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Application of level soft sets in decision making based on interval-valued fuzzy soft sets

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ABSTRACT

Molodtsov's soft set theory was originally proposed as a general mathematical tool for dealing with uncertainty. Research on (fuzzy) soft set based decision making has received much attention in recent years. This paper aims to give deeper insights into decision making involving interval-valued fuzzy soft sets, a hybrid model combining soft sets with interval-valued fuzzy sets. The concept called reduct fuzzy soft sets of interval-valued fuzzy soft sets is introduced. Using reduct fuzzy soft sets and level soft sets, flexible schemes for decision making based on (weighted) interval-valued fuzzy soft sets are proposed, and some illustrative examples are employed to show that the proposals presented here are not only more reasonable but more efficient in practical applications.

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1. Introduction

Complex problems involving vagueness (gradations in the notion of membership) and uncertainty (lack of information) are pervasive in many areas of modern technology. These practical problems arise in such diverse areas as economics, engineering, environmental science, social science, and medical science among others. While a wide variety of mathematical disciplines like probability theory, fuzzy set theory [1], rough set theory [2] and interval mathematics [3] are well known and often serve as useful mathematical approaches in modelling, each of them has its advantages as well as inherent limitations. One major weakness shared by these theories is possibly the inadequacy of the parameterization tool in them [4].

In 1999, Molodtsov [4] initiated *soft set theory* as a new mathematical tool for dealing with uncertainty which seems to be free from the inherent difficulties affecting the existing methods. He also demonstrated that the theory of soft sets has potential applications in various fields like the smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory, and measurement theory [5,4]. Later, Maji et al. [6] have further used soft sets to solve some decision making problems. Chen et al. [7] presented a new definition of soft set parameterization reduction to improve soft set based decision making in [6]. Kong et al. [8] introduced the definition of normal parameter reduction of soft sets and presented a heuristic algorithm to make normal parameter reduction.

The study of hybrid models combining soft sets with other mathematical structures is emerging as an active research topic of soft set theory (see [9–19]). Maji et al. [17] initiated *fuzzy soft sets*, a more generalized notion combining both fuzzy sets and soft sets. Roy and Maji [20] presented a novel method concerning object recognition from an imprecise multi-observer

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data so as to cope with decision making based on fuzzy soft sets. Then Kong et al. [21] revised the Roy–Maji method by considering "fuzzy choice values". Feng et al. [22] further discussed the application of fuzzy soft sets to decision making in an imprecise environment. They pointed out that the idea of choice values related to crisp soft sets based decision making in [6] is not fit to solve decision making problems involving fuzzy soft sets. Using a new concept called level soft sets, an adjustable approach to (weighted) fuzzy soft set based decision making was proposed in [22].

In many fuzzy decision making applications the related membership functions are extremely individual (dependent on experts' evaluation of alternatives) and thus cannot be lightly confirmed. It is more reasonable to give an interval-valued data to describe degree of membership; in other words, we can make use of interval-valued fuzzy sets which assign to each element an interval that approximates the "real" (but unknown) membership degree. In respond to this, Yang et al. [18] defined a hybrid model called *interval-valued fuzzy soft sets* and investigated some of their basic properties. They also presented an algorithm to solve decision making problems based on interval-valued fuzzy soft sets.

In this study, we follow the line of exploration in [22] and intend to give deeper insights into interval-valued fuzzy soft set based decision making discussed in [18]. The remainder of this paper is organized as follows: To facilitate our discussion, we first recall the standard (fuzzy) soft sets in Section 2. In Section 3, we briefly introduce Yang's interval-valued fuzzy soft sets and also initiate a new notion called reduct fuzzy soft sets. Section 4 is then devoted to a deep analysis on the application of interval-valued fuzzy soft set to fuzzy decision making. A counterexample is used to show that Yang et al.'s algorithm have some inherent limitations which arise from the improper use of (interval) fuzzy choice values. In Section 5, we revisit level soft sets and its application to fuzzy soft set based decision making. Based on reduct fuzzy soft sets and level soft sets, flexible schemes for decision making based on interval-valued fuzzy soft sets are proposed in Section 6. In Section 7, weighted interval-valued fuzzy soft sets is defined and applied to decision making problems in which all the decision criteria may not be of equal importance. Finally, we conclude the paper with a summary and outlook for further research in Section 8.

2. Preliminaries

Let U be an initial universe of objects and E_U (simply denoted by E) the universe of parameters related to the objects in U. The pair (U, E) will be called a *soft universe*. Let $\mathscr{P}(U)$ denote the power set of U and $A \subseteq E$. According to [9], the concept of soft sets is defined as follows.

Definition 2.1 (*[9]*)**.** A pair (F, A) is called a *soft set* over U, where F is a mapping given by $F : A \to \mathscr{P}(U)$.

By definition, a soft set (F, A) over the universe U can be regarded as a parameterized family of subsets of the universe U, which gives an approximate (soft) description of the objects in U. As pointed out by Molodtsov [4], for any parameter $\epsilon \in A$, the subset $F(\epsilon) \subseteq U$ may be considered as the set of ϵ-approximate elements in the soft set (F, A). It is worth noting that $F(\epsilon)$ may be arbitrary: some of them may be empty, and some may have nonempty intersection [4].

Combining fuzzy sets and soft sets, Maji et al. [17] initiated the following hybrid model called fuzzy soft sets, which can be seen as an extension of both fuzzy sets and crisp soft sets.

Definition 2.2 (*[20]*)**.** Let (U, E) be a soft universe and $A \subseteq E$. Let $\mathscr{F}(U)$ be the set of all fuzzy subsets in U. A pair $(\widetilde{F}, A)$ is called a *fuzzy soft set* over U, where $\widetilde{F}$ is a mapping given by $\widetilde{F} : A \to \mathscr{F}(U)$.

In this definition, fuzzy subsets are used as substitutes for the crisp subsets. Hence every soft set may be considered as a fuzzy soft set. Also it is obvious that a fuzzy set could be naturally viewed as a fuzzy soft set whose parameter set is a singleton. Generally speaking, $\widetilde{F}(\epsilon)$ is a fuzzy subset in U, $\forall \epsilon \in A$. Following the standard notations, $\widetilde{F}(\epsilon)$ can typically be written as $\widetilde{F}(\epsilon) = \{(x, \widetilde{F}(\epsilon)(x)) : x \in U\}$.

Definition 2.3 (*[20]*)**.** Let (U, E) be a soft universe and $A, B \subseteq E$. Let $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$ be two fuzzy soft sets over U. Then $(\widetilde{F}, A)$ is a *fuzzy soft subset* of $(\widetilde{G}, B)$, denoted $(\widetilde{F}, A) \widetilde{\subseteq} (\widetilde{G}, B)$, if $A \subseteq B$ and $\widetilde{F}(a)$ is a fuzzy subset of $\widetilde{G}(a)$ for all $a \in A$.

