Indag. Mathem., N.S., 19 (1), 135–150 March, 2008

On extensions of triangular norms on bounded lattices

by Susanne Saminger-Platz[a], Erich Peter Klement[a] and Radko Mesiar[b,c]

[a] *Department of Knowledge-Based Mathematical Systems, Johannes Kepler University, A-4040 Linz, Austria*
[b] *Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, Bratislava, Slovakia*
[c] *Institute for Research and Applications of Fuzzy Modeling, University of Ostrava, Czech Republic*

Communicated by Prof. M.S. Keane

ABSTRACT

Smallest and largest possible extensions of triangular norms on bounded lattices are discussed. As such ordinal and horizontal sum like constructions for t-norms on bounded lattices are investigated. Necessary and sufficient conditions for the lattice guaranteeing that the extension is again a t-norm are revealed.

1 INTRODUCTION

Many-valued logics are usually based on a bounded lattice $(L, \leqslant, 0, 1)$ of truth values [17,18,25,31,36,37], not necessarily being a chain (a first attempt in this direction is described in [17, Section 15.2], compare [4,12] and also the *para-consistent logic* in [8]). In such a case, the conjunction is interpreted by some triangular norm on L. The structure of t-norms (fulfilling the intermediate value property) is known in some special cases only (closed real intervals, especially the unit interval, finite chains), see [3,21]. In this paper we are interested in the problem of extending a t-norm acting on a (complete) sublattice of L to a t-norm acting on L, discussing the largest and smallest possible extensions. Although in many of the before mentioned cases the lattices involved tend to be distributive we will not make any additional assumptions on the lattice structure except for its boundedness.

Let $(L, \leqslant, 0, 1)$ be a bounded lattice. An operation $T : L^2 \to L$ which turns L into an ordered abelian semigroup with neutral element 1 will be called a *triangular norm* or, briefly, a *t-norm* on L [10]. In fact, (T, L) is a commutative integral

E-mails: susanne.saminger-platz@jku.at (S. Saminger-Platz), ep.klement@jku.at (E.P. Klement), mesiar@math.sk (R. Mesiar).

l-monoid [20] (compare also Examples 1.1–1.4 of commutative semigroups in [16]) if and only if $T : L^2 \to L$ is a triangular norm on L additionally fulfilling $T(x, y \vee z) = T(x, y) \vee T(x, z)$ for all $x, y, z \in L$.

Note that the structure of the lattice L heavily influences which and how many t-norms on L can be defined. However, on each bounded lattice L with $|L| > 2$ there are at least two t-norms, the minimum $\wedge$ and the drastic product $T_{\mathbf{D}}{}^L$ defined by

$$T_{\mathbf{D}}{}^L(x, y) = \begin{cases} x \wedge y & \text{if } 1 \in \{x, y\}, \\ 0 & \text{otherwise,} \end{cases}$$

which are also the greatest and smallest t-norms on the lattice L (if $|L| = 2$ then $\wedge$ and $T_{\mathbf{D}}{}^L$ coincide with the standard boolean conjunction).

Now consider a bounded sublattice $(S, \leqslant, a, b)$ of L and a t-norm $T^S : S^2 \to S$ on S. We are investigating the strongest and weakest possible extension of T^S leading to a t-norm T on the lattice L.

Inspired by ideas of Clifford [7] (in the context of ordinal sums of abstract semigroups) and [14,24,29,34,35] (ordinal sums of t-norms on the unit interval), define the binary operation $T^L_{T^S} : L^2 \to L$ by

$$(1.1) \qquad T^L_{T^S}(x, y) = \begin{cases} T^S(x, y) & \text{if } (x, y) \in S^2, \\ x \wedge y & \text{otherwise.} \end{cases}$$

More recently, similar constructions (*towers of irreducible hoops* [1,6]) have been applied to characterize BL-chains [18]. Evidently, $T^L_{T^S}$ is an extension of T^S. Moreover, if $T^L_{T^S}$ is a t-norm then it clearly is the strongest t-norm extending T^S.

In the following sections we shall investigate under which conditions, starting from an arbitrary t-norm T^S on some sublattice S, the extension $T^L_{T^S}$ always is a t-norm on L. We will show that the arbitrariness of the choice of T^S on S, for $T^L_{T^S}$ to be always a t-norm on L, leads to some restrictions on the structure of the sublattice S. As a consequence also to restrictions on the structure of L, in case that not only any choice of T^S but also any choice of S shall be admissible. Based on these results we further discuss the strongest extension of families of arbitrary t-norms on some corresponding families of arbitrary sublattices and a few further properties of triangular norms. Finally, we turn to the determination of the smallest possible extension $W^L_{T^S}$ of a t-norm T^S on a bounded and complete sublattice S.

2 S AND L WITH COMMON BOTTOM AND TOP ELEMENTS

Fix a bounded lattice $(L, \leqslant, 0, 1)$ and consider a bounded sublattice $(S, \leqslant, a, b)$ of L and a t-norm $T^S : S^2 \to S$ on S. Obviously, $T^L_{T^S}$ as defined by (1.1) is commutative and has neutral element 1. Since S is also a sublattice of $([a, b], \leqslant, a, b)$ with $[a, b] = \{x \in L \mid a \leqslant x \leqslant b\}$, we have

$$T^L_{T^S} = T^L_{T^{[a,b]}_{T^S}}.$$

i.e., we may first extend T^S to $[a,b]$ via (1.1) and repeat the same procedure to extend $T_{T^S}^{[a,b]}$ to L. Because of

$$T_{T^S}^{[a,b]} = T_{T^S}^{L}|_{[a,b]^2},$$

a necessary condition for $T_{T^S}^L$ to be a t-norm is that $T_{T^S}^{[a,b]}$ is a t-norm. Therefore, without loss of generality we may restrict ourselves first to sublattices of L having the same bottom and top element as L.

Proposition 2.1. *Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $(S, \leqslant, 0, 1)$ a sublattice of L. The following are equivalent:*

(i) *For all $(x, y) \in (S \setminus \{1\}) \times (L \setminus S)$ we have $x \wedge y \in \{0, x\}$ and for all $(x, y) \in (L \setminus S)^2$ it holds that $x \wedge y \in S \Rightarrow x \wedge y = 0$.*
(ii) *For each t-norm $T^S : S^2 \to S$ on S, the operation $T_{T^S}^L$ is a t-norm on L.*

Proof. To show necessity assume that condition (i) is fulfilled. It is immediate to see that $T_{T^S}^L$ defined by (1.1) is commutative and has neutral element 1. Since for each t-norm T we additionally have $T(x, y) \leqslant x \wedge y$, for the monotonicity of $T_{T^S}^L$ it suffices to check if $T_{T^S}^L(x, y) \leqslant T_{T^S}^L(x^*, y)$ for $x \leqslant x^*$ in case $x \notin S$, $x^* \in S$, and $y \in S \setminus \{1\}$.

If $x^* \neq 1$, then, because of condition (i), $x^* \wedge x = x \in \{0, x^*\} \subset S$, contradicting the assumption $x \notin S$. Therefore, $x^* = 1$, and we can conclude $T_{T^S}^L(x, y) = x \wedge y \in \{0, y\}$, $T_{T^S}^L(x^*, y) = T^S(x^*, y) = y$, and, obviously, $T_{T^S}^L(x, y) \leqslant y = T_{T^S}^L(x^*, y)$.

