "BRST" operator assigned to matrix-valued generators of adjoint representation, one can define the matrix-valued gauge field as a commutator of intrinsic "BRST" operator with an auxilliary "gauge" Fermion linear in the adjoint component of the Yang–Mills field. Thus, one has arrived at the intrinsic "BRST" representation to the matrix-valued gauge field.

In the present article, we study in detail the approach based on the intrinsic "BRST" representation. In the case of a Lie group we have shown that the new approach does reproduce exactly the standard formulation of the Yang–Mills theory. Then, we consider the case of a quasigroup/groupoid [15–22], where the structure coefficients of the intrinsic algebra of matrix-valued generators are matrix-valued themselves. In that case, we have found a natural counterpart to the Yang–Mills action. Finally, we consider the most general case of being the matrix-valued generators in the general involution among themselves.

2. Outline of the construction

Let $A_\mu(x)$ be a boson $N \times N$ matrix-valued vector field as defined by the formula

$$A_\mu(x) = [\mathcal{A}_\mu(x), Q], \quad [A_\mu(x), Q] = 0,$$
$$\varepsilon(A_\mu) = 0, \qquad \mathrm{gh}(A_\mu) = 0, \tag{2.1}$$

where Q is a nilpotent Fermion operator,

$$Q^2 = \frac{1}{2}[Q, Q] = 0, \quad \varepsilon(Q) = 1, \quad \mathrm{gh}(Q) = 1, \tag{2.2}$$

and Fermion vector field $\mathcal{A}_\mu(x)$ has ghost number -1,

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$$\varepsilon(\mathcal{A}_\mu)=1, \quad \mathrm{gh}(\mathcal{A}_\mu)=-1. \tag{2.3}$$

In more detail, Q is an $N\times N$ matrix-valued operator and, at the same time does depend on m Fermion canonical pairs of ghosts $(C^a,\mathcal{P}_a)$, $a=1,\dots,m$, $\varepsilon(C^a)=\varepsilon(\mathcal{P}_a)=1$, $\mathrm{gh}(C^a)=1$, $\mathrm{gh}(\mathcal{P}_a)=-1$,

$$[C^a,C^b]=0, \quad [C^a,\mathcal{P}_b]=\delta^a_b, \quad [\mathcal{P}_a,\mathcal{P}_b]=0. \tag{2.4}$$

We assume these operators to be realized as $n\times n$ matrices, so that in fact the Q is defined on tensor product of the original matrix arguments and the ones of ghosts in (2.4). The same status we do assume as to the $\mathcal{A}_\mu$ in (2.1), (2.3). The assumption of a matrix realization of ghosts, together with the relations (2.4), allows one to get simple expressions for traces of homogeneous $C\mathcal{P}$ normal ordered monomials with ghost number zero. In what follows, we will use the two simple examples,

$$\mathrm{tr}(C^a\mathcal{P}_b)=\frac{n}{2}\delta^a_b, \tag{2.5}$$

$$\mathrm{tr}(C^aC^b\mathcal{P}_c\mathcal{P}_d)=\frac{n}{4}\left(\delta^a_d\delta^b_c-\delta^a_c\delta^b_d\right). \tag{2.6}$$

These two formulae do follow from the general representation for ghost canonical pairs in terms of two conjugate sets of $n\times n$ gamma matrices,

$$2C^a=\gamma^a_+ + \gamma^a_-, \quad 2\mathcal{P}_a=(\gamma^b_+ - \gamma^b_-)g_{ba}, \tag{2.7}$$

where $g_{ab}=g_{ba}$ is a constant invertible metric, $g^{ab}=g^{ba}$ is its inverse, and the γ matrices do commute as

$$\gamma^a_\pm\gamma^b_\pm+(a\leftrightarrow b)=(\pm)2g^{ab}\,\mathbf{1}, \tag{2.8}$$

$$\gamma^a_\pm\gamma^b_\mp+\gamma^b_\mp\gamma^a_\pm=0. \tag{2.9}$$

It follows from (2.7) that (2.5), (2.6) do generalize to

$$\mathrm{tr}(X(C,\mathcal{P}))=\left(X\left(\frac{\partial}{\partial J},\frac{\partial}{\partial K}\right)\exp\left\{\frac{1}{2}K^aJ_a\right\}n\right)\Big|_{J=0,K=0}, \tag{2.10}$$

$$\varepsilon(J)=1, \quad \mathrm{gh}(J)=-1, \quad \varepsilon(K)=1,$$
$$\mathrm{gh}(K)=1, \quad \varepsilon(X)=0, \quad \mathrm{gh}(X)=0. \tag{2.11}$$