3. Interval-valued fuzzy soft sets

Consider the set $L^I = \{[a, b] : 0 \le a \le b \le 1\}$ and the order relation $\le_{L^I}$ given by:

$$[a_1, b_1] \le_{L^I} [a_2, b_2] \Leftrightarrow a_1 \le a_2, b_1 \le b_2, \quad \forall [a_1, b_1], [a_2, b_2] \in L^I.$$

Then $\mathscr{L}^I = (L^I, \le_{L^I})$ is a complete lattice [23].

An *interval-valued fuzzy set* on a universe U is a mapping $\mu : U \to L^I$. Let $\mu(x) = [\mu_*(x), \mu^*(x)]$, $\forall x \in U$. Then $\mu_*(x), \mu^*(x)$ are called the lower and upper degrees of membership of x in μ, respectively. The standard interpretation is that the interval $\mu(x) = [\mu_*(x), \mu^*(x)]$ is used to approximate the "real" but unknown degree of membership of x in μ.

Interval-valued fuzzy sets are a special case of L-fuzzy sets in the sense of Goguen [24] and a special case of type 2 fuzzy sets introduced by Zadeh [25]. The union, intersection and complement of interval-valued fuzzy sets can be obtained by canonically extending fuzzy set-theoretic operations to intervals (see [18]). The set of all interval-valued fuzzy sets on U is denoted by $\mathscr{I}(U)$.

Table 1
Interval-valued fuzzy soft set $\Im = (\widetilde{F}, A)$.

U	e_1	e_2	e_3	e_4	e_5	e_6
h_1	[0.3, 0.5]	[0.6, 0.7]	[0.6, 0.7]	[0.6, 0.7]	[0.3, 0.5]	[0.5, 0.8]
h_2	[0.6, 0.8]	[0.1, 0.3]	[0.8, 1.0]	[0.1, 0.3]	[0.7, 0.9]	[0.8, 1.0]
h_3	[0.5, 0.6]	[0.5, 0.6]	[0.2, 0.4]	[0.2, 0.4]	[0.5, 0.7]	[0.2, 0.5]
h_4	[0.2, 0.3]	[0.2, 0.4]	[0.0, 0.1]	[0.0, 0.1]	[0.2, 0.3]	[0.1, 0.3]
h_5	[0.8, 0.9]	[0.5, 0.7]	[0.1, 0.3]	[0.1, 0.3]	[0.8, 0.9]	[0.2, 0.5]
h_6	[0.8, 1.0]	[0.8, 1.0]	[0.5, 0.6]	[0.7, 0.8]	[0.2, 0.5]	[0.5, 0.6]

Definition 3.1 (*[18]*). Let (U, E) be a soft universe and $A \subseteq E$. A pair $\Im = (\widetilde{F}, A)$ is called an *interval-valued fuzzy soft set* (IVFS) over U, where $\widetilde{F}$ is a mapping given by $\widetilde{F} : A \to \mathscr{I}(U)$.

An interval-valued fuzzy soft set is a parameterized family of interval-valued fuzzy subsets of U. $\forall \epsilon \in A$, $\widetilde{F}(\epsilon)$ is called the *interval fuzzy value set* of the parameter ϵ. It is easy to see that fuzzy soft sets are a special case of IVFSs since interval-valued fuzzy sets are extensions of classical fuzzy sets.

We shall consider the following house purchase problem (see [18]) to illustrate an application of IVFSs to multi-criteria decision making.

Example 3.2 (*A House Purchase Problem*). Suppose that there are six houses under consideration, namely the universe $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$, and the parameter set $A = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, where e_i stand for "beautiful", "modern", "large", "cheap", "in good repair" and "in green surroundings" respectively. Suppose Mr. X wants to buy a house which satisfies the criteria in A to the utmost extent.

All the available information on these houses can be characterized by an interval-valued fuzzy soft set $\Im = (\widetilde{F}, A)$. The tabular representation of $(\widetilde{F}, A)$ is shown in Table 1. In Table 1, we can see that the precise evaluation for an alternative to satisfy a criterion is unknown while the lower and upper approximations of such an evaluation are given. For example, we cannot present the precise degree of how beautiful house h_1 is, however, house h_1 is at least beautiful on the degree of 0.3 and it is at most beautiful on the degree of 0.5.

As mentioned above, an IVFS can be viewed as a parameterized family of interval-valued fuzzy subsets of U. For each parameter, the corresponding interval fuzzy value set is an interval-valued fuzzy set; hence if a fuzzy set can be derived from each of these interval-valued fuzzy sets, then a fuzzy soft set can be induced by the IVFS in a natural way. The notion introduced below is an implementation of this idea.

Definition 3.3. Let (U, E) be a soft universe and $A \subseteq E$. Let $(\widetilde{F}, A)$ be an IVFS over U such that for all $a \in A$, $\widetilde{F}(a)$ is an interval-valued fuzzy set with $\widetilde{F}(a)(x) = [\widetilde{F}(a)_*(x), \widetilde{F}(a)^*(x)]$, $\forall x \in U$. Then the fuzzy soft set $(\widetilde{F}_-, A)$ over U given by

$$\widetilde{F}_-(a) = \{(x, \widetilde{F}(a)_*(x)) : x \in U\}, \quad \forall a \in A,$$

is called the *pessimistic reduct fuzzy soft set* (PRFS) of $(\widetilde{F}, A)$.

The semantic meaning of PRFSs can be construed as a pessimistic estimation of uncertain membership values, representing a prudent/pessimistic attitude. This is due to the observation that for all parameters a PRFS assigns to each object the "worst" value from the interval which approximates the "real" degree of membership. Such a pessimistic attitude works well in the cases where one needs to be very cautious, such as medical or space applications.

To fulfill various needs of decision making arising in different scenarios, we may consider other kinds of reduct fuzzy soft sets of IVFSs as follows.

Definition 3.4. The *optimistic reduct fuzzy soft set* (ORFS) of an interval-valued fuzzy soft set $(\widetilde{F}, A)$ is defined as a fuzzy soft set $(\widetilde{F}_+, A)$ over U such that

$$\widetilde{F}_+(a) = \{(x, \widetilde{F}(a)^*(x)) : x \in U\}, \quad \forall a \in A.$$

Note that ORFSs can naturally be interpreted as an optimistic estimation of uncertain membership values since an ORFS always assigns to each object the "best" degree of membership. This notion will be useful in situations where the decision maker adopts an optimistic attitude and hopes to relax the criteria in decision making.

