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On extensions of representations for compact Lie groups

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Abstract

Let H be a closed normal subgroup of a compact Lie group G such that G/H is connected. This paper provides a necessary and sufficient condition for every complex representation of H to be extendible to G, and also for every complex G-vector bundle over the homogeneous space G/H to be trivial. In particular, we show that the condition holds when the fundamental group of G/H is torsion free.

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1. Introduction

One of the classical problems in finite group theory is to characterize extensions of representations. We mean an extension of a representation in the following way: Given a normal subgroup H of a group G, a (complex) representation $\rho: H \to \mathrm{GL}(n,\mathbb{C})$ is called *extendible to* G if there exists a representation $\tilde{\rho}: G \to \mathrm{GL}(n,\mathbb{C})$ (called a G-*extension*) such that $\rho = \tilde{\rho}$ on H. It is to be noted that the dimension n is not changed, since ρ as a sub-representation is always contained in the restriction of the induced representation of ρ to H.

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In the case of finite G, it is well known that every complex irreducible representation of H, which is G-invariant under conjugation (see Section 2 for the definition), is extendible to G if the second group cohomology $H^2(G/H, \mathbb{C}^*)$ vanishes [5, Theorem 11.7]. On the other hand, the extension problem for infinite groups has not been extensively studied. In this article, we study the problem for compact Lie groups when G/H is connected. Our main result is a necessary and sufficient condition for every complex representation of H to be extendible to G. It is also shown that the condition is related to a topological invariant, the fundamental group of G/H.

For any group G, let G' denote the commutator subgroup of G.

Theorem 1.1. *Let G be a compact Lie group and H a closed normal subgroup such that G/H is connected. Then every complex representation of H is extendible to G if and only if H is a direct summand of $G'H$.*

Corollary 1.2. *Let G be a compact Lie group and H a closed normal subgroup such that G/H is connected. Then every complex representation of H is extendible to G if the fundamental group $\pi_1(G/H)$ is torsion free, or equivalently if $(G/H)'$ is simply connected.*

Our theorem provides a complete characterization of the triviality of complex G-vector bundles over the homogeneous space G/H. Let E be a complex G-vector bundle over G/H. We recall that E is *trivial* if it is isomorphic to the product bundle $G/H \times V$ for some complex G-module V. Since E is uniquely determined by the fiber at the identity element of G/H (say E_0), the bundle E is trivial if and only if E_0 as a complex representation of H is extendible to G. Theorem 1.1 leads us to the following corollary.

Corollary 1.3. *Let G be a compact Lie group and H a closed normal subgroup such that G/H is connected. Then every complex G-vector bundle over the homogeneous space G/H is trivial if and only if H is a direct summand of $G'H$.*

The existence of G-extensions plays an important role even in equivariant K-theory. Let X be a connected topological space with a compact Lie group G action. Let H be the normal subgroup of G which consists of all elements of G acting trivially on X. Then the projection $G \to G/H$ induces the canonical homomorphism $\phi : K_{G/H}(X) \to K_G(X)$ which sends a G/H-vector bundle over X to the same bundle viewed as a G-vector bundle with the trivial H-action.

On the other hand, suppose that every complex irreducible representation of H is extendible to G. Then there is an injective (not natural) group homomorphism $e : R(H) \to R(G)$ between two representation rings defined as follows. For each irreducible complex H-module U choose a G-extension U_G, and define $e([U]) = [U_G]$ where [ ] denote the classes in the representation rings. Then extend the definition of e to $R(H)$ so that it defines a homomorphism $R(H) \to R(G)$. For each complex G-module V we can associate the trivial complex G-vector bundle $\underline{V} = X \times V$, which defines the natural homomorphism $t : R(G) \to K_G(X)$. We now define a group homomorphism

$$\mu : R(H) \otimes K_{G/H}(X) \to K_G(X), \qquad (V, \xi) \mapsto t \circ e(V) \otimes \phi(\xi). \tag{1}$$