By inserting the doublet (exact) form, the first in (2.1), into the curvature form

$$G_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu], \tag{2.12}$$

we have

$$G_{\mu\nu}=[Q,\mathcal{G}_{\mu\nu}], \tag{2.13}$$

$$\mathcal{G}_{\mu\nu}=\partial_\mu\mathcal{A}_\nu-\partial_\nu\mathcal{A}_\mu+(\mathcal{A}_\mu,\mathcal{A}_\nu)_Q, \tag{2.14}$$

where in (2.14) the quantum antibracket $(X,Y)_Q$ is defined for any two operators X,Y, as [23]

$$2(X,Y)_Q=[X,[Q,Y]]-[Y,[Q,X]](-1)^{(\varepsilon_X+1)(\varepsilon_Y+1)}. \tag{2.15}$$

When deriving (2.14), we have used the general property

$$[[Q,X],[Q,Y]]=[Q,(X,Y)_Q]=$$
$$=([Q,X],Y)_Q-(X,[Q,Y])_Q(-1)^{\varepsilon_X}. \tag{2.16}$$

Notice that the quantum antibrackets do satisfy the Jacobi identity modulo a doublet (exact) form

$$(X,(Y,Z)_Q)_Q(-1)^{(\varepsilon_X+1)(\varepsilon_Z+1)}+\text{cyclic perm.}(X,Y,Z)=$$
$$=\frac{1}{2}[(X,Y,Z)_Q(-1)^{(\varepsilon_X+1)(\varepsilon_Z+1)},Q], \tag{2.17}$$

where $(X,Y,Z)_Q$ is the so-called quantum 3-antibracket,

$$3(X,Y,Z)_Q=$$
$$=-(-1)^{(\varepsilon_X+1)(\varepsilon_Z+1)}([X,(Y,Z)_Q](-1)^{[\varepsilon_X(\varepsilon_Z+1)+\varepsilon_Y]}+$$
$$+\text{cyclic perm.}(X,Y,Z)), \tag{2.18}$$

and so on [24] (see also [25–27]). The modified Leibnitz rule for quantum antibracket reads

$$(XY,Z)_Q-X(Y,Z)_Q-(X,Z)_QY(-1)^{\varepsilon_Y(\varepsilon_Z+1)}=$$
$$=\frac{1}{2}([X,Z][Y,Q](-1)^{\varepsilon_Z(\varepsilon_Y+1)}+[X,Q][Y,Z](-1)^{\varepsilon_Y}). \tag{2.19}$$

In terms of the curvature (2.13), the General “Yang–Mills” Lagrangian reads

$$\mathcal{L}=-\frac{1}{2}\mathrm{tr}(G_{\mu\nu}G_{\mu\nu})=-\frac{1}{2}\mathrm{tr}([Q,\mathcal{G}_{\mu\nu}][Q,\mathcal{G}_{\mu\nu}]). \tag{2.20}$$

Let us consider infinitesimal gauge transformations with an operator-valued Fermion “parameter” Ξ, $\varepsilon(\Xi)=1$, $\mathrm{gh}(\Xi)=-1$, $\Xi\to 0$,

$$\delta A_\mu=-(\partial_\mu[Q,\Xi]+[A_\mu,[Q,\Xi]]). \tag{2.21}$$

It follows from the first in (2.1), and (2.16), that the respective variation in $\mathcal{A}_\mu$ can be chosen in the form

$$\delta\mathcal{A}_\mu=-(\partial_\mu\Xi+(\mathcal{A}_\mu,\Xi)_Q). \tag{2.22}$$

Due to (2.14), (2.17), it follows that the respective variation in $\mathcal{G}_{\mu\nu}$ can be chosen in the form

$$\delta\mathcal{G}_{\mu\nu}=-(\mathcal{G}_{\mu\nu},\Xi)_Q. \tag{2.23}$$

Now, we have, as to the respective variation in (2.20)