Definition 3.5. The *neutral reduct fuzzy soft set* (NRFS) of an interval-valued fuzzy soft set $(\widetilde{F}, A)$ is defined as a fuzzy soft set $(\widetilde{F}_N, A)$ over U such that

$$\widetilde{F}_N(a) = \{(x, (\widetilde{F}(a)_*(x) + \widetilde{F}(a)^*(x))/2) : x \in U\}, \quad \forall a \in A.$$

The semantic explanation of NRFSs could be a neutral estimation of uncertain membership values, representing an attitude which is neither pessimistic nor optimistic. The following notion is a generalization of PRFSs, ORFSs and NRFSs.

Table 2
Pessimistic reduct fuzzy soft set of $\mathfrak{I} = (\widetilde{F}, A)$.

U	e_1	e_2	e_3	e_4	e_5	e_6
h_1	0.3	0.6	0.6	0.6	0.3	0.5
h_2	0.6	0.1	0.8	0.1	0.7	0.8
h_3	0.5	0.5	0.2	0.2	0.5	0.2
h_4	0.2	0.2	0.0	0.0	0.2	0.1
h_5	0.8	0.5	0.1	0.1	0.8	0.2
h_6	0.8	0.8	0.5	0.7	0.2	0.5

Definition 3.6. Let $\alpha, \beta \in [0, 1]$ and $\alpha + \beta = 1$. The vector $W = (\alpha, \beta)$ is called an *opinion weighting vector*. The fuzzy soft set $(\widetilde{F}_W, A)$ over U such that

$$\widetilde{F}_W(a) = \{(x, \alpha\widetilde{F}(a)_*(x) + \beta\widetilde{F}(a)^*(x)) : x \in U\}, \quad \forall a \in A,$$

is called the *weighted reduct fuzzy soft set* (WRFS) of the interval-valued fuzzy soft set $(\widetilde{F}, A)$ with respect to the opinion weighting vector W.

WRFSs can be used in more complex decision applications, such as the multi-criteria fuzzy group decision making (where the weights α, β are the percentage numbers of pessimistic and optimistic decision makers in the expert group), to express the amalgamation of both pessimistic and optimistic attitudes adopted by different decision makers.

The relationships of these different reduct fuzzy soft sets can formally be presented as follows.

Proposition 3.7. *Let $(\widetilde{F}, A)$ be an IVFS over U and let $(\widetilde{F}_W, A)$ be a weighted reduct fuzzy soft set with the opinion weighting vector $W = (\alpha, \beta)$. Then we have the following:*

(1) $(\widetilde{F}_W, A) = (\widetilde{F}_-, A)$ *if* $\alpha = 1$ *and* $\beta = 0$.
(2) $(\widetilde{F}_W, A) = (\widetilde{F}_+, A)$ *if* $\alpha = 0$ *and* $\beta = 1$.
(3) $(\widetilde{F}_W, A) = (\widetilde{F}_N, A)$ *if* $\alpha = \beta = 1/2$.
(4) $(\widetilde{F}_-, A) \widetilde{\subseteq} (\widetilde{F}_W, A) \widetilde{\subseteq} (\widetilde{F}_+, A)$.

Note that $\widetilde{\subseteq}$ denotes the fuzzy soft subset relation between two fuzzy soft sets given by Definition 2.3. The proof of this result can easily be obtained by related definitions, and so omitted here. To illustrate the notions presented above, let us reconsider the IVFS in Example 3.2.

Example 3.8 (*Reduct Fuzzy Soft Sets of IVFSs*)**.** Compute the PRFS $(\widetilde{F}_-, A)$, ORFS $(\widetilde{F}_+, A)$ and NRFS $(\widetilde{F}_N, A)$ of the interval-valued fuzzy soft set $\mathfrak{I} = (\widetilde{F}, A)$ shown in Table 1.

For the criterion $e_1 \in A$, the corresponding interval fuzzy value set is an interval-valued fuzzy set that can be written as follows:

$$\widetilde{F}(e_1) = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\ [0.3, 0.5] & [0.6, 0.8] & [0.5, 0.6] & [0.2, 0.3] & [0.8, 0.9] & [0.8, 1.0] \end{pmatrix}.$$

By Definitions 3.3–3.5, we have the following fuzzy sets:

$$\widetilde{F}_-(e_1) = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\ 0.3 & 0.6 & 0.5 & 0.2 & 0.8 & 0.8 \end{pmatrix},$$

$$\widetilde{F}_+(e_1) = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\ 0.5 & 0.8 & 0.6 & 0.3 & 0.9 & 1.0 \end{pmatrix},$$

$$\widetilde{F}_N(e_1) = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\ 0.4 & 0.7 & 0.55 & 0.25 & 0.85 & 0.9 \end{pmatrix}.$$

Similarly we can compute other fuzzy sets with respect to other parameters in A. The PRFS $(\widetilde{F}_-, A)$, ORFS $(\widetilde{F}_+, A)$ and NRFS $(\widetilde{F}_N, A)$ of the IVFS $\mathfrak{I} = (\widetilde{F}, A)$ are shown in Tables 2, 3 and 4 respectively.

4. Decision making based on IVFSs

Since fuzzy sets were introduced by Zadeh in 1965, fuzzy set theory has been applied in dealing with fuzzy decision making problems [26,27]. With the development of soft set theory, the application of fuzzy soft sets in solving decision making problems has been discussed by many researchers [22,21,20]. In [18], Yang et al. presented the following algorithm to solve fuzzy decision making problems based on interval-valued fuzzy soft sets. Here some modifications on notations and technical terms of the algorithm have been made to fit the context of our discussion.

Table 3
Optimistic reduct fuzzy soft set of $\mathfrak{I} = (\widetilde{F}, A)$.

U	e_1	e_2	e_3	e_4	e_5	e_6
h_1	0.5	0.7	0.7	0.7	0.5	0.8
h_2	0.8	0.3	1.0	0.3	0.9	1.0
h_3	0.6	0.6	0.4	0.4	0.7	0.5
h_4	0.3	0.4	0.1	0.1	0.3	0.3
h_5	0.9	0.7	0.3	0.3	0.9	0.5
h_6	1.0	1.0	0.6	0.8	0.5	0.6

Table 4
Neutral reduct fuzzy soft set of $\mathfrak{I} = (\widetilde{F}, A)$.

U	e_1	e_2	e_3	e_4	e_5	e_6
h_1	0.40	0.65	0.65	0.65	0.40	0.65
h_2	0.70	0.20	0.90	0.20	0.80	0.90
h_3	0.55	0.55	0.30	0.30	0.60	0.35
h_4	0.25	0.30	0.05	0.05	0.25	0.20
h_5	0.85	0.60	0.20	0.20	0.85	0.35
h_6	0.90	0.90	0.55	0.75	0.35	0.55

Table 5
Interval fuzzy choice values and scores of $\mathfrak{I} = (\widetilde{F}, A)$.