This homomorphism is an isomorphism. Indeed, the inverse is given as follows. Let $\mathrm{Irr}(H)$ denote the set of all isomorphism classes of complex irreducible representations of H. For each $[\chi] \in \mathrm{Irr}(H)$ choose a G-extension of χ, and let V_χ be the corresponding G-module to the chosen G-extension. For a complex G-vector bundle E over X, the canonical isomorphism

$$E \stackrel{\cong}{\to} \bigoplus_{[\chi]\in \mathrm{Irr}(H)} \underline{V_\chi} \otimes \mathrm{Hom}_H(\underline{V_\chi}, E)$$

induces a group homomorphism $K_G(X) \to R(H) \otimes K_{G/H}(X)$ which is the desired inverse (see [2, Section 2] for more general arguments). Therefore, we have a generalization of Proposition 2.2 in [6] which deals with the extreme case when G acts trivially on X.

Corollary 1.4. *Let G be a compact Lie group and H a closed normal subgroup such that G/H is connected. Let X be a connected G-space such that H acts trivially on X. If H is a direct summand of $G'H$, then the map $\mu : R(H) \otimes K_{G/H}(X) \to K_G(X)$ in* (1) *can be defined, and it is a group isomorphism.*

This article is organized as follows. In Section 2, we shall give some basic notions and then show that a complex irreducible representation of H, which is G-invariant under conjugation, induces an associated projective representation of G which may be viewed as a G-extension in the projective representation level. Section 3 is devoted to prove that every complex representation of H has a G-extension when G/H is connected and abelian. In Section 4, we shall proceed the study in the case that G/H is semisimple and connected. After showing that the extension problem can be reduced to this case, we shall prove Theorem 1.1.

2. Associated projective representations

Let G be a topological group and H a closed normal subgroup of G. By a (complex) *representation* of G we shall mean a continuous homomorphism of G into the general linear group $\mathrm{GL}(n, \mathbb{C})$ of nonsingular $n \times n$ matrices over the field $\mathbb{C}$ of complex numbers. A representation $\rho : H \to \mathrm{GL}(n, \mathbb{C})$ is called *extendible to G* if there exists a representation $\tilde{\rho} : G \to \mathrm{GL}(n, \mathbb{C})$ (called a *G-extension* of ρ) such that $\rho(h) = \tilde{\rho}(h)$ for all $h \in H$.

$$\begin{array}{ccc} H & \xrightarrow{\rho} & \mathrm{GL}\,(n,\mathbb{C}) \\ \big\downarrow & \overset{\tilde{\rho}}{\nearrow} & \\ G & & \end{array}$$

Moreover, it is enough to get a G-extension of ρ that there is a representation $\tilde{\rho} : G \to \mathrm{GL}(n, \mathbb{C})$ such that its restriction to H is isomorphic (or similar) to ρ, i.e., there exists a matrix $M \in \mathrm{GL}(n, \mathbb{C})$ such that $M^{-1}\tilde{\rho}(h)M = \rho(h)$ for all $h \in H$.

Given a representation $\rho : H \to \mathrm{GL}(n, \mathbb{C})$ the map ${}^g\chi : H \to \mathbb{C}$ defined by the conjugation ${}^g\chi(h) = \chi(g^{-1}hg)$ becomes a representation of H for each $g \in G$. We say that ρ is *G-invariant* if it is isomorphic to the conjugate representation ${}^g\rho$ for all $g \in G$, which is a necessary condition of ρ to be extendible to G.

In the following, we assume that a representation $\rho : H \to \mathrm{GL}(n, \mathbb{C})$ is irreducible and G-invariant. Then there exists a matrix $M_g \in \mathrm{GL}(n, \mathbb{C})$ for each $g \in G$ such that $M_g^{-1}\rho(h)M_g = {}^g\rho(h) = \rho(g^{-1}hg)$ for all $h \in H$. Since ρ is irreducible, the Schur's lemma implies that M_g is unique up to multiplication by nonzero constant in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. So we are able to define a function ρ^* of G into the projective linear group $\mathrm{PGL}(n, \mathbb{C}) = \mathrm{GL}(n, \mathbb{C})/\mathbb{C}^*$ by $\rho^*(g) = [M_g]$ for each $g \in G$, where $[M_g]$ denotes the image of M_g by the canonical projection π: $\mathrm{GL}(n, \mathbb{C}) \to \mathrm{PGL}\,(n, \mathbb{C})$.