$$\delta\mathcal{L}=-\mathrm{tr}([Q,\delta\mathcal{G}_{\mu\nu}][Q,\mathcal{G}_{\mu\nu}])=$$
$$=\mathrm{tr}([[Q,\mathcal{G}_{\mu\nu}],[Q,\Xi]][Q,\mathcal{G}_{\mu\nu}])=$$
$$=\mathrm{tr}([Q,\mathcal{G}_{\mu\nu}][Q,\Xi][Q,\mathcal{G}_{\mu\nu}]-[Q,\Xi][Q,\mathcal{G}_{\mu\nu}][Q,\mathcal{G}_{\mu\nu}])=$$
$$=0. \tag{2.24}$$

Here, in the second equality we have used (2.16) backward, and we have moved the last commutator to the leftmost position in the second term in the left-hand side of the last (fourth) equality. Thereby, we have confirmed explicitly that the Lagrangian (2.20) is gauge invariant. Thus, we have constructed a family of gauge-invariant classical theories of the type (2.20), closely related to the “general Yang–Mills theory”. Every of those classical theories can certainly be considered as a starting point as to apply the Hamiltonian BFV or Lagrangian BV quantization scheme, although we do not do that in the present article.

In what follows below through the article, we assume the operator Q as represented in $C\mathcal{P}$ normal form. In that case, it follows in terms of the quantum antibrackets, with no further assumptions,

$$(T_a,T_b)_Q=0, \tag{2.25}$$

$$(\mathcal{P}_a,\mathcal{P}_b)_Q+C^c(\mathcal{P}_c,\mathcal{P}_a,\mathcal{P}_b)_Q=U^c_{ab}\mathcal{P}_c, \tag{2.26}$$

$$2(\mathcal{P}_a,T_b)_Q=[T_a,T_b], \tag{2.27}$$

where we have denoted

$$T_a=[\mathcal{P}_a,Q], \quad U^c_{ab}=[[\mathcal{P}_a,[\mathcal{P}_b,Q]],C^c], \tag{2.28}$$

and the quantum 3-antibracket of the ghost momenta reads

$$(\mathcal{P}_a,\mathcal{P}_b,\mathcal{P}_c)_Q=[\mathcal{P}_a,[\mathcal{P}_b,[\mathcal{P}_c,Q]]]. \tag{2.29}$$

In turn, by commuting the Q with (2.26), we get

$$[T_a, T_b] + [Q\,, C^c(\mathcal{P}_c, \mathcal{P}_a, \mathcal{P}_b)_Q] = U^c_{ab}T_c + [Q\,, U^c_{ab}]\mathcal{P}_c. \tag{2.30}$$

If we assume that A^a_μ and Ξ^a are c-numbers, and

$$\mathcal{A}_\mu = A^a_\mu \mathcal{P}_a, \quad \Xi = \Xi^a \mathcal{P}_a \qquad (\text{Mod}\,[Q\,,\text{Anything}]) \tag{2.31}$$

then we get, due to the first in (2.1) and in (2.28),

$$A_\mu = A^a_\mu T_a, \quad [\Xi, Q] = \Xi^a T_a. \tag{2.32}$$

If, moreover, U^c_{ab} are c-numbers, and the metric,

$$\eta_{ab} = (Nn)^{-1}\,\text{tr}(T_aT_b), \tag{2.33}$$

is invertible, so that η^{ab} is its inverse, then we have for the field components

$$A^a_\mu = (Nn)^{-1}\,\text{tr}(A_\mu T_b)\eta^{ba}, \tag{2.34}$$

and, therefore, their gauge transformation presents

$$\delta A^a_\mu = -\partial_\mu \Xi^a - A^c_\mu \Xi^d (Nn)^{-1}\,\text{tr}([T_c, T_d]T_b)\eta^{ba}. \tag{2.35}$$

3. Yang–Mills theory generated by a compact semisimple Lie group

Let $t_a, a = 1, \ldots, m$, be $N \times N$ matrix-valued boson generators of a semisimple Lie group,

$$[t_a, t_b] = U^c_{ab}t_c, \quad \text{tr}(t_a) = 0, \tag{3.1}$$

where $U^c_{ab} = -U^c_{ba} = const$ are structure constants of the group. They satisfy the relations

$$U^c_{ab}U^e_{cd} + \text{cyclic perm.}(a, b, d) = 0, \quad U^b_{ab} = 0. \tag{3.2}$$

