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Renormalization group methods for a Mathieu equation with delayed feedback

Yunna Wu[a)] and Xu Xu[b)]
College of Mathematics, Jilin University, Changchun 130012, China

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Abstract This paper presents the application of the renormalization group (RG) methods to the delayed differential equation. By analyzing the Mathieu equation with time delay feedback, we get the amplitude and phase equations, and then obtain the approximate solutions by solving the corresponding RG equations. It shows that the approximate solutions obtained from the RG method are superior to those from the conventionally perturbation methods. [doi:10.1063/2.1306307]

Keywords renormalization group methods, RG equations, time delay, multi-scale method

The renormalization group (RG) method[1–3] is one of the unified tool for global asymptotic analysis of the singularly-perturbed differential equations.[4,5] Motivated by RG methods being widely used in solid state physics and other areas,[5,6] it has been applied to decuce approximating solutions of differential equations, with and without turning points, with fast and slow time scales, with boundary layers, etc. It is usually used to provide an option to classical perturbation methods, such as the multiple time scales method, the averaging method, and the WKB method.[4] Some numerical experimental from RG method agree with those from classical methods. Moreover, RG method introduces automatically the appropriate gauge functions of small parameter ε, and avoids the guess of appropriate form of functions. However, up to now, there are a little applications of RG method in the fields of engineering and the applied mechanics.[7–10] The aim of this paper is to apply RG method to solve the delayed differential equations by considering a Mathieu equation with time delay feedback.

We consider the following general differential equation

$$\begin{aligned} &\dot{x} = Ax + \varepsilon f(x), \\ &x(t_0) = x_0, \end{aligned} \tag{1}$$

where $f(x)$ is the nonlinear function, $\varepsilon \ll 1$, and t_0 is the initial time. The aim of this paper is to give approximate analytical solutions of Eq. (1) on time scales of $O(1/\varepsilon)$. The RG method consists of the following steps.[7]

(1) A naive perturbation expansion is derived for the solutions of the given system.

(2) All instances of the initial condition are removed by making a preparatory change of variables.

(3) Introduce an arbitrary time k in between t and initial time t_0.

(4) Remove terms involving $(k - t_0)$ by renormalizing the solution.

(5) Apply the RG condition $\mathrm{d}x/\mathrm{d}k|_{k=t} = 0$ to get the renormalized solution.

By using the RG method, we investigate the following Mathieu equation with delayed position feedback

$$\ddot{y} + ay = by(t - \tau) - 2\varepsilon y \cos t, \tag{2}$$

where $\varepsilon \ll a$, τ is time delay, $by(t - \tau)$ is the delayed position feedback, and $b = O(\varepsilon)$. The stability of solutions near $a = 1/4$ is firstly investigated without losing generality, and then the stability boundary in the (a, ε) plane is determined. By expanding a and y in powers of ε, we have

$$\begin{aligned} &a = 1/4 + a_1\varepsilon + a_2\varepsilon^2 + \cdots, \\ &y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots \end{aligned} \tag{3}$$

The aim of the paper is to determine the values of a_1, a_2, y_0, y_1, y_2, ...

Using multiple scale method, we can solve this problem. However, the multiple time scales $\tau_1 = \varepsilon t$, $\tau_2 = \varepsilon^2 t, \cdots$ are not enough to derive the coefficient a_2, and the hidden time scale $\sigma = \varepsilon^{3/2} t$ must be included. This is a typical shortcoming of multiple scale analysis.[1,2] However, some studies show that the RG method can automatically give the time scales $\sigma = \varepsilon^{3/2} t$, only staring with a straightforward perturbative expansion.[1,2] For simplicity, we only consider the first-order expansion.

Substituting $a = 1/4 + a_1\varepsilon$ and $y = y_0 + \varepsilon y_1$ into Eq. (2), one gets

$$\begin{aligned} &\ddot{y}_0 + 1/4 y_0 = 0, \\ &\ddot{y}_1 + 1/4 y_1 = c y_0(t - \tau) - 2y_0 \cos t - a_1 y_0, \end{aligned} \tag{4}$$

with $c = b/\varepsilon$. The solution of the first equation of Eq. (4) is

$$y_0 = R_0 \cos(t/2 + \theta_0), \tag{5}$$

where R_0 and θ_0 are constants depending on the initial conditions. Substituting Eq. (5) into Eq. (4) gets

$$y_1 = cR_0 t \sin\left(\frac{t}{2} - \frac{\tau}{2} + \theta_0\right) - R_0 t \sin\left(\frac{t}{2} - \theta_0\right) -$$

a)Email: wuyunnuo@163.com.
b)Corresponding author. Email: xuxu@jlu.edu.cn.

$$a_1 R_0 t \sin\left(\frac{t}{2}+\theta_0\right)+\frac{1}{2}R_0\cos\left(\frac{3t}{2}+\theta_0\right) \tag{6}$$

and

$$y = R_0\cos\left(\frac{1}{2}t+\theta_0\right)+ \varepsilon\Bigg(cR_0 t\sin\left(\frac{1}{2}t-\frac{1}{2}\tau+\theta_0\right)- R_0 t\sin\left(\frac{1}{2}t-\theta_0\right)-a_1R_0t\sin\left(\frac{1}{2}t+\theta_0\right)+ \frac{1}{2}R_0\cos\left(\frac{3}{2}t+\theta_0\right)\Bigg)+O(\varepsilon^2). \tag{7}$$