$$\begin{array}{ccc} H & \xrightarrow{\ \rho\ } & \mathrm{GL}(n,\mathbb{C}) \\ \Big\downarrow & & \Big\downarrow \pi \\ G & \xrightarrow{\ \rho^*\ } & \mathrm{PGL}(n,\mathbb{C}) \end{array}$$

Lemma 2.1. *Let G be a topological group and H a compact normal subgroup of G. Given a complex irreducible representation $\rho : H \to \mathrm{GL}(n, \mathbb{C})$ which is G-invariant, the function $\rho^* : G \to \mathrm{PGL}(n, \mathbb{C})$ defined above is a continuous homomorphism, called the projective representation of G associated with ρ. Moreover, the image of ρ^* is contained in $U(n)/S^1 \subset \mathrm{PGL}(n, \mathbb{C})$ if ρ is a unitary representation of H.*

Proof. It is immediate that ρ^* is a homomorphism. Since H is compact we may assume that ρ is a unitary representation of H, i.e., the image of ρ is contained in the unitary group $U(n)$. Then M_g is a constant multiple of a matrix in $U(n)$ so that $\rho^*(g)$ is contained in $U(n)/S^1$ for all $g \in G$. For the continuity of ρ^* it suffices to show that the graph of ρ^* in $G \times \mathrm{PGL}(n, \mathbb{C})$ is closed, since $U(n)/S^1$ is a compact Hausdorff space.

Consider the family of continuous maps $\Phi_h : G \times \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ for each $h \in H$ given by $(g, M) \mapsto \rho(h)M\rho(g^{-1}hg)^{-1}M^{-1}$. Then the set

$$\bigcap_{h \in H} \Phi_h^{-1}(I) = \bigcup_{g \in G} \{(g, M) \in G \times \mathrm{GL}(n, \mathbb{C}) \mid M \in \pi^{-1}(\rho^*(g))\},$$

is the inverse image of the graph of ρ^* in $G \times \mathrm{PGL}(n, \mathbb{C})$ by the canonical projection $1 \times \pi : G \times \mathrm{GL}(n, \mathbb{C}) \to G \times \mathrm{PGL}(n, \mathbb{C})$, which is obviously closed in $G \times \mathrm{GL}(n, \mathbb{C})$. Therefore, the graph of ρ^* is also closed in $G \times \mathrm{PGL}(n, \mathbb{C})$. $\square$

We may say that ρ is extendible to G in the projective representation level, since $\rho^*(h) = [\rho(h)]$ for all $h \in H$, i.e., $\rho^* = \pi \circ \rho$ on H.

$$\begin{array}{ccc} H & \xrightarrow{\rho} & \mathrm{GL}\,(n,\mathbb{C}) \\ \big\downarrow & \overset{\tilde{\rho}}{\nearrow} & \big\downarrow \pi \\ G & \xrightarrow[\rho^*]{} & \mathrm{PGL}\,(n,\mathbb{C}) \end{array}$$

Note that any G-extension (if exists) $\tilde{\rho}$ of ρ is a lifting homomorphism of ρ^*, i.e., $\rho^* = \pi \circ \tilde{\rho}$, since $\rho^*(g) = [\tilde{\rho}(g)]$ for all $g \in G$.

Remark. In case that G is finite, choose a transversal T containing e for H in G and set $M_e = I$, the identity matrix in $\mathrm{GL}(n,\mathbb{C})$. For each $t \in T$ and $h \in H$, the map $\rho' : G \to \mathrm{GL}(n,\mathbb{C})$ sending $th \mapsto M_t\rho(h)$ is a lifting (not necessarily homomorphism) of ρ^*, i.e., $\pi \circ \rho' = \rho^*$, and it determines a cocycle β in the second group cohomology $H^2(G/H, \mathbb{C}^*)$, which depends only on ρ. Moreover, ρ is extendible to G if and only if β is trivial, see [5, Theorem 11.7] for more details.