For proving the associativity, i.e., $T(x, T(y, z)) = T(T(x, y), z)$, it is obvious that it holds whenever either all $x, y, z \in S$ or all $x, y, z \in L \setminus S$ as well as if $0 \in \{x, y, z\}$ or $1 \in \{x, y, z\}$. Therefore, let us first assume that $x, y \notin S$ and $z \in S \setminus \{0, 1\}$. Then, $x \wedge z \in \{0, z\}$, $y \wedge z \in \{0, z\}$, and if $x \wedge y \in S$ then $x \wedge y = 0$ such that in all cases it follows

$$T\big(x, T(y, z)\big) = x \wedge y \wedge z = T\big(T(x, y), z\big).$$

Similar arguments can be applied in case $x, z \notin S$ and $y \in S \setminus \{0, 1\}$ resp. $y, z \notin S$ and $x \in S \setminus \{0, 1\}$.

In case that only one element involved is element of the sublattice, let us first assume that $x \notin S$ and $y, z \in S \setminus \{0, 1\}$, then $x \wedge y \in \{0, y\}$, $x \wedge z \in \{0, z\}$, $y \wedge z \in S \setminus \{1\}$, and $x \wedge T(y, z) \in \{0, T(y, z)\}$. Then the following can be argued: If $x \wedge y \wedge z = 0$ then associativity is trivially fulfilled. Otherwise, if $x \wedge y \wedge z = y \wedge z > 0$, such that $T(x, y) = y$ and therefore $T(T(x, y), z) = T(y, z)$ and $T(x, T(y, z)) = x \wedge T(y, z) = T(y, z)$ since $T(y, z) \leqslant y = x \wedge y \leqslant x$. Analogous arguments can be applied for proving the case $z \notin S$ and $x, y \in S \setminus \{0, 1\}$. Finally, it remains to show associativity for $y \notin S$ and $x, z \in S \setminus \{0, 1\}$. If $x \wedge y \wedge z = 0$, then again it is trivially fulfilled. Otherwise, necessarily $x \wedge y = x$ and $y \wedge z = z$, such that

$$T\big(x, T(y, z)\big) = T(x, y \wedge z) = T(x, z) = T(x \wedge y, z) = T\big(T(x, y), z\big).$$

Conversely, assume that $T^L_{T^S}$ is t-norm for each t-norm T^S on S and fix some $(x, y) \in (S \setminus \{1\}) \times (L \setminus S)$ such that $x \wedge y \notin \{0, x\}$.

If $x \wedge y \in S$ consider the t-norm T^S on S given by

$$T^S(u, v) = \begin{cases} 0 & \text{if } (u, v) \in ([0, x] \cap S)^2 \setminus \{(x, x)\}, \\ u \wedge v & \text{otherwise,} \end{cases}$$

and we obtain $T^L_{T^S}(T^L_{T^S}(x, x), y) = x \wedge y \neq 0 = T^L_{T^S}(x, T^L_{T^S}(x, y))$.

If $x \wedge y \notin S$ then

$$T^L_{T_\mathbf{D}^S}\big(T^L_{T_\mathbf{D}^S}(x, x), y\big) = 0 \neq x \wedge y = T^L_{T_\mathbf{D}^S}\big(x, T^L_{T_\mathbf{D}^S}(x, y)\big).$$

Moreover, fix some $(x, y) \in (L \setminus S)^2$ such that $x \wedge y = z \in S \setminus \{0\}$. Then

$$\begin{aligned} T^L_{T_\mathbf{D}^S}\big(T^L_{T_\mathbf{D}^S}(x, y), z\big) &= T^L_{T_\mathbf{D}^S}(z, z) \\ &= 0 < z = T^L_{T_\mathbf{D}^S}(x, z) = T^L_{T_\mathbf{D}^S}\big(x, T^L_{T_\mathbf{D}^S}(y, z)\big). \end{aligned}$$

Since in all cases the associativity is violated, this proves that (ii) implies (i). □

Note that condition (i) equivalently expresses that for all $x \in S \setminus \{1\}$ and for all $y \in L \setminus S$ either $x \wedge y = 0$ or $x \leqslant y$ is fulfilled and for all $x \in S \setminus \{0, 1\}$ and all $y, z \in L \setminus S$, such that $x \leqslant y$ and $x \leqslant z$, also $y \wedge z \in L \setminus S$.

3 EXTENSION OF T-NORMS ON AN ARBITRARY INTERVAL

First consider a fixed subinterval $[a, b]$ of a bounded lattice $(L, \leqslant, 0, 1)$ and an arbitrary t-norm $T^{[a,b]}$ on $[a, b]$. We want to check under which conditions on the interval $[a, b]$ (and on the lattice L) the operation $T^L_{T^{[a,b]}}$ constructed by (1.1) will be a t-norm on L (see also Theorem 4.8 in [33]). Recall that the open interval $]a, b[$ is defined by $[a, b] \setminus \{a, b\}$. Moreover, if $]a, b[= \emptyset$, then $T^{[a,b]} = \wedge$ and also $T^L_{T^{[a,b]}} = \wedge$ clearly being a t-norm on L, so without loss of generality we can restrict in the sequel to subintervals $[a, b]$ with $]a, b[\neq \emptyset$ only.

Proposition 3.1. *Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $[a, b]$ a subinterval of L with $]a, b[\neq \emptyset$. The following are equivalent:*

(i) $\{x \in L \mid \exists y \in]a, b[: x \leqslant y \text{ or } x \geqslant y\} = [0, a] \cup [a, b] \cup [b, 1]$.
(ii) *For each t-norm $T^{[a,b]} : [a, b]^2 \to [a, b]$ on $[a, b]$, the operation $T^L_{T^{[a,b]}}$ is a t-norm on L.*

Proof. Note that condition (i) expresses that whenever some lattice element x, not necessarily from $[a, b]$, is comparable to an interior element of the subinterval, then it must be comparable to both boundaries of the subinterval, i.e., to a as well as to b.

Now assume that condition (i) is fulfilled. For the monotonicity of $T^L_{T^{[a,b]}}$ it suffices to check the case $x \notin [a, b]$, $\{y, z\} \subseteq [a, b]$ and $x \leqslant y$. If $x < a$ then

$$T^L_{T^{[a,b]}}(x, z) = x \wedge z \leqslant a \wedge z = a \leqslant T^{[a,b]}(y, z) = T^L_{T^{[a,b]}}(y, z).$$

If x and a are incomparable, then $x \notin [0, a] \cup [a, b] \cup [b, 1]$, i.e., x is incomparable to all $u \in [a, b[$, such that $y = b$ and

$$T^L_{T^{[a,b]}}(x, z) = x \wedge z \leqslant z = T^{[a,b]}(y, z) = T^L_{T^{[a,b]}}(y, z).$$

Similarly, the associativity of $T^L_{T^{[a,b]}}$ can be checked case by case. We illustrate the case of $x \in L$ being incomparable to a and $y, z \in [a, b]$. We prove the associativity for this case by a series of properties:

Since x is incomparable to a, it is incomparable to all $u \in [a, b[$ and therefore it follows that, necessarily $x \wedge v \notin [a, b]$ for all $v \in [a, b]$.