U	h_1	h_2	h_3	h_4	h_5	h_6
c_i	$c_1 = [2.9, 3.9]$	$c_2 = [3.1, 4.3]$	$c_3 = [2.1, 3.2]$	$c_4 = [0.7, 1.5]$	$c_5 = [2.5, 3.6]$	$c_6 = [3.5, 4.5]$
r_i	$r_1 = 5.0$	$r_2 = 8.6$	$r_3 = -4.0$	$r_4 = -22.6$	$r_5 = 0.8$	$r_6 = 12.2$

4.1. Yang et al.'s algorithm based on interval fuzzy choice values

Algorithm 1 ([18]).

1. Input the set of interval-valued fuzzy soft sets (say $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$).
2. Input the parameter set P consisting of preferred decision criteria.
3. Compute the corresponding resultant interval-valued fuzzy soft set $(\widetilde{H}, P)$ from $(\widetilde{F}, A)$ and $(\widetilde{G}, B)$.
4. $\forall h_i \in U$, compute the "interval fuzzy choice value" c_i for each house h_i such that

$$c_i = [c_i^-, c_i^+] = \left[\sum_{p \in P} \widetilde{H}_-(p)(h_i), \sum_{p \in P} \widetilde{H}_+(p)(h_i) \right],$$

where $(\widetilde{H}_-, P)$, $(\widetilde{H}_+, P)$ are PRFS and ORFS of $(\widetilde{H}, P)$ respectively.
5. $\forall h_i \in U$, compute the score r_i of h_i such that

$$r_i = \sum_{h_j \in U} \left((c_i^- - c_j^-) + (c_i^+ - c_j^+) \right).$$

6. The optimal decision is to choose any one of the objects $h_k \in U$ such that $r_k = \max_{h_i \in U} \{r_i\}$.

The use of Algorithm 1 was illustrated in [18] by a concrete example (see Section 4 of [18] for details). Let us simply sketch out that example here. In the example, the corresponding resultant interval-valued fuzzy soft set $(\widetilde{H}, P)$ (obtained in the third step of Algorithm 1) is just the IVFS $\mathfrak{I} = (\widetilde{F}, A)$ shown in Table 1. By Algorithm 1 we compute the interval fuzzy choice value c_i and the score r_i for all $h_i \in U$. The corresponding results are shown in Table 5, from which we see that h_6 is the best choice according to Algorithm 1, since it has the maximum score $r_6 = 12.2$.

4.2. A counterexample

The following example will show that Algorithm 1 may not be successfully applied to some fuzzy decision making problems. Suppose that $\mathfrak{J} = (\widetilde{G}, B)$ is an interval-valued fuzzy soft set and Table 6 gives its tabular representation with choice values and scores. From Table 6, we can see the scores of o_1 and o_2 are the same, namely $r_1 = r_2 = 0$, whence by Algorithm 1 both of them could be selected as the best choice. Nevertheless, in general we should choose o_2 as the optimal alternative because o_2 relatively satisfies more decision criteria than o_1.

4.3. The inherent drawbacks of fuzzy choice value based method

The main problem for Yang et al.'s algorithm stems from the use of interval fuzzy choice values. As it was pointed out by Yang et al. [18], Algorithm 1 is based on the comparison of interval fuzzy choice values of different alternatives; hence the

Table 6
Interval-valued fuzzy soft set $\Im = (\widetilde{G}, B)$ with interval fuzzy choice values and scores.

U	e_1	e_2	e_3	e_4	e_5	Choice value (c_i)	Score (s_i)
o_1	[0.5, 0.7]	[0.5, 0.6]	[0.4, 0.6]	[0.2, 0.3]	[0.3, 0.4]	$c_1 = [1.9, 2.6]$	$r_1 = 0$
o_2	[0.1, 0.2]	[0.6, 0.8]	[0.5, 0.7]	[0.3, 0.5]	[0.3, 0.5]	$c_2 = [1.8, 2.7]$	$r_2 = 0$

optimal decision is to select the alternative with the highest interval fuzzy choice value. Yang et al.'s algorithm can be seen as an interval-valued fuzzy counterpart of the decision making approach in [21] based on "fuzzy choice values" (see the declaration at the beginning of Section 4 in [18]). Actually it is easy to see that in Algorithm 1 score r_i is the total difference in fuzzy choice values of both PRFS (i.e., $\sum_{h_j \in U}(c_i^- - c_j^-)$) and ORFS (i.e., $\sum_{h_j \in U}(c_i^+ - c_j^+)$) between the alternative h_i and all other alternatives $h_j \in U$. Such a fuzzy choice value based method, however, has some inherent drawbacks as pointed out by the authors in [22]. Hence Yang et al.'s algorithm inevitably have some inherent limitations. Thorough analysis regarding the improper use of fuzzy choice values in decision making problems is summarized as follows (see [22] for more details):

Consider the decision making problem based on crisp soft sets in [6]. In this crisp case, choice value of an alternative precisely represents the number of the criteria satisfied by the alternative. Hence the object, say h_k with the maximum choice value c_k should be selected as the optimal alternative. That is, the optimal decision is made due to the fact that

$$c_k = \max_{h_i \in U}\{c_i\} = \max_{h_i \in U} |\{a_j \in A : h_i \models a_j\}|,$$

where $h_i \models a_j$ means "criterion a_j is satisfied by alternative h_i".

The above idea initiated in [6] was not followed in dealing with fuzzy decision making problems. Roy and Maji [20], on the other hand, proposed another method to solve fuzzy soft set based decision making problems. In the case of fuzzy decision making, satisfaction of an alternative with respect to a criterion is gradual and characterized by the membership degree given by the fuzzy soft set under consideration. Since the so-called "fuzzy choice value" given by Kong et al. [21] is just the sum of membership degrees of an alternative with respect to all attributes, fuzzy choice value cannot be interpreted as the number of the criteria satisfied by the alternative. It can only be seen as a synthesized measure to estimate each object via a fusion of all attributes. However, this direct addition of all the membership values with respect to different attributes is not always reasonable; in fact, this is sometimes just like the addition of height and weight.

To cope with the complexity of decision making involving fuzzy soft sets, it would be more reasonable to compare the membership values of two objects with respect to a common attribute so as to determine which one relatively satisfies the criterion better. This is indeed the basic idea underlying the method introduced by Roy and Maji [20].