3. Extensions when G/H is connected abelian

In this section, we shall prove that every complex representation of H is extendible to G when G/H is compact, connected, and abelian, that is a torus. We begin with a general result on extensions of representations in the special case when $G = SH$ for some closed subgroup S of G.

Lemma 3.1. *Let G be a compact topological group such that $G = SH$ for a closed subgroup S and a closed normal subgroup H of G. Then a complex representation $\rho : H \to \mathrm{GL}(n,\mathbb{C})$ is extendible to G if and only if there exists a representation $\varphi : S \to \mathrm{GL}(n,\mathbb{C})$ such that*

(1) *$\varphi = \rho$ on $S \cap H$, and*
(2) *$\varphi(s)^{-1}\rho(h)\varphi(s) = \rho(s^{-1}hs)$ for all $s \in S$ and $h \in H$.*

Proof. The necessity is obvious so we prove the sufficiency. Define a function $\tilde{\rho} : G \to \mathrm{GL}(n,\mathbb{C})$ by $\tilde{\rho}(sh) = \varphi(s)\rho(h)$ for $s \in S$ and $h \in H$. It is immediate that $\tilde{\rho} = \rho$ on H. In this proof, we shall use the symbols s, s' and h, h' for elements in S and H, respectively.

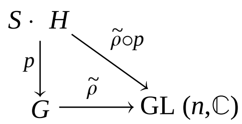
Claim: *$\tilde{\rho}$ is well defined.* If $sh = s'h' \in G$, then $(s')^{-1}s = h'h^{-1} \in S \cap H$. Then condition (1) *implies that* $\varphi(s')^{-1}\varphi(s) = \rho(h')\rho(h)^{-1}$ and thus $\tilde{\rho}(sh) = \varphi(s)\rho(h) = \varphi(s')\rho(h') = \tilde{\rho}(s'h')$.

Claim: *$\tilde{\rho}$ is a homomorphism.* For $sh, s'h' \in G$, condition (2) implies that

$$\begin{aligned} \tilde{\rho}((s'h')(sh)) &= \varphi(s')\varphi(s)\rho(s^{-1}h's)\rho(h) \\ &= \varphi(s')\varphi(s)\varphi(s)^{-1}\rho(h')\varphi(s)\rho(h) \\ &= \tilde{\rho}(s'h')\tilde{\rho}(sh), \end{aligned}$$

since $(s'h')(sh) = (s's)(s^{-1}h's)h$ and $s^{-1}h's \in H$.

Claim: $\tilde{\rho}$ is continuous. The map $p: S \times H \to G$ sending $(s,t) \mapsto st$ is a continuous surjection. Since both S and H are compact, p is a closed map so that G has the quotient topology induced by p.



Then the continuity of $\tilde{\rho}$ follows from the universal property of the identification map p since the composition $\tilde{\rho} \circ p: S \times H \to \mathrm{GL}(n,\mathbb{C})$ sending $(s,t) \mapsto \varphi(s)\rho(t)$ is continuous. □

Remark. In case that ρ is irreducible, condition (2) in Lemma 3.1 implies that φ is a lifting homomorphism of the associated projective representation ρ^* (defined in the previous section) over S, i.e., $\pi \circ \varphi = \rho^*$ on S. On the other hand, any lifting homomorphism φ of ρ^* over S satisfies condition (2).

Our main concern in this paper is to study extensions of representations when G is a compact Lie group and H is a closed normal subgroup of G such that G/H is connected. In this case, every complex representation ρ of H is G-invariant. Indeed, for each $g \in G$, there is a continuous path g_t in G from g to an element $h \in H$ since every connected component of G contains an element of H. Then the path g_t induces a continuous family of conjugate representations ${}^{g_t}\rho$ so that all representations ${}^{g_t}\rho$ are isomorphic (see [3, Lemma 38.1] for more general result). In particular, ${}^{g}\rho = {}^{g_0}\rho$ and $\rho = {}^{h}\rho = {}^{g_1}\rho$ are isomorphic.