Further, for all $v \in [a, b[$ it holds that $x \wedge v = x \wedge a$: If $v = a$, this is obviously true, therefore assume that $v \in]a, b[$. In order to guarantee that $x \wedge v \notin [a, b]$ it follows from $x \wedge v \leqslant v < b$ that necessarily $x \wedge v \leqslant a$ and further $x \wedge v \leqslant a \wedge x \leqslant v \wedge x$ due to the monotonicity and the idempotency of $\wedge$.

Based on these properties we can now conclude for the associativity of some $x \in L$ being incomparable to a and some $y, z \in [a, b]$ with $y \wedge z \in [a, b[$:

$$\begin{aligned} T\big(x, T(y, z)\big) &= x \wedge T(y, z) = x \wedge a = x \wedge a \wedge z \\ &= T(x \wedge a, z) = T(x \wedge y, z) = T\big(T(x, y), z\big). \end{aligned}$$

If $y = z = b$, then $T(x, T(y, z)) = T(x, b) = x \wedge b = T(x \wedge b, b) = T(T(x, y), z)$ which concludes the case. The remaining cases for showing the associativity of T can be checked analogously, thus showing that (i) implies (ii). Clearly, $T^L_{T^{[a,b]}}$ is commutative and has 1 as a neutral element.

Conversely, let $x \in L$ be incomparable to b and comparable to some $u \in]a, b[$, i.e., $x \geqslant u$, which implies $b \wedge x \in]a, b[$. Then

$$\begin{aligned} u &= T_{\mathbf{D}}{}^{[a,b]}(b, u) = T^L_{T_{\mathbf{D}}{}^{[a,b]}}(b, x \wedge u) \\ &= T^L_{T_{\mathbf{D}}{}^{[a,b]}}\big(b, T^L_{T_{\mathbf{D}}{}^{[a,b]}}(x, u)\big) = T^L_{T_{\mathbf{D}}{}^{[a,b]}}\big(T^L_{T_{\mathbf{D}}{}^{[a,b]}}(b, x), u\big) \\ &= T^L_{T_{\mathbf{D}}{}^{[a,b]}}(b \wedge x, u) = T_{\mathbf{D}}{}^{[a,b]}(b \wedge x, u) = a \end{aligned}$$

contradicting $u \in]a, b[$ and showing that the incomparability of x to b implies the incomparability to all elements of $]a, b[$. In complete analogy we can show that the incomparability of x to a implies the incomparability to all elements of $]a, b[$ by proving a contradiction to $T(T(u, u), x) = T(u, T(u, x))$ in case that x is comparable to some $u \in]a, b[$, i.e., in case $x \leqslant u$ and choosing $T^{[a,b]} = T_{\mathbf{D}}{}^{[a,b]}$, thus completing the proof that (ii) implies (i). □

Corollary 3.2. *Let $(L, \leqslant, 0, 1)$ be a bounded lattice, $(S, \leqslant, a, b)$ a bounded sublattice of L and $T^S : S^2 \to S$ a t-norm on S. Assume that for each $(x, y) \in (S \setminus \{b\}) \times ([a, b] \setminus S)$ we have $x \wedge y \in \{a, x\}$, that for each $(x, y) \in ([a, b] \setminus S)^2$ it follows that $x \wedge y \in S$ implies $x \wedge y = a$, and that, in case $]a, b[\neq \emptyset$, condition* (i) *in Proposition* 3.1 *holds. Then $T^L_{T^S}$ is a t-norm on L.*

Note that the conditions in Proposition 3.1 heavily depend on the interval $[a, b]$ and on the lattice L. Now we look for conditions on L only guaranteeing that for each subinterval each t-norm can be extended to a t-norm on L.

Recall that a bounded poset $(X, \leqslant, 0, 1)$ is called a *horizontal sum* of the bounded posets $((X_\imath, \leqslant_\imath, 0, 1))_{\imath\in I}$ if $X = \bigcup_{\imath\in I} X_\imath$ with $X_\imath \cap X_\jmath = \{0, 1\}$ whenever $i \neq j$, and $x \leqslant y$ if and only if there is an $i \in I$ such that $\{x, y\} \subseteq X_\imath$ and $x \leqslant_\imath y$ (compare, e.g., horizontal sums of effect algebras [32]). A non-trivial example of a bounded lattice which is a horizontal sum of chains is given by

$$L = \{(-1, -1), (1, 1), (-x, 1 - x), (x, x - 1) \mid x \in]0, 1[\}$$

equipped with the product order on $\mathbb{R}^2$.

Proposition 3.3 ([33]). *Let $(L, \leqslant, 0, 1)$ be a bounded lattice. The following are equivalent:*

(i) *L is a horizontal sum of chains.*
(ii) *For all $x, y \in L$: $\{x \wedge y, x \vee y\} \subseteq \{0, x, y, 1\}$.*
(iii) *For each subinterval $[a, b]$ of L and each t-norm $T^{[a,b]} : [a, b]^2 \to [a, b]$ on $[a, b]$, the operation $T^L_{T[a,b]}$ is a t-norm on L.*

4 T-NORMS ON HORIZONTAL SUMS OF CHAINS

Until now we have considered one subinterval of the bounded lattice $(L, \leqslant, 0, 1)$ only. However, Proposition 3.3 can be generalized to a system of pairwise disjoint intervals.

Definition 4.1. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and I some index set. Further, let $(\,]a_\imath, b_\imath[\,)_{\imath\in I}$ be a family of pairwise disjoint subintervals of L and $(T^{[a_\imath,b_\imath]})_{\imath\in I}$ a family of t-norms on the corresponding intervals $[a_\imath, b_\imath]$. Then the $\wedge$*-extension* $T : L^2 \to L$, denoted $T = (\langle [a_\imath, b_\imath], T_\imath\rangle)_{\imath\in I}$, is given by

$$T(x, y) = \begin{cases} T^{[a_\imath,b_\imath]}(x, y) & \text{if } (x, y) \in [a_\imath, b_\imath]^2, \\ x \wedge y & \text{otherwise.} \end{cases} \tag{4.1}$$

Corollary 4.2 ([33]). *Let $(L, \leqslant, 0, 1)$ be a bounded lattice. The following are equivalent:*

(i) *L is a horizontal sum of chains.*
(ii) *For each family of pairwise disjoint subintervals $(\,]a_\imath, b_\imath[\,)_{\imath\in I}$ of L and for each family of t-norms $(T^{[a_\imath,b_\imath]})_{\imath\in I}$ on the corresponding intervals $[a_\imath, b_\imath]$ the $\wedge$-extension $(\langle [a_\imath, b_\imath], T_\imath\rangle)_{\imath\in I}$ defined by (4.1) is a t-norm on L.*

As an immediate consequence of Corollary 4.2 we obtain the ordinal sum construction [14,24,29,34,35] for t-norms on the unit interval (see also [22] for a full investigation of the relationship with the concept of Clifford [7]) and on any chain.