Due to the first in (3.1) and (3.2), the following operator

$$Q = C^a t_a + \frac{1}{2}C^bC^aU^c_{ab}\mathcal{P}_c, \tag{3.3}$$

does satisfy (2.2). Vise versa, the nilpotency condition (2.2) does imply the algebra of the first (3.1) and (3.2). Now, let us choose the operator $\mathcal{A}_\mu(x)$ in the form

$$\mathcal{A}_\mu(x) = A^a_\mu(x)\mathcal{P}_a. \tag{3.4}$$

It follows then from (2.1)

$$A_\mu(x) = A^a_\mu(x)T_a, \quad T_a = [\mathcal{P}_a, Q] = t_a + C^bU^c_{ba}\mathcal{P}_c, \tag{3.5}$$

$$\text{tr}(A_\mu(x)T_b) = A^a_\mu(x)\,\text{tr}(T_aT_b). \tag{3.6}$$

In turn, due to the first in (3.1) and (3.2), it follows for the second in (3.5)

$$[T_a, T_b] = U^c_{ab}T_c, \quad \text{tr}(T_a) = \frac{Nn}{2}U^b_{ba} = 0. \tag{3.7}$$

Then, we have

$$\text{tr}(T_aT_d) = n\,\text{tr}(t_at_d) + N\,\text{tr}\left(C^bU^c_{ba}\mathcal{P}_cC^eU^f_{ed}\mathcal{P}_f\right) = \\ = Nn\left[N^{-1}\,\text{tr}(t_at_d) + \frac{1}{4}U^b_{ca}U^c_{bd}\right], \tag{3.8}$$

where (2.4), (2.5), (2.6) have been used. Thus, we have reproduced the well-known Yang–Mills Lagrangian

$$\mathcal{L} = -\frac{1}{2Nn}\,\text{tr}(G_{\mu\nu}(x)G_{\mu\nu}(x)) = -\frac{1}{2Nn}G^a_{\mu\nu}(x)G^b_{\mu\nu}(x)\,\text{tr}(T_aT_b), \tag{3.9}$$

where the Yang–Mills curvature (stress tensor) has the usual form

$$G_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)] = G^a_{\mu\nu}(x)T_a, \tag{3.10}$$

$$G^a_{\mu\nu}(x) = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + A^b_\mu(x)A^c_\nu(x)U^a_{bc}. \tag{3.11}$$

Ghost-extended generators, similar to the second in (3.5), have been first introduced in string theory [28,29], and then generalized and studied systematically in [30,31], being called as "BRST-invariant constraints".

4. The quasigroup/groupoid case

Now, let us consider a more general situation of quasigroup/groupoid, where the structure coefficient of the algebra are matrix-valued operators rather then constants. In that case we have

$$[t_a, t_b] = U^c_{ab}t^c, \quad U^c_{ab} = -U^c_{ba}, \quad \text{tr}(U^c_{ab}t_c) = 0, \tag{4.1}$$

$$(U^c_{ab}U^e_{cd} - [t_d, U^e_{ab}]) + \text{cyclic perm.}(a, b, d) = 0, \tag{4.2}$$

$$([U^c_{ab}, U^f_{de}] - (c \leftrightarrow f)) + \text{cyclic perm.}(a, b, d, e) = 0, \tag{4.3}$$

where we have denoted

$$X_{abcd} + \text{cyclic perm.}(a, b, c, d) = S^{hgfe}_{abcd}X_{efgh}, \tag{4.4}$$

$$4!S^{hgfe}_{abcd} = \partial_a\partial_b\partial_c\partial_dC^hC^gC^fC^e, \quad \partial_a = \frac{\partial}{\partial C^a}. \tag{4.5}$$

Due to these relations (4.1)–(4.3) the operator (3.3) with the matrix-valued structure operators U^c_{ab} does satisfy the nilpotency (2.2). The quasigroup/groupoid is the most general case of generators, where the operator Q (3.3) linear in ghost momenta does satisfy the nilpotency (2.2).