We then introduce the renormalization variables R and θ as $R_0 = R+R\gamma\varepsilon$ and $\theta_0=\theta+\lambda\varepsilon$. For an arbitrary time k between 0 and t, we have $t = t-k+k$, where 0 is assumed to be the initial time. Substituting it into Eq. (7), one gets

$$y = R\cos\left(\frac{t}{2}+\theta\right)+R\gamma\varepsilon\cos\left(\frac{t}{2}+\theta\right)- R\lambda\varepsilon\sin\left(\frac{t}{2}+\theta\right)+\varepsilon R\Bigg[c(t-k)\cdot \sin\left(\frac{t}{2}-\frac{\tau}{2}+\theta\right)+ck\left(\sin\left(\frac{t}{2}+\theta\right)\cos\frac{\tau}{2}- \cos\left(\frac{t}{2}+\theta\right)\sin\frac{\tau}{2}\right)-(t-k)\sin\left(\frac{t}{2}-\theta\right)- k\left(\sin\left(\frac{t}{2}+\theta\right)\cos 2\theta-\cos\left(\frac{t}{2}+\theta\right)\sin 2\theta\right)- a_1(t-k)\sin\left(\frac{t}{2}+\theta\right)-a_1k\sin\left(\frac{t}{2}+\theta\right)+ \frac{1}{2}\cos\left(\frac{3t}{2}+\theta\right)\Bigg]+O(\varepsilon^2). \tag{8}$$

In order to eliminate the terms as $k\sin(t/2+\theta)$ and $k\cos(t/2+\theta)$, we let

$$\gamma - ck\sin\frac{1}{2}\tau+k\sin 2\theta = 0, \tag{9}$$

$$\lambda - ck\cos\frac{1}{2}\tau+k\cos 2\theta+a_1k = 0. \tag{10}$$

Thus, Eq. (8) can be written as

$$y = R\cos\left(\frac{t}{2}+\theta\right)+\varepsilon R\Bigg[c(t-k)\cdot \sin\left(\frac{t}{2}-\frac{\tau}{2}+\theta\right)-(t-k)\sin\left(\frac{t}{2}-\theta\right)- a_1(t-k)\sin\left(\frac{t}{2}+\theta\right)+\frac{1}{2}\cos\left(\frac{3t}{2}+\theta\right)\Bigg]. \tag{11}$$

Since we can not find k in the original problem, the solution must be independent of k. Therefore, we have $\mathrm{d}y/\mathrm{d}k|_{k=t}=0$ for any t. Then we have

$$\frac{\partial y}{\partial R}\frac{\mathrm{d}R}{\mathrm{d}k}+\frac{\partial y}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}k}+\frac{\partial y}{\partial k}=0, \tag{12}$$

where

$$\left.\frac{\partial y}{\partial R}\right|_{k=t} = \cos\left(\frac{1}{2}t+\theta\right)+\frac{1}{2}\varepsilon\cos\left(\frac{3}{2}t+\theta\right), \tag{13}$$

$$\left.\frac{\partial y}{\partial \theta}\right|_{k=t} = -R\sin\left(\frac{1}{2}t+\theta\right)- \frac{1}{2}\varepsilon R\sin\left(\frac{3}{2}t+\theta\right), \tag{14}$$

$$\left.\frac{\partial y}{\partial k}\right|_{k=t} = \varepsilon R\sin\left(\frac{t}{2}+\theta\right)\left(\cos 2\theta+a_1-c\cos\frac{\tau}{2}\right)+ \varepsilon R\cos\left(\frac{t}{2}+\theta\right)\left(c\sin\frac{\tau}{2}-\sin 2\theta\right). \tag{15}$$

Note that the second terms of the right side of Eqs. (13) and (14) are $O(\varepsilon)$. Therefore we approximately have the following RG equation

$$\frac{\mathrm{d}R}{\mathrm{d}k} = \varepsilon R\sin 2\theta - \varepsilon cR\sin\frac{1}{2}\tau, \tag{16}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}k} = \varepsilon(\cos 2\theta + a_1)-\varepsilon c\cos\frac{1}{2}\tau. \tag{17}$$

To analyze the stability of steady-state solution, we let the right side of Eq. (17) be zero and get

$$\begin{aligned}&\cos 2\theta = c\cos\frac{\tau}{2}-a_1,\\ &\sin 2\theta = \pm\sqrt{1-\left(c\cos\frac{\tau}{2}-a_1\right)^2}.\end{aligned} \tag{18}$$