Let ρ be a complex irreducible representation of H. Since ρ is always G-invariant, the associated projective representation ρ^* exists by Lemma 2.1. To get a G-extension of ρ we shall first find a closed subgroup S of G such that $G = SH$, and then construct a lifting homomorphism φ of ρ^* over S (so that condition (2) is satisfied). Finally, modifying φ a little to satisfy condition (1) we may get a G-extension of ρ.

Lemma 3.2. *Let G be a compact Lie group and H a closed normal subgroup such that $G/H \cong S^1$. Then there exists a circle subgroup S of G such that $G = SH$ and $S \cap H$ is finite cyclic.*

Proof. Let G_0 denote the identity component of G. Since the canonical projection $p: G \to G/H$ is open and closed, $p(G_0)$ is a connected component of G/H so that $p(G_0) = G/H$. It is well known in Lie group theory [4, Theorem 6.15] that $G_0 = Z_0 G_0'$, where Z_0 is the identity component of the center of G_0, which is a torus and G_0' is the commutator subgroup of G_0. Then $G_0' \subset G_0 \cap H \subset H$ since $G/H = G_0/(G_0 \cap H)$ is abelian, and thus $p(Z_0) = G/H$. Using the isomorphism $G/H \cong U(1)$ we may view $p|_{Z_0}$ as a one-dimensional unitary representation of the torus Z_0. It is elementary in representation theory that there exists a circle subgroup $S \subset Z_0$ such that $p(S) = G/H$. Therefore, $G = SH$ and, furthermore, the proper subgroup $S \cap H$ of the circle group S is finite cyclic. □

Lemma 3.3. *Let T be a maximal torus in $U(n)$. Then the exact sequence $0 \to S^1 \to T \to T/S^1 \to 0$ splits. Here S^1 is identified with the subgroup of $U(n)$ consisting of constant multiples zI for $z \in S^1 \subset \mathbb{C}$ where I denotes the identity matrix.*

Proof. Since any maximal torus T in $U(n)$ is conjugate to the subgroup $\Delta(n) \subset U(n)$ of diagonal matrices

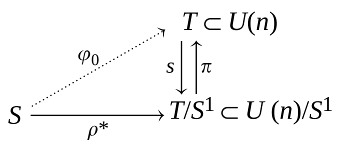
$$D(z_1, \ldots, z_n) = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix}, \quad z_i \in S^1,$$

it suffices to show that the exact sequence $0 \to S^1 \to \Delta(n) \to \Delta(n)/S^1 \to 0$ splits. But the splitting is immediate because of the homomorphism $\Delta(n) \to S^1$ mapping a diagonal matrix $D(z_1, \ldots, z_n)$ to the constant multiple $z_1 I \in S^1$. □

Proposition 3.4. *Let G be a compact Lie group and H a closed normal subgroup such that $G/H \cong S^1$. Then every complex representation of H is extendible to G.*

Proof. Let $\rho : H \to \mathrm{GL}(n, \mathbb{C})$ be a given representation. Since H is compact, we may assume that all the images of ρ are contained in $U(n) \subset \mathrm{GL}(n, \mathbb{C})$. Moreover, it is enough to prove the case that ρ is irreducible. Since $G/H \cong S^1$ is connected, ρ is G-invariant so that the associated projective representation $\rho^* : G \to U(n)/S^1 \subset \mathrm{PGL}(n, \mathbb{C}) = \mathrm{GL}(n, \mathbb{C})/\mathbb{C}^*$ exists by Lemma 2.1. From Lemma 3.2 we can choose a circle subgroup S of G such that $G = SH$ and $S \cap H$ is finite cyclic.

We shall find a lifting homomorphism $\varphi_0 : S \to U(n)$ of ρ^* over S. Since $\rho^*(S)$ is compact, connected, and abelian, it is a torus in $U(n)/S^1$. Note that every maximal torus in $U(n)/S^1$ has the form T/S^1 for some maximal torus T of $U(n)$ [1, Theorem 2.9, Chapter IV]. Choose a maximal torus T of $U(n)$ such that $\rho^*(S) \subset T/S^1$. By Lemma 3.3 the exact sequence $0 \to S^1 \to T \xrightarrow{\pi} T/S^1 \to 0$ splits, i.e., the canonical projection $\pi : T \to T/S^1$ has a continuous section (homomorphism) $s : T/S^1 \to T$ such that the composition $\pi \circ s$ is the identity map of T/S^1. Then $\varphi_0 = s \circ \rho^*|_S$ is a desired lifting homomorphism of ρ^* over S.