Moreover, applying consecutively Proposition 2.1 and Corollary 4.2 we obtain the following general result:

Proposition 4.3. *Let $(L, \leqslant, 0, 1)$ be a bounded lattice which is a horizontal sum of some family $(L_k)_{k\in K}$ of chains, and let $(S_\imath, \leqslant, a_\imath, b_\imath)_{\imath\in I}$ be a family of bounded sublattices such that the sets $S^*_{\imath,k}$ defined by $S^*_{\imath,k} =]a_\imath, b_\imath[\cap L_k$ are pairwise disjoint.*

If for each $i \in I$ and for each $(x, y) \in (S_\imath \setminus \{b_\imath\}) \times ([a_\imath, b_\imath] \setminus S_\imath)$ we have $x \wedge y \in \{a_\imath, x\}$, then for each family $(T^{S_\imath})_{\imath\in I}$ of t-norms on the corresponding sublattices $S_\imath$ the function $T^L : L^2 \to L$ given by

$$T(x, y) = \begin{cases} T^{S_\imath}(x, y) & \text{if } (x, y) \in {S_\imath}^2, \\ x \wedge y & \text{otherwise,} \end{cases}$$

is a t-norm on L.

Proof. If for all $i \in I$ and all $k \in K$, $S^*_{\imath,k} = \emptyset$, it follows that $S_\imath = \{0, 1\}$ for all $i \in I$ and therefore $T = \wedge$. Otherwise, assume that for some $i \in I$ and for some $k \in K$, $S^*_{\imath,k} \neq \emptyset$. It remains to show that for all $(x, y) \in ([a_\imath, b_\imath] \setminus S_\imath)^2$ it follows that $x \wedge y \in S_\imath$ implies $x \wedge y = a_\imath$.

In case $a_\imath \neq 0$ or $b_\imath \neq 1$, for any $(x, y) \in (S^*_{\imath,k} \setminus S_\imath)^2$ it follows that $x \wedge y \in \{x, y\}$ such that $x \wedge y \notin S_\imath$. In case $a_\imath = 0$ and $b_\imath = 1$ it might be that $x \in S^*_{\imath,k} \setminus S_\imath \subset [a_\imath, b_\imath] \setminus S_\imath$ and $y \in S^*_{\imath,l} \setminus S_\imath \subset [a_\imath, b_\imath] \setminus S_\imath$ with $k \neq l$, however, then $x \wedge y = 0 = a_\imath \in S_\imath$ follows immediately. Because of Proposition 2.1 we can further conclude that $T|_{([a_\imath, b_\imath] \cap L_k)^2}$ is a t-norm on the bounded sublattice (subinterval) $([a_\imath, b_\imath] \cap L_k, \leqslant, a_\imath, b_\imath)$ of L. Notice that in case $a_\imath = 0$ and $b_\imath = 1$, $[a_\imath, b_\imath] \cap L_k = L_k$ and otherwise $[a_\imath, b_\imath] \cap L_k = [a_\imath, b_\imath]$. $\square$

The special structure of horizontal sums allows us to represent each t-norm as the $\wedge$-extension of its restrictions to the summands:

Proposition 4.4. *Let $(L, \leqslant, 0, 1)$ be a bounded lattice which is a horizontal sum of some family $(L_k)_{k\in K}$ of bounded lattices. Then a binary operation $T : L^2 \to L$ is a t-norm on L if and only if $T = (\langle L_k, T|_{{L_k}^2}\rangle)_{k\in K}$.*

Proposition 4.4 allows us to give a representation of certain types of t-norms on horizontal sums of chains, thus generalizing the representation theorem [19,24,29, 30] of continuous t-norms on the unit interval and of t-norms on finite chains [26] fulfilling the intermediate value property by means of additive generators. Recall that a t-norm $T : L^2 \to L$ on L fulfills the *intermediate value property* if it satisfies that for all $x, y, z \in L$ with $x \leqslant y$ and for each $u \in [T(x, z), T(y, z)]$, there is a $v \in [x, y]$ such that $T(v, z) = u$.

Corollary 4.5. *Let $(L, \leqslant, 0, 1)$ be a bounded lattice which is a horizontal sum of bounded chains $(C_k)_{k\in K}$, where each chain C_k is either finite or isomorphic to* *a non-trivial compact subinterval of the real line. If $T: L^2 \to L$ is a t-norm on L fulfilling the intermediate value property, then there exist a family $(S_\iota, \leqslant, a_\iota, b_\iota)_{\iota\in I}$ of subchains of L satisfying the hypothesis of Proposition* 4.3 *and a family of continuous, strictly decreasing real-valued functions $(t_\iota : S_\iota \to [0, \infty])_{\iota \in I}$ satisfying $t_\iota(b_\iota) = 0$ such that for each $(x, y) \in L^2$*

$$T(x, y) = \begin{cases} t_\iota^{-1}(\min(t_\iota(x) + t_\iota(y), t_\iota(a_\iota))) & \textit{if } (x, y) \in S_\iota^{\,2}, \\ x \wedge y & \textit{otherwise.} \end{cases}$$

Proof. From Proposition 4.4 we know $T = (\langle C_k, T|_{C_k^2}\rangle)_{k\in K}$. If C_k is finite then the t-norm $T|_{C_k^2}$ fulfills the intermediate value property, and the existence of the subchains S_ι of C_k and the functions $t_\iota : S_\iota \to [0, \infty]$ with the desired properties follows from [26,27]. If C_k is isomorphic to a non-trivial compact subinterval of the real line, then $(C_k, T|_{C_k^2})$ is isomorphic to an I-semigroup [13], and the result follows from [29] (compare also [21,30]). □

Due to the well-known structure of t-norms on the real unit interval and on finite chains fulfilling the intermediate value property (in the latter case such t-norms are uniquely determined by their non-trivial idempotent elements), we are able to construct all such t-norms on bounded lattices which are horizontal sums of non-trivial compact subintervals of the real line and finite chains [21,26,29].

As an immediate consequence, the number of t-norms on a finite lattice L which is a horizontal sum of chains which fulfill the intermediate value property is given by $2^{|L|-2}$ (compare the result of [26] for divisible t-norms on finite chains). Observe that the minimum $\wedge$ always satisfies the hypothesis of Corollary 4.5 (the index set I being empty in this case) whereas, e.g., for $L = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ the drastic product ${T_\mathbf{D}}^L$ does not fulfill the intermediate value property.

Further note that in Corollary 4.5 the hypothesis that the infinite chains involved there be isomorphic to non-trivial compact subintervals of the real line cannot be relaxed, in general. Take the chain $(L, \leqslant)$ with $L =]0, 1[^2 \cup \{(0, 0), (1, 1)\}$ and $\leqslant$ being the lexicographic order. Then the function $T: L^2 \to L$ given by $T((x_1, y_1), (x_2, y_2)) = (x_1 y_1, x_2 y_2)$ is a t-norm which fulfills the intermediate value property and is not representable as a $\wedge$-extension of some t-norm possessing an additive generator since the semigroup $(L \setminus \{(0, 0)\}, T|_{(L\setminus\{(0,0)\})^2})$ is Archimedean and cancellative, but has anomalous pairs (e.g., $(0.5, 0.6)$ and $(0.5, 0.5)$), compare [2,28] and see [15] for the corresponding notions and related results. Moreover, take the chain $(L, \leqslant)$ with $L = \{-1\} \cup [0, 1]$ and $\leqslant$ the standard order on the real line. Then the function $T: L^2 \to L$ defined by

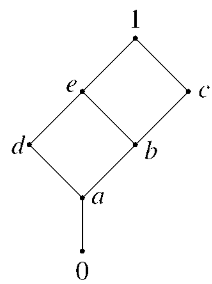
$$T(x, y) = \begin{cases} x + y - 1 & \text{if } x + y \geqslant 1, \\ -1 & \text{otherwise,} \end{cases}$$

is an Archimedean t-norm fulfilling the intermediate value property but with no additive generator.