If one defines a counterpart to the BRST-invariant generators, the second in (3.5), with operator-valued U^c_{ab}, then the respective algebra

$$[T_a, T_b] = U^c_{ab}T_c + [Q\,, U^c_{ab}]\mathcal{P}_c, \tag{4.6}$$

$$\eta_{ab} = (Nn)^{-1}\,\text{tr}(T_aT_b) = \\ = N^{-1}\,\text{tr}\left[\left(t_a + \frac{1}{2}U^c_{ca}\right)\left(t_b + \frac{1}{2}U^d_{db}\right) + \frac{1}{4}U^c_{da}U^d_{cb}\right], \tag{4.7}$$

does involve the ghost momenta $\mathcal{P}_a$ to serve as new generators with their own semi-Abelian subalgebra,

$$[\mathcal{P}_a, \mathcal{P}_b] = 0, \quad [\mathcal{P}_a, T_b] = U^c_{ab}\mathcal{P}_c = (\mathcal{P}_a, \mathcal{P}_b)_Q\,. \tag{4.8}$$

Notice that if one commutes the Q with (4.6), one gets no further consequences. So, only the doubled set of generators,

$$\mathcal{T}_A = \{T_a; \mathcal{P}_a\}, \tag{4.9}$$

does have a closed algebra. However, the general formulation of Section 2 appears capable to operate efficiently even in such a complicated situation, as a part of the most general case to be considered below.

5. The most general case

Now, let us consider the most general involution (4.1), without assuming the conditions (4.2), (4.3). In that case, one should seek for a solution to the operator Q in the form of a ghost power series expansion of the form

$$Q = C^at_a + \frac{1}{2}C^bC^aU^c_{ab}\mathcal{P}_c + \frac{1}{12}C^cC^bC^aU^{ed}_{abc}\mathcal{P}_d\mathcal{P}_e + \cdots. \tag{5.1}$$

We do assume the following irreducibility condition for the generators t_a to be satisfied:

$$Z^a \neq 0, \quad Z^a t_a = 0 \quad \Rightarrow \quad Z^a = Z^{cb}\Pi^a_{bc}, \tag{5.2}$$

where $Z^{cb} = -Z^{bc}$ are arbitrary, and we have denoted

$$\Pi^a_{bc} = \frac{1}{2}(t_b \delta^a_c - (b \leftrightarrow c) - U^a_{bc}), \tag{5.3}$$

so that there holds identically

$$\Pi^a_{bc} t_a = \frac{1}{2}([t_b, t_c] - U^a_{bc} t_a) = 0. \tag{5.4}$$

In case the Z^a, in the first and second in (5.2), have some extra free indices, these indices are inherited by the Z^{ab}, in the third in (5.2), together with their symmetry properties, if any. Now, commute the first in (4.1) with t_d and then sum up the cyclic permutations (a, b, d). By using (4.1) again, one gets

$$Y^e_{abd} t_e = 0, \tag{5.5}$$

where Y^e_{abd} just denotes the left-hand side in (4.2). Due to the irreducibility (5.2), one gets

$$Y^e_{abd} = -U^{fg}_{abd}\Pi^e_{gf}, \tag{5.6}$$

which is exactly the relation that does follow from (2.2) to the $(CCC\mathcal{PP})$ order. In this way, one is able, in principle, to show, order by order, that there formally exist all the structure operators in the series expansion (5.1).

In the case of a Lie group, where the generators t_a and T_a do satisfy the same algebra, there exists a natural counterpart to (5.4) in terms of T_a, that is

$$\tilde{\Pi}^c_{ab} = T_a \delta^c_b - (a \leftrightarrow b) - U^c_{ab}, \quad \tilde{\Pi}^c_{ab} T_c = 0, \tag{5.7}$$

which extends naturally the irreducibility concept as to the ghost-extended generators T_a. Then, we have the following relation to hold

$$\tilde{\Pi}^c_{ab} = \Pi^c_{ab} + (C^d U^e_{da}\mathcal{P}_e \delta^c_b - (a \leftrightarrow b)). \tag{5.8}$$