Let t_0 denote a generator of the finite cyclic group $S \cap H$. Since $\pi \circ \varphi_0 = \rho^* = \pi \circ \rho$ on $S \cap H$, $\varphi_0(t_0) = \xi \rho(t_0)$ for some constant $\xi \in S^1 \subset \mathbb{C}^*$. Note that ξ is an nth root of unity, where n is the order of $S \cap H$. So it is possible to choose a one-dimensional unitary representation τ of the circle group S such that $\tau(t_0) = \xi^{-1}$. Then the unitary representation $\varphi = \tau \otimes \varphi_0$ satisfies the conditions (1) and (2) in Lemma 3.1. □

Corollary 3.5. *Let G be a compact Lie group and H a closed normal subgroup such that G/H is connected and abelian. Then every complex representation of H is extendible to G.*

Proof. Since G/H is compact, connected, and abelian, it is isomorphic to a torus. So we have a finite chain of subgroups

$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_{n-1} \lhd H_n = G$$

such that H_i is normal in H_{i+1} and $H_{i+1}/H_i \cong S^1$. Applying Proposition 3.4 inductively, any representation of H is extendible to G. □

4. Extensions when G/H is connected

In this section, we consider the general case, so G/H will be assumed to be connected (not necessarily abelian). In this case, the commutator subgroup $(G/H)' = G'H/H$ of G/H is semisimple connected [4, Theorem 6.18]. The following proposition reduces the extension problem to the case that G/H is semisimple and connected.

Proposition 4.1. *Let G be a compact Lie group and H a closed normal subgroup of G such that G/H is connected. A complex representation of H is extendible to G if and only if it is extendible to $G'H$.*

Proof. The necessity is obvious, and the sufficiency follows from Corollary 3.5 since the factor group $G/G'H \cong (G/H)/(G'H/H) = (G/H)/(G/H)'$ is compact, connected, and abelian, that is a torus. □

In the case that G/H is semisimple connected, the following result is well-known in Lie group theory (see for instance, [4, Proposition 6.14]).

Lemma 4.2. *Let G be a compact Lie group and H a closed normal subgroup such that G/H is semisimple and connected. Then there is a semisimple connected closed normal subgroup S in G such that $G = SH$ and the map $S \times H \to G$ sending $(s,h) \mapsto sh$ is a homomorphism with a discrete kernel isomorphic to $S \cap H$.*

Remark. Proposition 6.14 in [4] deals with the case when G is connected. However, the same proof holds even if G is not connected, since G/H is connected. Moreover, we can find the fact in the proof that S is semisimple and connected.

The following result implies that the existence of a G-extension when G/H is semisimple and connected is completely determined by the restriction of a given representation to $S \cap H$.

Proposition 4.3. *Under the hypotheses of Lemma 4.2, a complex irreducible representation ρ of H is extendible to G if and only if ρ is trivial on $S \cap H$, i.e., $\rho(g) = I$, the identity matrix, for all $g \in S \cap H$.*

Proof. It is immediate that S commutes with H, since the map $S \times H \to G$ sending $(s,h) \mapsto sh$ is a homomorphism. To prove the sufficiency, it is enough to choose the trivial representation φ of S, i.e., $\varphi(s) = I$ for all $s \in S$. Since S commutes with H, the two conditions (1) and (2) in Lemma 3.1 are satisfied immediately.

On the other hand, suppose $\tilde{\rho}$ is a G-extension of ρ. Since S commutes with H, we have $\tilde{\rho}(s)^{-1}\rho(h)\tilde{\rho}(s) = \rho(h)$ for all $s \in S$ and $h \in H$. Then the Schur's lemma implies that $\tilde{\rho}(s)$ is constant for all $s \in S$, so we may view the restriction $\tilde{\rho}|_S$ as a one-dimensional complex representation of S. Since semisimple Lie groups have no nontrivial abelian factor group, the trivial representation is the unique one-dimensional complex representation of S. Therefore, $\tilde{\rho}$ is trivial on S, in particular, on $S \cap H$. □

Remark. Note that the number of G-extensions (if exist) is exactly one, since every G-extension should be trivial on S.