A lattice $(L, 0, 1, \leqslant)$ equipped with some t-norm $T : L^2 \to L$ is called *divisible* [20] if for all $x, y \in L$ with $y \leqslant x$ there exists some $z \in L$ such that $y = T(x, z)$ (compare also the natural ordering of groupoids in [15]). Note that the divisibility of a t-norm T is, in general, a weaker property than its intermediate value property as the following example shows.

Example 5.1. Consider the bounded lattice $(L, \leqslant, 0, 1)$ with $L = \{0, 1, a, b, c, d, e\}$ as displayed and define $T : L^2 \to L$ by

T	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	a
b	0	0	0	0	0	a	b
c	0	0	0	a	a	b	c
d	0	0	0	a	0	a	d
e	0	0	a	b	a	d	e
1	0	a	b	c	d	e	1



Then T is a t-norm on L which is divisible but does not fulfill the intermediate value property (it suffices to choose for $x = b$, $y = e$ and $z = c$).

However, for chains the intermediate value property and divisibility coincide. Moreover, the intermediate value property of a t-norm T on a lattice L which is a horizontal sum is equivalent to the intermediate value property of T restricted to the summands of L. Thus the requirement of the intermediate value property for T in Corollary 4.5 can be relaxed to divisibility.

Further note that in many-valued logics, the algebraic background is mostly a residuated lattice $(L, 0, 1, \leqslant, *, \to)$, where $* : L^2 \to L$ is a t-norm on L. The t-norm $*$ modelling the conjunction operator and the operator $\to : L^2 \to L$ modelling the implication operator form adjoint operators linked to each other by the adjunction relation

$$x * y \leqslant z \quad \text{if and only if} \quad x \leqslant y \to z$$

for all $x, y, z \in L$. Note that a such residuated lattice is divisible if and only if

$$x * (x \to y) = x \wedge y \tag{5.1}$$

for all $x, y \in L$ [20]. Observe that (5.1) is preserved by ordinal sums. However, this is not more true for horizontal sums of chains. To see this, consider any finite bounded lattice $(L, 0, 1, \leqslant)$. Choosing $* = \wedge$, then $(L, 0, 1, \leqslant, \wedge, \to)$ is residuated if and only if it is distributive, i.e., it does not contain as a sublattice a non-trivial 5-point horizontal sum [5,23]. Thus the only non-trivial horizontal sum of chains which yields a residuated lattice $(L, 0, 1, \leqslant, \wedge, \to)$ is the four-point diamond lattice which is also the product of two chains with two elements. Note that for this lattice all t-norms T fulfill the intermediate value property and therefore divisibility, but (5.1) is only fulfilled for $* = \wedge$.

6 WEAKEST POSSIBLE EXTENSION

It was mentioned already in the beginning that the $\wedge$-extension of some t-norm T^S on some bounded sublattice S as given by (1.1) is the strongest possible extension of T^S. We have shown that guaranteeing that the $\wedge$-extension is a t-norm independent of the choice of the t-norm T^S (and the sublattice S) demands rather restrictive conditions on the underlying lattice. Quite different is the situation when looking for the weakest possible extension of T^S on some single sublattice S.

Definition 6.1. Let $(L, \leqslant, 0, 1)$ be a bounded lattice, $(S, \leqslant, a, b)$ a complete and bounded sublattice, and T^S a t-norm on the corresponding sublattice S. Then define $T^{S\cup\{0,1\}}: (S \cup \{0, 1\})^2 \to (S \cup \{0, 1\})$ by

$$(6.1) \qquad T^{S\cup\{0,1\}}(x, y) := \begin{cases} x \wedge y & \text{if } 1 \in \{x, y\}, \\ 0 & \text{if } 0 \in \{x, y\}, \\ T(x, y) & \text{if } (x, y) \in S^2. \end{cases}$$

Further define $W^L_{T^S}: L^2 \to L$ by

$$(6.2) \qquad W^L_{T^S}(x, y) := \begin{cases} x \wedge y & \text{if } 1 \in \{x, y\}, \\ T^{S\cup\{0,1\}}(x^*, y^*) & \text{otherwise}, \end{cases}$$

with $x^* = \sup\{z \mid z \leqslant x, z \in S \cup \{0, 1\}\}$.

Lemma 6.2. *Let $(L, \leqslant, 0, 1)$ be a bounded lattice and assume some complete, bounded sublattice $(S, \leqslant, a, b)$. Let T^S be a t-norm on the corresponding sublattice S. Then $T^{S\cup\{0,1\}}: (S \cup \{0, 1\})^2 \to (S \cup \{0, 1\})$ defined by (6.1) is a t-norm on $S \cup \{0, 1\}$. Moreover, it is the unique t-norm extension of T^S from S to $S \cup \{0, 1\}$.*

Proof. In case $\{0, 1\} \subseteq S$, then $T^{S\cup\{0,1\}} = T^S$. Moreover, clearly this "extension" is unique. For all other cases, it is immediate that $T^{S\cup\{0,1\}} = T^{S\cup\{0,1\}}_{T^S}$, i.e., $T^{S\cup\{0,1\}}$ coincides with the strongest possible extension provided by means of (1.1) such that indeed $T^{S\cup\{0,1\}}$ is a t-norm. For any extension T' of T^S to the sublattice $S \cup \{0, 1\}$ which is also a t-norm it holds that $T'(x, y) = T^S(x, y)$ for any $(x, y) \in S^2$. Moreover, $T'(x, 0) = T'(0, x) = 0 = T^{S\cup\{0,1\}}(x, 0) = T^{S\cup\{0,1\}}(0, x)$ for any $x \in S \cup \{0, 1\}$, and $T'(x, 1) = T'(1, x) = x = T^{S\cup\{0,1\}}(x, 1) = T^{S\cup\{0,1\}}(1, x)$ for any $x \in S \cup \{0, 1\}$, showing that $T^{S\cup\{0,1\}}$ is the unique and as such the weakest and strongest possible t-norm extension of T^S on $S \cup \{0, 1\}$. $\square$

Proposition 6.3. *Let $(L, \leqslant, 0, 1)$ be a bounded lattice and assume some complete, bounded sublattice $(S, \leqslant, a, b)$. Let T^S be a t-norm on the corresponding sublattice S. Then $W^L_{T^S}: L^2 \to L$ defined by (6.2) is a t-norm on L.*

Proof. First note that in case some x is smaller or incomparable to all elements of S, then $x^* = 0$. If x is greater than some element in S, then $x^* \in S$ since S is a complete sublattice. Moreover, if $x \in S$, then $x^* = x \in S$. Since in any case $x^* \leqslant x$ it is guaranteed that W_{TS}^L is well defined.

Moreover, for any $x, y \in L \setminus \{0, 1\}$ it holds that $W_{TS}^L(x, y) \in S \cup \{0, 1\}$ and therefore $W_{TS}^L(x, y)^* = W_{TS}^L(x, y)$. It is immediate to see that W_{TS}^L has neutral element 1 and that it is symmetric.