6. Natural canonical equivalence

As to the nilpotency condition (2.2), one can always subject the operator Q to an arbitrary canonical transformation [32]

$$Q \;\rightarrow\; Q' = \exp\{sG\} Q \exp\{-sG\}, \tag{6.1}$$

where s is a boson parameter, and G is a matrix-valued and ghost dependent generator,

$$\varepsilon(G) = 0, \quad \mathrm{gh}(G) = 0. \tag{6.2}$$

We have

$$[Q', Q'] = 0, \quad \varepsilon(Q') = 1, \quad \mathrm{gh}(Q') = 1, \tag{6.3}$$

$$\partial_s Q' = [G, Q'], \quad Q'\big|_{s=0} = Q, \quad \partial_s = \frac{\partial}{\partial s}, \tag{6.4}$$

$$Q' = C^a t'_a + \frac{1}{2} C^b C^a\, U'^{\,c}_{ab}\, \mathcal{P}_c + \dots, \tag{6.5}$$

$$G = G_0 + C^a\, G^b_a\, \mathcal{P}_b + \dots, \tag{6.6}$$

with all matrix-valued structure coefficients. It follows from (6.3) that the all the primed structure coefficients of the primed Q' satisfy the same equations as their unprimed counterparts do. In turn, it follows from (6.4)

$$\partial_s t'_a = [G_0, t'_a] + G^b_a t'_b, \quad t'_a\big|_{s=0} = t_a, \tag{6.7}$$

$$\partial_s U'^{\,c}_{ab} = [G_0, U'^{\,c}_{ab}] + (G^d_a\, U'^{\,c}_{db} - (a \leftrightarrow b)) - U'^{\,d}_{ab}\, G^c_d + G^{de}_{ab}\, U'^{\,c}_{ed},$$
$$U'^{\,c}_{ab}\big|_{s=0} = U^c_{ab}, \tag{6.8}$$

and so on. Here in (6.8), G^{de}_{ab} are structure coefficients as to the order $CC\mathcal{PP}$ in G (6.6). These equations do determine the transformation law as to all the structure coefficients in (6.5). In particular, the G_0 does determine the canonical transformation in the original matrix-valued sector. In turn, the G^b_a do determine the actual rotations as to the basis of the original generators. In turn, the latter two transformations, as induced to the next structure coefficient U^c_{ab}, are determined by the equation (6.8), and so on in (6.5). Our main conjecture claims that the natural arbitrariness (6.1) is maximal, if the irreducibility (5.2) holds for primed basis of the generators t'_a, as well. In that case, canonical transformations (6.1) are capable to interpolate between the most general generator and Abelian ones.

If one rewrites the (5.4) in the form with enumerated indices,

$$\Pi^{b_1}_{a_1 a_2} t_{b_1} = 0, \tag{6.9}$$

due to the nilpotency (2.2), it becomes rather obvious that there exists a chain of recursive relations extending (6.9) as

$$\Pi^{b_n \dots b_1}_{a_1 \dots a_{n+1}} \Pi^{c_{n-1} \dots c_1}_{b_1 \dots b_n} = 0, \quad n = 2, \dots, \tag{6.10}$$

where the n-th Π (with n uppercases) is constructed of the first $n+1$ structure coefficients in (5.1). That chain of recursive relations extends naturally the irreducibility concept as to higher structure coefficients. As an example, we demonstrate the case $n = 2$:

$$\Pi^{b_2 b_1}_{a_1 a_2 a_3} = \frac{1}{2}(\Pi^{b_2}_{a_1 a_2}\delta^{b_1}_{a_3} - (b_1 \leftrightarrow b_2)) +$$
$$+ \text{cyclic perm.}(a_1, a_2, a_3) - \tilde{U}^{b_2 b_1}_{a_1 a_2 a_3}, \tag{6.11}$$

where we have also used the relation

$$\Pi^{b_2}_{a_1 a_2}\Pi^{c_1}_{b_2 a_3} + \text{cyclic perm.}(a_1, a_2, a_3) = -\tilde{U}^{b_2 b_1}_{a_1 a_2 a_3}\Pi^{c_1}_{b_1 b_2}, \tag{6.12}$$

$$(U - \tilde{U})^{b_2 b_1}_{a_1 a_2 a_3}\Pi^{c_1}_{b_1 b_2} =$$
$$= (U^{b_1}_{a_1 a_2} t_{b_1}\delta^{c_1}_{a_3} + t_{a_2} U^{c_1}_{a_1 a_3}) + \text{cyclic perm.}(a_1, a_2, a_3). \tag{6.13}$$