5. Using level soft sets in fuzzy soft set based decision making

Decision making relies on the evaluation of all available alternatives with respect to certain criteria. Some of these problems are essentially humanistic and thus subjective in nature (e.g. human understanding and vision systems). In general, there does not exist a unique/uniform principle for making the optimal decision. To overcome all the above difficulties, we should use an adjustable framework to solve fuzzy soft set based decision making problems. A proposal of flexible feature was initiated by the authors in [22], using the following novel concept called level soft sets.

5.1. Level soft sets

Definition 5.1 (*[22]*)**.** Let (U, E) be a soft universe, $A \subseteq E$ and $t \in [0, 1]$. The *t-level soft set* of a fuzzy soft set $\mathfrak{S} = (\widetilde{F}, A)$ over U is a soft set $L(\mathfrak{S}; t) = (F_t, A)$ defined by

$$F_t(a) = L(\widetilde{F}(a); t) = \{x \in U : \widetilde{F}(a)(x) \geq t\}, \quad \forall a \in A.$$

Level soft sets can be viewed as "soft generalizations" of classical level (cut) sets in fuzzy set theory. In this notion, $t \in [0, 1]$ serves as a given threshold on membership degrees. For real-life applications, this threshold is chosen by a decision maker; it represents the personal requirement on the level of membership degrees. It is natural to say that an alternative satisfies a criterion if it meets the desirable level required by a decision maker.

Example 5.2. Consider the neutral reduct fuzzy soft set $\mathfrak{S}_N = (\widetilde{F}_N, A)$ (see Table 4) of the IVFS $\Im = (\widetilde{F}, A)$ in Example 3.2. Put $t = 0.5$, let us compute the 0.5-level soft set of the fuzzy soft set $(\widetilde{F}_N, A)$. For the parameter e_1, we should consider the 0.5-level set of the fuzzy set $\widetilde{F}_N(e_1)$, namely the following crisp subset of U:

$$L(\widetilde{F}_N(e_1), 0.5) = \{h_2, h_3, h_5, h_6\},$$

which says that prudently speaking h_2, h_3, h_5, h_6 are beautiful houses at least at the level of membership degrees $t = 0.5$.

Table 7
Level soft set $L(\mathfrak{S}_N; 0.5)$.

U	e_1	e_2	e_3	e_4	e_5	e_6
h_1	0	1	1	1	0	1
h_2	1	0	1	0	1	1
h_3	1	1	0	0	1	0
h_4	0	0	0	0	0	0
h_5	1	1	0	0	1	0
h_6	1	1	1	1	0	1

By taking other parameters into consideration, we obtain the 0.5-level sets of the fuzzy sets $\widetilde{F}(e_i)$ $(2 \le i \le 6)$ as follows:

$L(\widetilde{F}_N(e_2), 0.5) = \{h_1, h_3, h_5, h_6\}$,
$L(\widetilde{F}_N(e_3), 0.5) = \{h_1, h_2, h_6\}$,
$L(\widetilde{F}_N(e_4), 0.5) = \{h_1, h_6\}$,
$L(\widetilde{F}_N(e_5), 0.5) = \{h_2, h_3, h_5\}$,
$L(\widetilde{F}_N(e_6), 0.5) = \{h_1, h_2, h_6\}$.

Then the 0.5-level soft set of $\mathfrak{S}_N$ is a crisp soft set $L(\mathfrak{S}_N; 0.5) = (F_{0.5}, A)$ over U, where $F_{0.5} : A \to \mathscr{P}(U)$ is a set-valued mapping defined by $F_{0.5}(e_i) = L(\widetilde{F}_N(e_i), 0.5)$ for all $e_i \in A$. Table 7 gives the tabular representation of the level soft set $L(\mathfrak{S}_N; 0.5)$.

In Definition 5.1, the level (or threshold) assigned to each parameter is always a constant value $t \in [0, 1]$. In many decision making problems, however, different thresholds should be imposed on different decision parameters. To address this issue, we need the following notion.

Definition 5.3 (*[22]*)**.** Let (U, E) be a soft universe and $A \subseteq E$. Let $\mathfrak{S} = (\widetilde{F}, A)$ be a fuzzy soft set over U and let $\lambda : A \to [0, 1]$ be a fuzzy set in A called a *threshold fuzzy set*. The level soft set of $\mathfrak{S} = (\widetilde{F}, A)$ with respect to λ is a soft set $L(\mathfrak{S}; \lambda) = (F_\lambda, A)$ defined by

$$F_\lambda(a) = L(\widetilde{F}(a); \lambda(a)) = \{x \in U : \widetilde{F}(a)(x) \ge \lambda(a)\}, \quad \forall a \in A.$$

In other words, a function instead of a constant number is considered as the threshold on membership values. Let $\hat{t}$ denote the constant fuzzy set in A given by $\hat{t}(a) = t$ for all $a \in A$. Evidently it holds that $L(\mathfrak{S}; \hat{t}) = L(\mathfrak{S}; t)$, whence t-level soft sets can also be viewed as level soft sets with respect to $\hat{t}$.

Definition 5.4 (*[28]*)**.** An *aggregation operator* is a function $F : \bigcup_{n\in\mathbb{N}}[0, 1]^n \to [0, 1]$ such that:

(1) $F(x_1, \ldots, x_n) \le F(y_1, \ldots, y_n)$ whenever $x_i \le y_i, \ \forall i$;
(2) $F(t) = t, \ \forall t \in [0, 1]$;
(3) $F(0, \ldots, 0) = 0, F(1, \ldots, 1) = 1$.

Each aggregation operator F can be represented by a family of n-ary operators $f_n : [0, 1]^n \to [0, 1]$ given by $f_n(x_1, \ldots, x_n) = F(x_1, \ldots, x_n)$. We can relate aggregation operators to level soft sets in the following way:

Definition 5.5. Let (U, E) be a soft universe with $U = \{x_1, \ldots, x_n\}$ and $A \subseteq E$. Let $\mathfrak{S} = (\widetilde{F}, A)$ be a fuzzy soft set over U and let $G : \bigcup_{n\in\mathbb{N}}[0, 1]^n \to [0, 1]$ be an aggregation operator. The level soft set of $\mathfrak{S}$ with respect to G is a soft set $L(\mathfrak{S}; G) = (F_G, A)$ defined by

$$F_G(a) = \{x_i \in U : \widetilde{F}(a)(x_i) \ge G(\widetilde{F}(a)(x_1), \ldots, \widetilde{F}(a)(x_n))\}, \quad \forall a \in A.$$

By considering different aggregation operators we can obtain a wide class of formulations for the level soft set in Definition 5.5. For instance, if $G =$ mid, namely the arithmetic mean, then this aggregation operator combined with the fuzzy soft set $\mathfrak{S} = (\widetilde{F}, A)$ can induce a fuzzy set $\widetilde{\mathrm{mid}}_{\mathfrak{S}} : A \to [0, 1]$ given by

$$\widetilde{\mathrm{mid}}_{\mathfrak{S}}(a) = \mathrm{mid}(\widetilde{F}(a)(x_1), \ldots, \widetilde{F}(a)(x_n)) = \frac{1}{n}\sum_{i=1}^{n}\widetilde{F}(a)(x_i), \quad \forall a \in A.$$

Now it follows that $L(\mathfrak{S}; \mathrm{mid}) = L(\mathfrak{S}; \widetilde{\mathrm{mid}}_{\mathfrak{S}})$; this is called the *mid-level soft set* of $\mathfrak{S}$ in [22].