Let us next focus on its monotonicity. Therefore, assume some $x, x', y \in L$ such that $x \leqslant x'$ and let us show $T(x, y) \leqslant T(x', y)$. Since $x \leqslant x'$, also $x^* \leqslant (x')^*$. Whenever $1 \in \{x, y\}$, monotonicity is trivially fulfilled. Therefore, assume that $x' = 1$ but $x \neq 1$ and $y \neq 1$, then

$$W_{TS}^L(x, y) = T^{S \cup \{0,1\}}\big(x^*, y^*\big) \leqslant x^* \wedge y^* \leqslant y = W_{TS}^L(1, y).$$

And, finally, for all other $x, x', y \in L$ it holds that

$$W_{TS}^L(x, y) = T^{S \cup \{0,1\}}\big(x^*, y^*\big) \leqslant T^{S \cup \{0,1\}}\big(\big(x'\big)^*, y^*\big) = W_{TS}^L\big(x', y\big).$$

It remains to prove associativity, i.e., $W_{TS}^L(x, W_{TS}^L(y, z)) = W_{TS}^L(W_{TS}^L(x, y), z)$ for all $x, y, z \in L$. Whenever all $x, y, z \in S$, $1 \in \{x, y, z\}$, or $0 \in \{x^*, y^*, z^*\}$, this holds immediately. However, for all remaining cases, we have $W_{TS}^L(x, y) = T^{S \cup \{0,1\}}(x^*, y^*) = T^{S \cup \{0,1\}}(x^*, y^*)^*$, such that

$$\begin{aligned} W_{TS}^L\big(W_{TS}^L(x, y), z\big) &= T^{S \cup \{0,1\}}\big(T^{S \cup \{0,1\}}\big(x^*, y^*\big), z^*\big) \\ &= T^{S \cup \{0,1\}}\big(x^*, T^{S \cup \{0,1\}}\big(y^*, z^*\big)\big) = W_{TS}^L\big(x, W_{TS}^L(y, z)\big) \end{aligned}$$

proving associativity and thus that W_{TS}^L is indeed a t-norm on L. □

Proposition 6.4. *Let $(L, \leqslant, 0, 1)$ be a bounded lattice and assume some complete, bounded sublattice $(S, \leqslant, a, b)$. Let T^S be a t-norm on the corresponding sublattice S. Then $W_{TS}^L : L^2 \to L$ defined by (6.2) is the smallest possible t-norm extension of T^S on L.*

Proof. Assume that T' is a t-norm extension of T^S on L. For all $(x, y) \in (S \cup \{0, 1\})^2$, $T'(x, y) = W_{TS}^L(x, y)$. Next, consider that either $x \notin S \cup \{0, 1\}$ or $y \notin S \cup \{0, 1\}$, then $x^* \leqslant x$ and $y^* \leqslant y$, and further

$$T'(x, y) \geqslant T'\big(x^*, y^*\big) = T^{S \cup \{0,1\}}\big(x^*, y^*\big) = W_{TS}^L(x, y)$$

such that W_{TS}^L is indeed the smallest possible t-norm extension of T^S on L. □

So far we have considered one complete sublattice S of the bounded lattice $(L, \leqslant, 0, 1)$ only. Next, we aim at a generalization in case of families of complete sublattices and corresponding t-norms.

Definition 6.5. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and I some index set. Further, let $(S_\imath, \leqslant, a_\imath, b_\imath)_{\imath\in I}$ be a family of complete and bounded sublattices of L such that the family $(\,]a_\imath, b_\imath[\,)_{\imath\in I}$ consists of pairwise disjoint subintervals of L. Finally, let $(T^{S_\imath})_{\imath\in I}$ be a family of t-norms on the corresponding sublattices $S_\imath$. Then define $W^L_{T^{S_\imath}} : L^2 \to L$ by

$$(6.3) \qquad W^L_{T^{S_\imath}}(x, y) := \begin{cases} x \wedge y & \text{if } 1 \in \{x, y\}, \\ T^{S_\imath \cup \{0,1\}}(x^*_\imath, y^*_\imath) & \text{otherwise,} \end{cases}$$

with $x^*_\imath = \sup\{z \mid z \leqslant x, z \in S_\imath \cup \{0, 1\}\}$ and define $W : L^2 \to L$ by

$$(6.4) \qquad W(x, y) := \sup_{\imath\in I} W^L_{T^{S_\imath}}(x, y).$$

Note that, by definition, W is a symmetric and monotone operation on L which has neutral element 1. However, further restrictions on the family of sublattices have to be applied in order to guarantee that W is indeed an extension of arbitrary t-norms $T^{S_\imath}$ on the sublattices $S_\imath$.

Proposition 6.6. *Let $(L, \leqslant, 0, 1)$ be a bounded lattice and I some index set. Further, let $(S_\imath, \leqslant, a_\imath, b_\imath)_{\imath\in I}$ be a family of complete sublattices of L such that the family $(\,]a_\imath, b_\imath[\,)_{\imath\in I}$ consists of pairwise disjoint subintervals of L. Further assume that for all $i, j \in I$ with $i \neq j$ it holds that*

(i) *if $x \in S_j$ then $x^*_\imath \notin S_\imath \setminus \{a_\imath, b_\imath\}$, i.e., $x^*_\imath \in \{0, a_\imath, b_\imath\}$,*
(ii) *if $x \in S_j \setminus \{b_j\}$ and $x^*_\imath = a_\imath$, then $(a_j)^*_\imath \geqslant a_\imath$, and*
(iii) *if $x \in S_j \setminus \{b_j\}$ and $x^*_\imath = b_\imath$, then $(a_j)^*_\imath = b_\imath$.*

Then for all t-norms $T^{S_\imath}$ on $S_\imath$ and for all t-norms T^{S_j} on S_j with $i \neq j$ it holds that $W^L_{T^{S_\imath}}(x, y) \leqslant T^{S_j}(x, y)$ for all $(x, y) \in S^2_j$ and $W^L_{T^{S_j}}(x, y) \leqslant T^{S_\imath}(x, y)$ for all $(x, y) \in S^2_\imath$, i.e.,

$$W^L_{T^{S_\imath}}|_{S_j^2} \leqslant T^{S_j} \quad \textit{and} \quad W^L_{T^{S_j}}|_{S_\imath^2} \leqslant T^{S_\imath}.$$

Moreover, W given by (6.4) is a monotone and symmetric extension of each $T^{S_\imath}$, i.e., $W|_{S^2_\imath} = T^{S_\imath}$ for all $i \in I$, which has neutral element 1.