One can resolve for the $\tilde{U}$ operators,

$$(U - \tilde{U})^{b_2 b_1}_{a_1 a_2 a_3} =$$
$$= \left[\frac{1}{2}(\delta^{b_2}_{a_1} t_{a_2}\delta^{b_1}_{a_3} - (b_1 \leftrightarrow b_2)) + \text{cyclic perm.}(a_1, a_2, a_3)\right], \tag{6.14}$$

to get the following explicit solution

$$\Pi^{b_2 b_1}_{a_1 a_2 a_3} =$$
$$= \left[t_{a_1}\frac{1}{2}(\delta^{b_2}_{a_2}\delta^{b_1}_{a_3} - (b_1 \leftrightarrow b_2)) + \text{cyclic perm.}(a_1, a_2, a_3)\right] -$$
$$- \left[\frac{1}{2}(U^{b_2}_{a_1 a_2}\delta^{b_1}_{a_3} - (b_1 \leftrightarrow b_2)) + \text{cyclic perm.}(a_1, a_2, a_3)\right] -$$
$$- U^{b_2 b_1}_{a_1 a_2 a_3}. \tag{6.15}$$

7. Note added in proof

Here we claim that the standard Faddeev–Popov measure can also be naturally reformulated in terms of the generators T_a (2.28), by using the representation similar to (2.1) as applied to the Nakanishi–Lautrup matrix valued fields Π (Lagrange multipliers for gauge fixing functions), as well as to the ghost and antighost

Faddeev–Popov matrix valued field $B, \bar{B}$. Then, in the case of the Lorentz gauge, $\partial_\mu A_\mu = 0$, the gauge fixing part of the total Lagrangian reads

$$(Nn)^{-1}\,\mathrm{tr}(\Pi\,\partial_\mu A_\mu + (\partial_\mu \bar{B})(\partial_\mu B + [A_\mu, B])), \tag{7.1}$$

where all fields take their values in the T-algebra,

$$A_\mu = A^a_\mu T_a, \quad \Pi = \Pi^a T_a, \quad B = B^a T_a, \quad \bar{B} = \bar{B}^a T_a. \tag{7.2}$$

The (7.1) is invariant under the standard BRST transformations

$$\delta A_\mu = (\partial_\mu B + [A_\mu, B])\mu, \quad \delta B = \frac{1}{2}[B, B]\mu,$$
$$\delta \bar{B} = -\Pi\mu, \quad \delta\Pi = 0. \tag{7.3}$$

If the T_a do satisfy a Lie algebra, the BRST invariance holds in a straightforward way, with all the coefficients in (7.2) (field components) being c-numbers. However, in the quasigroup/groupoid case, one should allow for these coefficients to be matrix valued, in general. Then, we have from (2.1) and the first in (2.28) and (2.31)

$$A_\mu = A^a_\mu T_a + [Q, A^a_\mu]\mathcal{P}_a, \tag{7.4}$$

and similar formulae for all other fields. The form of the second term here is quite similar to the one of the second term in (4.6), that makes unclosed the algebra of the generators T_a alone. The doubled generators $\mathcal{T}_A$, (4.9), do satisfy the closed involution

$$[\mathcal{T}_A, \mathcal{T}_B] = \mathcal{U}^C_{AB}\mathcal{T}_C, \tag{7.5}$$

with the structure coefficients $\mathcal{U}^C_{AB}$ given explicitly in the relations (4.6), (4.8).

Any operator $\mathcal{X}$ of the form similar to (7.4),

$$\mathcal{X} = \mathcal{X}^A\mathcal{T}_A, \quad \mathcal{X}^A = \{X^a(-1)^{\varepsilon_{\mathcal{X}}}; [Q, X^a]\}, \tag{7.6}$$

with "matrix valued" coefficients $\mathcal{X}^A$, belongs to the closed doubled $\mathcal{T}$-algebra. The latter makes all the commutators entering (7.1), (7.3) well defined as taking their values within the same closed doubled $\mathcal{T}$-algebra. Notice that the (7.6) rewrites in the natural form

$$\mathcal{X} = [Q, X^a\mathcal{P}_a], \tag{7.7}$$

maintained under commuting of two operators of the form (7.7), due to the ghost number conservation.