Similarly, if $G = \max$, we can define a fuzzy set $\widetilde{\max}_{\mathfrak{S}} : A \to [0, 1]$ by

$$\widetilde{\max}_{\mathfrak{S}}(a) = \max_{1\le i\le n} \widetilde{F}(a)(x_i), \quad \forall a \in A.$$

The level soft set of $\mathfrak{S}$ with respect to $\widetilde{\max}_{\mathfrak{S}}$ (or equivalently max) is called the *top-level soft set* of $\mathfrak{S}$ in [22].

Note that by *mid-level decision rule* we shall mean considering the aggregation operator "mid" and the mid-level soft set in decision making; in a similar way, *top-level decision rule* indicates that the top-level soft set will be used for making the optimal decision [22].

5.2. An adjustable approach using level soft sets

The following algorithm was proposed by the authors in [22] for handling fuzzy soft set based decision making problems.

Algorithm 2 (*[22]*).

1. Input the (resultant) fuzzy soft set $\mathfrak{S} = (\widetilde{F}, A)$.
2. Input a threshold fuzzy set $\lambda : A \to [0, 1]$ (or give a threshold value $t \in [0, 1]$; or choose the mid-level decision rule; or choose the top-level decision rule) for decision making.
3. Compute the level soft set $L(\mathfrak{S}; \lambda)$ of $\mathfrak{S}$ with respect to the threshold fuzzy set λ (or the t-level soft set $L(\mathfrak{S}; t)$; or the mid-level soft set $L(\mathfrak{S}; \mathrm{mid})$; or the top-level soft set $L(\mathfrak{S}; \max)$).
4. Present the level soft set $L(\mathfrak{S}; \lambda)$ (or $L(\mathfrak{S}; t)$; or $L(\mathfrak{S}; \mathrm{mid})$; or $L(\mathfrak{S}; \max)$) in tabular form and compute the choice value c_i of o_i, $\forall i$.
5. The optimal decision is to select o_k if $c_k = \max_i c_i$.
6. If k has more than one value then any one of o_k may be chosen.

Notice that in the last step of the above algorithm, one may go back to the second step and change the threshold (or decision rule) that he once used so as to adjust the final optimal decision, especially when there are too many "optimal choices".

The basic idea behind Algorithm 2 is to solve fuzzy soft set based decision making problems by using level soft sets initiated in [22]. This approach is completely different from the methods in [20,21]. Level soft sets serve as bridges between fuzzy soft sets and crisp soft sets. By considering level soft sets, actually we need not work directly on fuzzy soft sets in decision making, but only deal with the crisp soft sets derived from them after choosing certain thresholds. Hence in this case, the problem concerning reasonableness of "fuzzy choice values" simply does not arise.

In other words, the choice value of an alternative in the level soft set represents the number of the criteria satisfied by the alternative *at certain level of membership degrees*. Hence the optimal decision will be made according to

$$c_k = \max_{h_i \in U}\{c_i\} = \max_{h_i \in U} |\{a_j \in A : h_i \mid\!\approx_\lambda a_j\}|,$$

where c_i is the choice value in the level soft set $L(\mathfrak{S}; \lambda)$ and $h_i \mid\!\approx_\lambda a_j$ means "criterion a_j is satisfied by alternative h_i at the level given by the threshold fuzzy set λ".

Remarks on advantages of Algorithm 2 as well as some examples to illustrate its application to fuzzy decision making can be found in [22]. Note also that this method can be easily extended to the case where each of the criteria has an importance weight (see Section 5 of [22]).

6. Flexible schemes for decision making based on IVFSs

The wide variety of possible relationships among the alternatives in decision making problems motivates our interest in seeking flexible/adjustable methods that can be used to model these various possibilities. In this section we shall propose flexible schemes for decision making based on interval-valued fuzzy soft sets. Actually we shall show that by considering appropriate reduct fuzzy soft sets and level soft sets, IVFS based decision making can be reduced to much simpler treatment of only crisp soft sets.

Algorithm 3. 1. Input the (resultant) interval-valued fuzzy soft set $\mathfrak{I} = (\widetilde{F}, A)$.
2. Input an opinion weighting vector $W = (\alpha, \beta)$ and compute the WRFS $\mathfrak{S}_W = (\widetilde{F}_W, A)$ of the IVFS $\mathfrak{I} = (\widetilde{F}, A)$ with respect to W.
3. Input a threshold fuzzy set $\lambda : A \to [0, 1]$ (or give a threshold value $t \in [0, 1]$; or choose an aggregation operator G).
4. Compute the level soft set $L(\mathfrak{S}_W; \lambda)$ of $\mathfrak{S}_W$ with respect to the threshold fuzzy set λ (or the t-level soft set $L(\mathfrak{S}_W; t)$; or the level soft set $L(\mathfrak{S}_W; G)$).
5. Present the level soft set $L(\mathfrak{S}_W; \lambda)$ (or $L(\mathfrak{S}_W; t)$; or $L(\mathfrak{S}_W; G)$) in tabular form and compute the choice value c_i of o_i, $\forall i$.
6. The optimal decision is to select o_k if $c_k = \max_i c_i$.
7. If k has more than one value then any one of o_k may be chosen.

Remark 6.1. It should be pointed out that the opinion weighting vector is given by the user (decision maker). Note also that the opinion weighting vector could be adjusted by the user (decision maker) interactively to fulfill the real needs better. For instance, if in the last step there are too many "optimal choices", then one can go back to Step 2 (or Step 3) to modify opinion weighting vector (or threshold/aggregation operator) so as to focus on a smaller set of optimal choices.

In essence, the mentioned algorithm simply represents the following idea: fuzzy decision making problems based on IVFSs can efficiently be solved by considering appropriate reduct fuzzy soft sets and level soft sets. By using Algorithm 3, in

Table 8
Typical schemes for interval-valued fuzzy soft set based decision making.