Proof. Without loss of generality fix some t-norms $T^{S_\imath}$, T^{S_j} on $S_\imath$ resp. S_j with $i, j \in I$, $i \neq j$, and let $(x, y) \in S^2_j$. Then $x^*_\imath, y^*_\imath \in \{0, a_\imath, b_\imath\}$. If $x^*_\imath = 0$ or $y^*_\imath = 0$, it follows immediately that $W^L_{T^{S_\imath}}(x, y) = 0 \leqslant T^{S_j}(x, y)$. If $x^*_\imath = b_\imath$ or $y^*_\imath = b_\imath$, then $(a_j)^*_\imath = b_\imath$ such that

$$W^L_{T^{S_\imath}}(x, y) \leqslant b_\imath \leqslant a_j \leqslant T^{S_j}(x, y).$$

Finally, for $x^*_\imath = y^*_\imath = a_\imath$, necessarily $(a_j)^*_\imath \geqslant a_\imath$, such that we can conclude

$$W^L_{T^{S_\imath}}(x, y) = a_\imath \leqslant (a_j)^*_\imath \leqslant a_j \leqslant T^{S_j}(x, y).$$

Therefore, for all $j \in I$ and for all $(x, y) \in S_j^2$,

$$\sup_{i \in I, i \neq j} W^L_{T^{S_i}}(x, y) \leqslant T^{S_j}(x, y)$$

and moreover, since $W^L_{T^{S_j}}|_{S_j^2} = T^{S_j}$, $W(x, y) = \sup_{i \in I} W^L_{T^{S_i}}(x, y) = T^{S_j}(x, y)$ showing that W is indeed an extension of T^{S_j}. $\square$

Further note that the supremum of arbitrary t-norms on a lattice L need not be a t-norm in general, compare also [11]. However, for particular and important classes of lattices the operation W as defined by (6.4) is associative, i.e., is a t-norm.

Example 6.7. Let $(L_i, \leqslant, 0_i, 1_i)$, $i \in \{1, \ldots, n\}$, $n \in \mathbb{N}$, be arbitrary complete and bounded lattices and consider their product lattice $L = \prod_{i=1}^n L_i$. Then for each $i \in \{1, \ldots n\}$, $S_i = \{(0_1, \ldots, x_i, \ldots, 0_n) \mid x_i \in L_i\}$ is a complete and bounded sublattice of L. Moreover, for each t-norm T_i on L_i, the function $T^{S_i} : S_i^2 \to S_i$ defined by

$$T^{S_i}(\mathbf{x}, \mathbf{y}) = \big(0_1, \ldots, T_i(x_i, y_i), \ldots, 0_n\big)$$

denotes a t-norm on S_i. Therefore, $W : L^2 \to L$ as defined by (6.4) can be computed as

$$W(\mathbf{x}, \mathbf{y}) = \sup_{i=1,\ldots,n} W^L_{T^{S_i}}(\mathbf{x}, \mathbf{y}) = \big(T_1(x_1, y_1), \ldots, T_n(x_n, y_n)\big)$$

and is a t-norm on L for arbitrary t-norms T_i on L_i.

Example 6.8. Let $(L, \leqslant, 0, 1)$ be a bounded lattice. Further, let $(\,]a_i, b_i[\,)_{i \in I}$ be a family of pairwise disjoint, non-empty subintervals of L with $(I, \preceq)$ a linearly ordered index set such that

- $(\{[a_i, b_i[\mid i \in I\}) \cup \{\{1\}\}$ forms a partition of L and
- whenever $i \prec j$ then $x \leqslant y$ for all $x \in [a_i, b_i]$ and for all $y \in [a_j, b_j]$,

i.e., L is a so-called ordinal sum of partially ordered sets $([a_i, b_i[, \leqslant)$, $i \in I$, and $(\{1\}, \leqslant)$, see e.g. [9]. Let J be a finite subset of I, i.e., $J = \{i_1, \ldots, i_n\} \subseteq I$ for some $n \in \mathbb{N}$, such that $i_1 < i_2 < \cdots < i_n$ and as a consequence $a_{i_1} < a_{i_2} < \cdots < a_{i_n}$. Additionally define $a_{i_{n+1}} := 1$.

Finally, let $(T^{[a_{i_j}, b_{i_j}]})_{i_j \in J}$ be a family of t-norms on the corresponding intervals $[a_{i_j}, b_{i_j}]$. Then, $([a_{i_j}, b_{i_j}], \leqslant)_{i_j \in J}$ forms a family of complete and bounded sublattices of L for which the requirements of Proposition 6.6 hold such that W defined by (6.4) can be computed as

$$W(x, y) = \begin{cases} x \wedge y & \text{if } 1 \in \{x, y\}, \\ T^{[a_{i_j}, b_{i_j}]}(x, y) & \text{if } (x, y) \in [a_{i_j}, b_{i_j}]^2, \\ x \wedge y \wedge b_{i_j} & \text{if } a_{i_j} \leqslant x \wedge y < a_{i_{j+1}} \text{ and } b_{i_j} \leqslant x \vee y < 1, \\ 0 & \text{if } x < a_{i_1} \text{ or } y < a_{i_1}. \end{cases}$$

Moreover, W is associative, i.e., a t-norm on L.

Remark 6.9. Note that for $J = I$, it holds that $a_{i_1} = 0$ and, for all $i_j \in J, b_{i_j} = a_{i_{j+1}}$ and therefore $x \wedge y \wedge b_{i_j} = x \wedge y$ whenever $x \wedge y \in [a_{i_j}, b_{i_j}]$ and $b_{i_j} \leqslant x \vee y$. As a consequence, for $J = I$, the weakest extension W and the strongest extension T as defined by (4.1) of t-norms $(T^{[a_i,b_i]})_{i\in I}$ on corresponding intervals $[a_i, b_i]$ coincide.

In case $J \subsetneqq I$, always $W \neq T$ such that the present example provides another way of obtaining t-norms on chains, in particular on $[0, 1]$, which extend t-norms $(T^{[a_i,b_i]})_{i\in I}$ on corresponding intervals $[a_i, b_i]$. Note further that for $L = [0, 1]$, the weakest extension W is right-continuous whenever all $(T^{[a_i,b_i]})_{i\in I}$ are right-continuous.

In case of chains the previous result can even be strengthened.

Proposition 6.10. *Let $(L, \leqslant, 0, 1)$ be a chain. Further, let $(\,]a_i, b_i[\,)_{i\in I}$ be a family of pairwise disjoint, non-empty subintervals of L and $(T^{[a_i,b_i]})_{i\in I}$ a family of t-norms on the corresponding subintervals with $(I, \preceq)$ a linearly ordered index set. Then W defined by (6.4) is associative, i.e., a t-norm on L.*

Proof. From Proposition 6.6 we can conclude that W is a monotone and symmetric extension of each $T^{[a_i,b_i]}$ which has neutral element 1. Next choose arbitrary $x, y, z \in L$. In case $1 \in \{x, y, z\}$ the associativity of W holds trivially, therefore assume that $1 \notin \{x, y, z\}$. In case $x \wedge y \wedge z \in [a_i, b_i]$ for some $i \in I$ we can conclude that

$$
\begin{aligned}
W\big(W(x, y), z\big) &= T^{[a_i,b_i]}\big(T^{[a_i,b_i]}(x \wedge b_i, y \wedge b_i), z \wedge b_i\big) \\
&= T^{[a_i,b_i]}\big(x \wedge b_i, T^{[a_i,b_i]}(y \wedge b_i, z \wedge b_i)\big) \\
&= W\big(x, W(y, z)\big).
\end{aligned}
$$

If $m = x \wedge y \wedge z \in L \setminus \bigcup_{i\in I} [a_i, b_i]$, then for all $i \in I$ such that $b_i < m$ it holds that

$$
W^L_{T^{[a_i,b_i]}}\big(W^L_{T^{[a_i,b_i]}}(x, y), z\big) = W^L_{T^{[a_i,b_i]}}\big(x, W^L_{T^{[a_i,b_i]}}(y, z)\big) = b_i.
$$

If $a_i > m$, then $W^L_{T^{[a_i,b_i]}}(W^L_{T^{[a_i,b_i]}}(x, y), z) = W^L_{T^{[a_i,b_i]}}(x, W^L_{T^{[a_i,b_i]}}(y, z)) = 0$. As a consequence $W(W(x, y), z) = W(x, W(y, z)) = \sup\{b_i \mid b_i < m\}$. □

Proposition 6.10 can further be extended to horizontal sums of chains.