Substituting Eq. (18) into Eq. (16) gets

$$R = \bar{R}\exp\left(\pm\sqrt{1-\left(c\ \cos\frac{\tau}{2}-a_1\right)^2}- c\ \sin\frac{\tau}{2}\right)\varepsilon t, \tag{19}$$

where $\bar{R}$ is constant. Substituting $k=t$ into Eq. (11) gets the asymptotic expansions of solutions

$$y = R\cos\left(\frac{t}{2}+\theta\right)+\varepsilon R\frac{1}{2}\cos\left(\frac{3t}{2}+\theta\right), \tag{20}$$

where R and θ satisfy Eqs. (18) and (19), respectively. It is clear that

$$Z \equiv \pm\sqrt{1-\left(c\ \cos\frac{\tau}{2}-a_1\right)^2}-c\ \sin\frac{\tau}{2}=0 \tag{21}$$

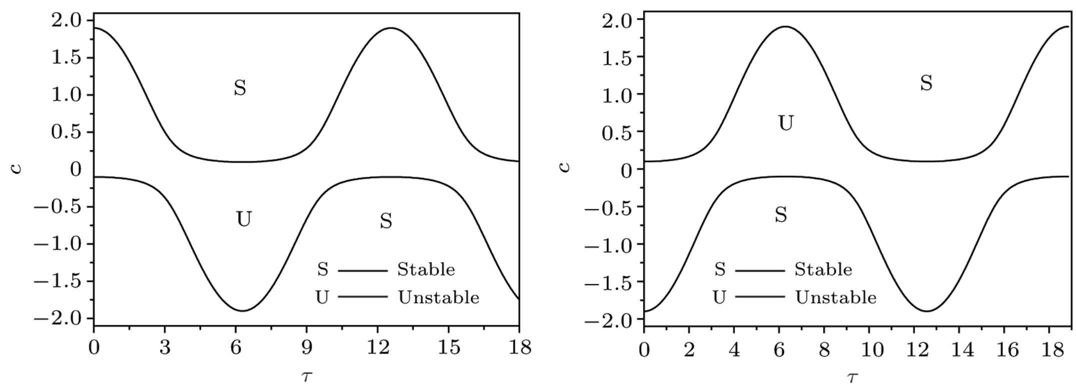
determines the boundary of stability. For the given c and τ, we can get the critical value of a_1.

(1) If $c=0$, Eq. (2) becomes the classic Mathieu equation. System is stable for $|a_1|>1$; $|a_1|<1$ indicates the divergent solutions; $|a_1|=1$ is the boundary of stability.[2]

(2) If $\tau=0$, $|a_1-c|=1$ is the boundary of stability.

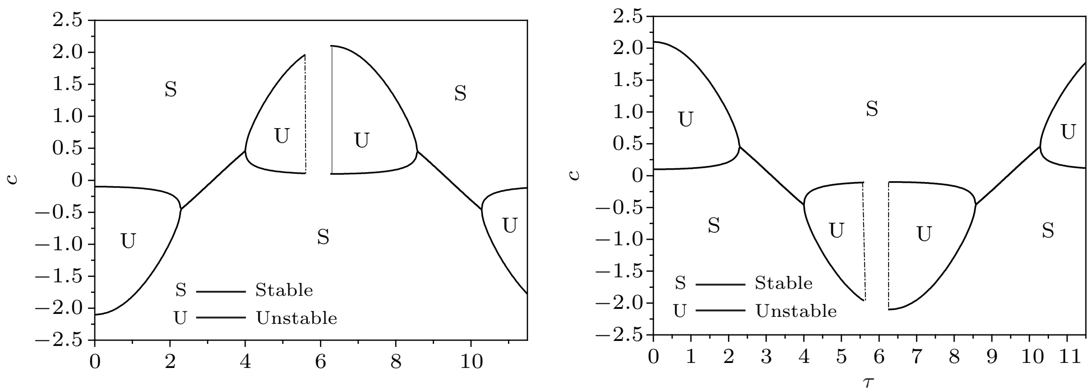
(3) If $A\equiv 1-(c\cos(\tau/2)-a_1)^2<0$, $-c\sin(\tau/2)<0$ indicates the system stability, and $-c\sin(\tau/2)>0$ indicates the system instability.

(4) If $A>0$, then $Z>0$ indicates the stability, while $Z<0$ indicates the instability.



(a) $a_1 = 0.9$ (b) $a_1 = -0.9$

Fig. 1. Stability regions in $c - \tau$ plane with $|a_1| < 1$.



(a) $a_1 = -1.1$ (b) $a_1 = 1.1$

Fig. 2. Stable and unstable regions in $c - \tau$ with $|a_1| > 1$.

Figure 1 illustrates the stability regions in $c-\tau$ with $|a_1| < 1$. It is clear that when $c = 0$, system is unstable which is consistent with that in Ref. 2 with $|a_1| < 1$. Figure 2 illustrates that the stable and unstable regions in $c-\tau$ with $|a_1| > 1$. It is easy to see from these two figures that the delayed position feedbacks can effectively stabilize the system.

Using RG method, we derive the approximate stead solutions of the Mathieu equation with time delay feedback. It shows that the RG method does not need to guess the appropriate gauge functions, and it is superior to some classic perturbation methods. Moreover, the real computation indicates that RG method has higher computational efficiency compared to some classical perturbation methods.

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