Scheme name	Reduct fuzzy soft set	Threshold fuzzy set	Level soft set
Opt-Mid	ORFS $\mathfrak{S}_+ = (\widetilde{F}_+, A)$	$\widetilde{\mathrm{mid}}_{\mathfrak{S}_+}$	$L(\mathfrak{S}_+, \mathrm{mid})$
Opt-Top	ORFS $\mathfrak{S}_+ = (\widetilde{F}_+, A)$	$\widetilde{\max}_{\mathfrak{S}_+}$	$L(\mathfrak{S}_+, \max)$
Neu-Mid	NRFS $\mathfrak{S}_N = (\widetilde{F}_N, A)$	$\widetilde{\mathrm{mid}}_{\mathfrak{S}_N}$	$L(\mathfrak{S}_N, \mathrm{mid})$
Neu-Top	NRFS $\mathfrak{S}_N = (\widetilde{F}_N, A)$	$\widetilde{\max}_{\mathfrak{S}_N}$	$L(\mathfrak{S}_N, \max)$
Pes-Mid	PRFS $\mathfrak{S}_- = (\widetilde{F}_-, A)$	$\widetilde{\mathrm{mid}}_{\mathfrak{S}_-}$	$L(\mathfrak{S}_-, \mathrm{mid})$
Pes-Top	PRFS $\mathfrak{S}_- = (\widetilde{F}_-, A)$	$\widetilde{\max}_{\mathfrak{S}_-}$	$L(\mathfrak{S}_-, \max)$

Table 9
Level soft set $L(\mathfrak{S}_-, \max)$ with choice values.

U	e_1	e_2	e_3	e_4	e_5	e_6	Choice value (c_i)
h_1	0	0	0	0	0	0	$c_1 = 0$
h_2	0	0	1	0	0	1	$c_2 = 2$
h_3	0	0	0	0	0	0	$c_3 = 0$
h_4	0	0	0	0	0	0	$c_4 = 0$
h_5	1	0	0	0	1	0	$c_5 = 2$
h_6	1	1	0	1	0	0	$c_6 = 3$

Table 10
Optimistic reduct fuzzy soft set $\mathfrak{T}_+ = (\widetilde{G}_+, B)$ of $\mathfrak{J} = (\widetilde{G}, B)$.

U	e_1	e_2	e_3	e_4	e_5
o_1	0.7	0.6	0.6	0.3	0.4
o_2	0.2	0.8	0.7	0.5	0.5

decision making process we actually do not work directly on interval-valued fuzzy soft sets, but only need to deal with the related reduct fuzzy soft sets and finally the crisp level soft sets derived from them after choosing certain thresholds. This makes our algorithm simpler in computational complexity and thus easier for application in real-life problems.

Great flexibility and modelling power of Algorithm 3 lies in the fact that there are a large variety of opinion weighting vectors (reduct fuzzy soft sets), thresholds as well as aggregation operators that can be used to find the optimal alternatives. Some special cases are worth noting: Table 8 gives some typical schemes that arise from Algorithm 3 by considering the ORFS (or NRFS; or PRFS) of an IVFS $\mathfrak{J} = (\widetilde{F}, A)$ combined with the aggregation operator $G = \mathrm{mid}$ (or $G = \max$). Notice also that these typical schemes can further be combined to give out more complex hybrid schemes. For instance, one may consider the hybrid scheme "Opt-Top AND Pes-Top" in which the level soft set $L(\mathfrak{S}_+, \max) \Cap L(\mathfrak{S}_-, \max)$ will be used for decision making. Here $\Cap$ denotes the bi-intersection of soft sets defined in [10].

To illustrate the use of Algorithm 3, let us reconsider the application of IVFSs in [18] (see also Example 3.2 in Section 3). The IVFS under consideration is $\mathfrak{J} = (\widetilde{F}, A)$ as in Table 1. Assume that the decision maker is very prudent and he intends to select the object fulfilling the highest standard in most aspects as his optimal choice. Hence in this case it will be very reasonable for us to use the "Pes-Top" scheme shown in Table 8 to solve this fuzzy decision making problem. According to the scheme we choose, at first the pessimistic reduct fuzzy soft set of $\mathfrak{J}$ should be considered, namely the fuzzy soft set $\mathfrak{S}_- = (\widetilde{F}_-, A)$ as in Table 2. Next, we shall consider the aggregation operator $G = \max$, which induces a threshold fuzzy set as follows:

$$\widetilde{\max}_{\mathfrak{S}_-} = \begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ 0.8 & 0.8 & 0.8 & 0.7 & 0.8 & 0.8 \end{pmatrix}.$$

Then we can obtain the top-level soft set of $\mathfrak{S}_-$, namely the crisp soft set $L(\mathfrak{S}_-, \max)$ shown in Table 9. From the choice values also listed in Table 9, we shall select h_6 as the best house since it has the maximum choice value $c_6 = 3$. Although this result coincides with what was obtained by using Algorithm 1 in [18], it is clear that the calculation is greatly simplified by using Algorithm 3 (see also Table 5 and the discussion in Section 4.1).

Let us now use Algorithm 3 to handle the decision making problem mentioned in Section 4.2, which cannot be successfully solved by Algorithm 1. Thus the IVFS under consideration is $\mathfrak{J} = (\widetilde{G}, B)$ shown in Table 6. Furthermore we shall assume that the "Opt-Top" scheme is to be used here. Hence we should first consider the ORFS of $\mathfrak{J}$, i.e., $\mathfrak{T}_+ = (\widetilde{G}_+, B)$ as in Table 10. It is easily seen that the aggregation operator $G = \max$ will induce a threshold fuzzy set as below:

$$\widetilde{\max}_{\mathfrak{T}_+} = \begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \\ 0.7 & 0.8 & 0.7 & 0.5 & 0.5 \end{pmatrix}.$$

Then the corresponding top-level soft set $L(\mathfrak{T}_+, \max)$ is shown in Table 11. From the choice values listed in Table 11, we can see that o_2 should be selected as the optimal alternative; this is just what we shall get by common-sense reasoning.

Table 11
Level soft set $L(\mathfrak{T}_+, \max)$ with choice values.

U	e_1	e_2	e_3	e_4	e_5	Choice value (c_i)
o_1	1	0	0	0	0	$c_1 = 1$
o_2	0	1	1	1	1	$c_2 = 4$

Table 12
Weighted fuzzy soft set $\mathfrak{S}_1 = (\widetilde{F}_1, A, f_w)$.

U	$e_1, w_1 = 0.5$	$e_2, w_2 = 0.7$	$e_3, w_3 = 0.8$
x_1	0.4	0.7	0.6
x_2	0.2	0.8	0.4
x_3	0.1	0.5	0.5
x_4	0.6	0.4	0.6

Table 13
Weighted fuzzy soft set $\mathfrak{S}_2 = (\widetilde{F}_2, A, f_w)$.