Corollary 4.4. *Let G be a compact Lie group and H a closed normal subgroup such that G/H is semisimple and connected. Every complex representation of H is extendible to G if and only if H is a direct summand of G, i.e., $G \cong S \times H$ for some subgroup S of G.*

Proof. The sufficiency is obvious so we prove the necessity. If H is not a direct summand of G, then $S \cap H$ in Lemma 4.2 contains a nontrivial element, say s_0. Since a faithful representation of H always exists [1, Theorem 4.1, Chapter III], we can choose an irreducible sub-representation ρ of H such that $\rho(s_0)$ is not trivial. Then ρ does not extend to a representation of G by Proposition 4.3. □

We shall now prove the main result in this paper. For the second statement of Theorem 1.1, we need the following lemma giving a relation between the normal subgroup $S \cap H$ in Lemma 4.2 and the fundamental group of G/H.

Lemma 4.5. *Under the hypotheses of Lemma* 4.2, *there exists a surjective homomorphism $\pi_1(G/H) \to S \cap H$.*

Proof. Since $S/(S \cap H) = G/H$, the restriction of the canonical projection $p : G \to G/H$ on S is surjective and its kernel $S \cap H$ is discrete. It follows that $p|_S$ is a covering homomorphism of G/H. From the uniqueness of the universal covering homomorphism $\tilde{q} : \widetilde{G/H} \to G/H$, there exists a covering homomorphism $q : \widetilde{G/H} \to S$ such that the diagram

$$\begin{array}{ccc} \widetilde{G/H} & \xrightarrow{q} & S \\ {\scriptstyle \tilde{q}}\downarrow & \swarrow {\scriptstyle p|_S} & \\ G/H & & \end{array}$$

commutes (compare with [4, Proposition 9.12]). Since $S \cap H = \ker p|_S = q(\ker \tilde{q})$ and $\ker \tilde{q}$ is isomorphic to $\pi_1(G/H)$, we have a surjective homomorphism of $\pi_1(G/H)$ onto $S \cap H$. □

Theorem 1.1 (rephrased). *Let G be a compact Lie group and H a closed normal subgroup such that G/H is connected. Then every complex representation of H is extendible to G if and only if H is a direct summand of $G'H$.*

Proof. Since the factor group $G'H/H = (G/H)'$ is semisimple and connected, the theorem follows immediately from Proposition 4.1 and Corollary 4.4. □

Proof of Corollary 1.2. We claim that $\mathrm{Tor}(\pi_1(G/H))$, the torsion subgroup of $\pi_1(G/H)$, is isomorphic to $\pi_1((G/H)')$. Denote by T the torus $(G/H)/(G/H)'$. Then the homotopy exact sequence of the fibration $(G/H)' \to G/H \to T$ implies that $\pi_1(G/H) \cong \pi_1((G/H)') \oplus \pi_1(T)$, since the second homotopy group of a compact Lie group vanishes, see [1, Proposition 7.5, Chapter V]. Since $(G/H)'$ is semisimple, $\pi_1((G/H)')$ is finite [1, Remark 7.13, Chapter V] so that it is isomorphic to $\mathrm{Tor}(\pi_1(G/H))$ as we claimed. Therefore, the condition of $\pi_1(G/H)$ being torsion free is equivalent to $(G/H)'$ being simply connected.

By Lemmas 4.2 and 4.5, $G'H = SH$ for some semisimple connected closed normal subgroup S in $G'H$ and there is a surjective homomorphism $\pi_1(G'H/H)=\pi_1((G/H)') \to S \cap H$. Therefore, if $(G/H)'$ is simply connected, then $\pi_1((G/H)') = S \cap H = \{e\}$ so that H is a direct summand of $G'H$. □

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