Corollary 6.11. *Let $(L, \leqslant, 0, 1)$ be a bounded lattice which is a horizontal sum of some family $(L_k)_{k\in K}$ of chains. Further, let $(\,]a_i, b_i[\,)_{i\in I}$ be a family of pairwise disjoint, non-empty subintervals of L and $(T^{[a_i,b_i]})_{i\in I}$ a family of t-norms on the corresponding intervals $[a_i, b_i]$. Then, for any $k \in K$, $W\,|_{L_k^2}$ is a t-norm on L_k and of the form as described in Proposition 6.10. In case $x \in L_k \setminus \{1\}$ and $y \in L_l \setminus \{1\}$ with $k \neq l$, $W(x, y) = 0$, and in case $1 \in \{x, y\}$, $W(x, y) = x \wedge y$, such that W is also a t-norm on L.*

ACKNOWLEDGEMENTS

The authors would like to thank Prof. Duistermaat for his valuable comments on the first version of this contribution. Susanne Saminger-Platz has been on a sabbatical year at the Dipartimento di Matematica "Ennio De Giorgi", Università del Salento (Italy), when writing and investigating parts of this paper. She therefore gratefully acknowledges the support by the Austrian Science Fund (FWF) in the framework of the Erwin Schrödinger Fellowship J 2636-N15. Radko Mesiar was also supported by the grants MSM6198898701 and VEGA 1/3006/06.

REFERENCES

[1] Agliano P., Montagna F. – Varieties of BL-algebras I: general properties, J. Pure Appl. Algebra **181** (2003) 129–131.

[2] Alimov N.G. – On ordered semigroups, Izv. Akad. Nauk SSSR **14** (1950) 569–576 (in Russian).

[3] Alsina C., Frank M., Schweizer B. – Associative Functions: Triangular Norms and Copulas, World Scientific, Singapore, 2006.

[4] Belnap N.D. – A useful four-valued logic, in: J.M. Dunn, G. Epstein (Eds.), Modern Uses of Multiple-Valued Logic, Reidel, Dordrecht, 1977, pp. 8–37.

[5] Birkhoff G. – Lattice Theory, American Mathematical Society, Providence, RI, 1973.

[6] Busaniche M. – Decomposition of BL-chains, Algebra Universalis **524** (2004) 519–525.

[7] Clifford A.H. – Naturally totally ordered commutative semigroups, Amer. J. Math. **76** (1954) 631–646.

[8] da Costa N.C.A. – On the theory of inconsistent formal systems, Notre Dame J. Formal Logic **15** (1974) 497–510.

[9] Davey B.A., Priestley H.A. – Introduction to Lattices and Order, 2nd ed., Cambridge University Press, Cambridge, 2002.

[10] De Cooman G., Kerre E.E. – Order norms on bounded partially ordered sets, J. Fuzzy Math. **2** (1994) 281–310.

[11] Durante F., Mesiar R., Papini P.L. – The lattice-theoretic structure of the sets of triangular norms and semi-copulas, Nonlinear Anal. **69** (2007) 46–52.

[12] Dunn J.M. – Intuitive semantics for first-degree entailments and 'coupled trees', Philos. Studies **29** (1976) 149–168.

[13] Faucett W.M. – Compact semigroups irreducibly connected between two idempotents, Proc. Amer. Math. Soc. **6** (1955) 741–747.

[14] Frank M.J. – On the simultaneous associativity of $F(x, y)$ and $x + y - F(x, y)$, Aequationes Math. **19** (1979) 194–226.

[15] Fuchs L. – Partially Ordered Algebraic Systems, Pergamon Press, Oxford, 1963.

[16] Fuchs L., Reis R. – On lattice-ordered commutative semigroups, Algebra Universalis **50** (2003) 341–357.

[17] Gottwald S. – A Treatise on Many-Valued Logic, Studies in Logic and Computation, Research Studies Press, Baldock, 2001.

[18] Hájek P. – Metamathematics of Fuzzy Logic, Kluwer Academic Publishers, Dordrecht, 1998.

[19] Hofmann K.H., Lawson J.D. – Linearly ordered semigroups, in: Historic Origins and A.H. Clifford's Influence, London Math. Soc. Lecture Notes, vol. 231, 1996, pp. 15–39.

[20] Höhle U. – Commutative, residuated ℓ-monoids, in: Theory and Decision Library Series B: Mathematical and Statistical Methods, vol. 32, Kluwer Academic Publishers, Dordrecht, 1995, pp. 53–106.

[21] Klement E.P., Mesiar R., Pap E. – Triangular Norms, Trends in Logic. Studia Logica Library, vol. 8, Kluwer Academic Publishers, Dordrecht, 2000.

[22] Klement E.P., Mesiar R., Pap E. – Triangular norms as ordinal sums of semigroups in the sense of A.H. Clifford, Semigroup Forum **65** (2002) 71–82.

[23] Lidl R., Pilz G. – Applied Abstract Algebra, Springer, New York, 1998.

[24] Ling C.M. – Representation of associative functions, Publ. Math. Debrecen **12** (1965) 189–212.
[25] Łukasiewicz J. – Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls, Comptes Rendus Séances Société des Sciences et Lettres Varsovie cl. III **23** (1930) 51–77.
[26] Mayor G., Torrens J. – On a class of operators for expert systems, Int. J. Intell. Syst. **8** (1993) 771–778.
[27] Mayor G., Torrens J. – Triangular norms on discrete settings, in: Klement E.P., Mesiar R. (Eds.), Logical Algebraic, Analytic, and Probabilistic Aspects of Triangular Norms, Elsevier, Amsterdam, 2005, pp. 189–230.
[28] Mesiarová A., Viceník P. – Cancellative t-norms, in: Proceedings EUSFLAT 2003 (Zittau), 2003, pp. 289–291.
[29] Mostert P.S., Shields A.L. – On the structure of semi-groups on a compact manifold with boundary, Ann. of Math., II. Ser. **65** (1957) 117–143.
[30] Paalman-de Miranda A.B. – Topological Semigroups, Mathematical Centre Tracts, vol. 11, Matematisch Centrum, Amsterdam, 1964.
[31] Post E.L. – Introduction to a general theory of elementary propositions, Amer. J. Math. **43** (1921) 163–185.
[32] Riečanová Z. – Pastings of MV-effect algebras, Internat. J. Theoret. Phys. **43** (2004) 1875–1883.
[33] Saminger S. – On ordinal sums of triangular norms on bounded lattices, Fuzzy Sets and Systems **157** (2006) 1403–1416.
[34] Schweizer B., Sklar A. – Associative functions and abstract semigroups, Publ. Math. Debrecen **10** (1963) 69–81.
[35] Schweizer B., Sklar A. – Probabilistic Metric Spaces, North-Holland, New York, 1983.
[36] Takano M. – Strong completeness of lattice-valued logic, Arch. Math. Logic **41** (2002) 497–505.
[37] Titani S. – A lattice-valued set theory, Arch. Math. Logic **38** (1999) 395–421.

(Received December 2005)