U	$e_1, w_1 = 0.5$	$e_2, w_2 = 0.7$	$e_3, w_3 = 0.8$
x_1	0.4	0.8	0.1
x_2	0.6	0.4	0.2
x_3	0.5	0.9	0.1
x_4	0.3	0.6	0.4

7. WIVFS based decision making

In the preceding section we have investigated the application of interval-valued fuzzy soft sets in decision making problems. A further representational capability can be added by associating with each parameter e_j a value $w_j \in [0, 1]$ called its weight. In the case of multi-criteria decision making, these weights can be used to represent the different importance of the concerned criteria. The following concept could provide a mathematical framework for modelling the IVFS based decision making problems in which all the decision criteria may not be of equal importance.

Definition 7.1. Let (U, E) be a soft universe and $A \subseteq E$. A *weighted interval-valued fuzzy soft set* (WIVFS) over U is a triple $\mathcal{W} = (\widetilde{F}, A, f_w)$ where $(\widetilde{F}, A)$ is an interval-valued fuzzy soft set over U, and $f_w : A \to [0, 1]$ is a *weight function* specifying the weight $w_j = f_w(e_j)$ for each parameter $e_j \in A$.

Recall that in [6] the *weighted table* of a crisp soft set was defined to have entries $d_{ij} = w_j \times h_{ij}$ instead of 0 and 1 only, where h_{ij} are the entries in the tabular representation of the crisp soft set and w_j are weights. The *weighted choice value* $\overline{c}_i$ of an object o_i is calculated as $\overline{c}_i = \sum_j d_{ij}$.

The flexible decision making schemes established in the preceding section can now be easily extended to deal with the case where each of the criteria has an importance weight. In response to this new framework weighted choice values should be used as substitutes for (ordinary) choice values in the process of decision making. It is easy to see that a revised version of Algorithm 3 for WIVFS based decision making can be obtained in a natural way. This idea will be illustrated by the following example:

Example 7.2. Assume that $U = \{x_1, x_2, x_3, x_4\}$ is the universe consisting of four machines. Suppose the parameter set $A = \{e_1, e_2, e_3\}$, i.e., we have three criteria to evaluate the performance of these machines. Suppose a firm wants to buy one such machine depending on its performance; hence the decision maker of the firm wants to buy a machine which satisfies the criteria in A to the utmost extent. Furthermore we shall assume that each of these criteria is associated with a weight w_i indicating its importance considered by the decision maker. Here we shall let $w_1 = 0.5$, $w_2 = 0.7$ and $w_3 = 0.8$.

Let there be two specialists $\{S_1, S_2\}$ to evaluate the performance of these machines for the firm. It is possible that their evaluation results to the same criterion are not the same to one another. Their evaluation results can be formulated into two weighted fuzzy soft sets $\mathfrak{S}_1 = (\widetilde{F}_1, A, f_w)$ and $\mathfrak{S}_2 = (\widetilde{F}_2, A, f_w)$ with tabular representations shown in Tables 12 and 13, respectively. Combining two independent evaluation results we can obtain a weighted interval-valued fuzzy soft set $\mathcal{W} = (\widetilde{F}, A, f_w)$ shown in Table 14.

Here we assume that the decision maker will use the "Neu-Mid" scheme in Table 8 to determine the best machine. Hence we should first consider the NRFS of $\mathcal{W}$, i.e., a weighted fuzzy soft set $\mathfrak{W}_N = (\widetilde{F}_N, A, f_w)$ shown in Table 15. The aggregation operator $G = \text{mid}$ will induce a threshold fuzzy set as below:

$$\widetilde{\text{mid}}_{\mathfrak{W}_N} = \begin{pmatrix} e_1 & e_2 & e_3 \\ 0.3875 & 0.6375 & 0.3625 \end{pmatrix}.$$

Then the corresponding mid-level soft set $L(\mathfrak{W}_N, \text{mid})$ is shown in Table 16. Then the firm will buy machine x_4 since it has the highest weighted choice value.

Table 14
Weighted interval-valued fuzzy soft set $\mathcal{W} = (\widetilde{F}, A, f_w)$.

U	$e_1, w_1 = 0.5$	$e_2, w_2 = 0.7$	$e_3, w_3 = 0.8$
x_1	[0.4, 0.4]	[0.7, 0.8]	[0.1, 0.6]
x_2	[0.2, 0.6]	[0.4, 0.8]	[0.2, 0.4]
x_3	[0.1, 0.5]	[0.5, 0.9]	[0.1, 0.5]
x_4	[0.3, 0.6]	[0.4, 0.6]	[0.4, 0.6]

Table 15
Neutral reduct fuzzy soft set $\mathfrak{W}_N = (\widetilde{F}_N, A, f_w)$ of $\mathcal{W} = (\widetilde{F}, A, f_w)$.

U	$e_1, w_1 = 0.5$	$e_2, w_2 = 0.7$	$e_3, w_3 = 0.8$
x_1	0.40	0.75	0.35
x_2	0.40	0.60	0.30
x_3	0.30	0.70	0.30
x_4	0.45	0.50	0.50

Table 16
Level soft set $L(\mathfrak{W}_N, \text{mid})$ with weighted choice values.

U	$e_1, w_1 = 0.5$	$e_2, w_2 = 0.7$	$e_3, w_3 = 0.8$	Weighted choice value ($\bar{c}_i$)
x_1	1	1	0	$\bar{c}_1 = 1.2$
x_2	1	0	0	$\bar{c}_2 = 0.5$
x_3	0	1	0	$\bar{c}_3 = 0.7$
x_4	1	0	1	$\bar{c}_4 = 1.3$

8. Conclusion

This paper can be viewed as a continuation of the study of Feng et al. [22]. Here we have further discussed the application of interval-valued fuzzy soft sets to solve decision making problems. We contributed to this research direction by proposing flexible schemes for decision making based on (weighted) interval-valued fuzzy soft sets. In fact, we have shown that by considering appropriate reduct fuzzy soft sets and level soft sets, IVFS based decision making can be reduced to the much simpler treatment of crisp soft sets; at the same time a wide variety of reduct fuzzy soft sets and level soft sets (available in the decision process) could result in great flexibility and modelling power as well. Hence our algorithm is simpler in computational complexity and thus easier to be applied in real-life applications.

To extend this work, one may possibly use (weighted) interval-valued fuzzy soft sets to address multi-criteria group decision making problems. Moreover, it is interesting to further investigate level soft set approach to decision making based on other extensions of fuzzy soft set theory.

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