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Appendix A. Generating equations for the quantum antibracket algebra

Here we include in short the generating equations for the quantum antibracket algebra [24]. Let us introduce an operator-valued exponential

$$U = \exp\{\lambda^a f_a\}, \quad U|_{\lambda=0} = 1, \tag{A.1}$$

where $\{f_a, a = 1, 2, \dots\}$, is a chain of operators, $\varepsilon(f_a) = \varepsilon_a$, and λ^a are parameters, $\varepsilon(\lambda^a) = \varepsilon_a$. Introduce the U-transformed Q-operator,

$$\tilde{Q} = UQU^{-1}, \quad \tilde{Q}^2 = 0. \tag{A.2}$$

We have

$$\partial_a\tilde{Q} = [R_a, \tilde{Q}], \quad R_a = (\partial_a U)U^{-1},$$
$$\partial_a = \frac{\partial}{\partial\lambda^a}, \quad \tilde{Q}|_{\lambda=0} = Q, \tag{A.3}$$
$$\partial_a R_b - \partial_b R_a(-1)^{\varepsilon_a\varepsilon_b} = [R_a, R_b]. \tag{A.4}$$

The Lie equation (A.3) and the Maurer–Cartan equation (A.4) do serve as the generating equations for quantum antibrackets. Here we present explicitly only the case of quantum 2-antibracket. It follows from (A.3) by λ differentiating, that

$$-\partial_a\partial_b\tilde{Q}(-1)^{\varepsilon_b} + \frac{1}{2}[(\partial_a R_b + \partial_b R_a(-1)^{\varepsilon_a\varepsilon_b})(-1)^{\varepsilon_b}, \tilde{Q}] =$$
$$= \frac{1}{2}\left([R_a, [\tilde{Q}, R_b]] - (a \leftrightarrow b)(-1)^{(\varepsilon_a+1)(\varepsilon_b+1)}\right) = (R_a, R_b)_{\tilde{Q}}. \tag{A.5}$$

It follows from (A.5) at $\lambda = 0$,

$$-(\partial_a\partial_b\tilde{Q})(-1)^{\varepsilon_b}|_{\lambda=0} = (f_a, f_b)_Q, \tag{A.6}$$

where we have used

$$(\partial_a R_b)|_{\lambda=0} = \frac{1}{2}[f_a, f_b]. \tag{A.7}$$

It follows in a similar way that higher λ derivatives of $\tilde{Q}$ do yield all higher quantum antibrackets,

$$(f_{a_1}, \dots, f_{a_n})_Q = -\mathrm{Sym}([f_{a_1}, \dots, [f_{a_n}, Q]\dots])(-1)^{E_n}, \tag{A.8}$$

where we have denoted

$$E_n = \sum_{k=1}^{[n/2]}\varepsilon_{a_{2k}}, \tag{A.9}$$

$$\mathrm{Sym}(X_{a_1\dots a_n}) = S^{b_n\dots b_1}_{a_1\dots a_n}X_{b_1\dots b_n},$$
$$n!S^{b_n\dots b_1}_{a_1\dots a_n} = \partial_{a_1}\dots\partial_{a_n}\lambda^{b_n}\dots\lambda^{b_1}. \tag{A.10}$$

It has also been shown in [24], how these equations enable one to derive the modified Jacobi relations for subsequent higher quantum antibrackets.

Notice, in conclusion, that there exists a nice interpretation of the quantum antibracket algebra via the so-called differential polarization [20]. In particular, being B an arbitrary boson operator, one can 3-times commute that B with the nilpotency equation (2.2), to get the relation

$$6(B, (B, B)_Q)_Q = [(B, B, B)_Q, Q], \quad \varepsilon(B) = 0. \tag{A.11}$$

Then, by choosing B in the form

$$B = \alpha X + \beta Y + \gamma Z, \tag{A.12}$$

with parameters α, β, γ of the same Grassmann parities as the ones of the operators X, Y, Z, respectively, are, one applies to (A.11) the differential operator

$$\partial_\alpha\partial_\beta\partial_\gamma(-1)^{(\varepsilon_\alpha+1)(\varepsilon_\gamma+1)+\varepsilon_\beta}, \tag{A.13}$$

to get exactly the relation (2.17).

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