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Enumeration of noncrossing trees on a circle

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Abstract

We consider several enumerative problems concerning labelled trees whose vertices lie on a circle and whose edges are rectilinear and do not cross.

1. Introduction

Take n points on a circle labelled counterclockwise from 1 to n and consider graphs whose vertices are the given points and whose edges are rectilinear and do not cross. Call them *noncrossing graphs*. The problem of counting such graphs according to n and to the number m of edges was already studied by Kirkman and Cayley in the last century, and more recently by Watson [12] and Domb and Barrett [3]. The last reference also considers the enumeration of *connected* non-crossing graphs and contains recurrence formulae for computing them. Specializing when $m=n-1$, one can, in principle, compute the number t_n of non-crossing *trees* (*nc-trees* for short) on n points on a circle (see Fig. 1).

However, the problem of counting nc-trees was not considered explicitly in [3]. In the work of Dulucq and Penaud [4] one can find (among other interesting results related to configurations of chords on a circle and to the decomposition of permutations) a simple combinatorial proof of the following basic fact.

Theorem 1.1 (Dulucq and Penaud [4]). *The number of nc-trees on $n+1$ points is equal to the number of ternary trees with n internal vertices.*

If we let σ_n be the number of ternary trees with n internal vertices, it is known (see [7]) that $\sigma_n=(1/(2n+1))\binom{3n}{n}$, a generalized Catalan number. These numbers, although not as ubiquitous as the ordinary Catalan numbers, also appear in the solution of several combinatorial problems: dissections of a convex polygon into quadrilaterals by

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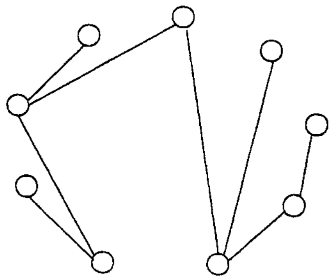


Fig. 1. A non-crossing tree.

means of diagonals, ways of associating a ternary operation to a string of symbols, lattice-paths below the line $y = 2x$, and some others.

Later we will make use of the fact that the generating function $S = \sum \sigma_n z^n$ satisfies the equation

$$S - 1 = zS^3.$$

It follows that, if we define $T = \sum t_n z^n$ then $T = zS$, and we have the following immediate corollary to Theorem 1.1.

Corollary 1.2. *The number of nc-trees on n points is equal to*

$$t_n = \frac{1}{2n-1}\binom{3n-3}{n-1}$$

and the generating function $T = \sum t_n z^n$ *satisfies* $zT - z^2 = T^3$.

On the other hand, we would like to remark that the enumeration of non-crossing trees on point configurations, other than a circle, has been studied elsewhere [6].

The goal of this paper is to further the enumerative study of non-crossing trees. In Section 2, we study the enumeration of non-crossing trees according to the degree of a fixed vertex. In Sections 3–6, we enumerate the following: unicyclic non-crossing graphs, bipartite non-crossing trees, non-crossing forests, and unlabelled non-crossing trees. In some cases we find closed-form formulae, while in others we obtain algebraic equations for the corresponding generating functions and deduce asymptotic estimates from them. The paper concludes with some remarks and open problems.

2. Enumeration according to the degree of a node

Let $t(n,d)$ be the number of trees on n vertices in which a given vertex (say number 1) has degree d. The following lemma expresses $t(n,d)$ as a convolution on the numbers t_n.

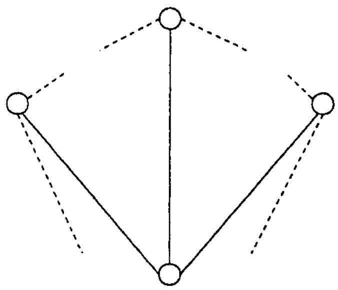


Fig. 2. Proof of Lemma 2.1.

Lemma 2.1. *The numbers $t(n,d)$ can be written in terms of the numbers t_n as follows*:

$$t(n,d) = \sum_{\substack{i_1+\cdots+i_{2d}=n+d-1 \\ i_1,\ldots,i_{2d}\geqslant 1}} t_{i_1} t_{i_2} \cdots t_{i_{2d}}.$$

Proof. Let T be an nc-tree on n vertices in which vertex 1 has degree d and is joined to vertices $k_1 < \cdots < k_d$. It is clear that, for every $i = 1,\ldots,d-1$, the subgraph induced by T on the vertex set $\{k_i,\ldots,k_{i+1}\}$ is the disjoint union of two nc-trees. Also, the subgraphs induced on $\{2,\ldots,k_1\}$ and $\{k_d,\ldots,n\}$ are both nc-trees. This makes a total of $2d$ trees of sizes, say, $i_1, i_2, \ldots, i_{2d}$ and it is easily checked that $i_1+\cdots+i_{2d} = n+d-1$. Fig. 2 illustrates this fact for $d = 3$.

Moreover, a family of $2d$ nc-trees on the corresponding vertex sets determines a unique nc-tree T in which $\delta(1) = d$. This proves the formula. □

In order to obtain a closed formula for $t(n,d)$, we introduce the generating functions $T_d = \sum t(n,d) z^n$, for $d > 0$. The main result will be an application of the Lagrange–Bürmann inversion formula (see [2]).

Lemma 2.2 (Lagrange–Bürmann). *Let $s(z)$ be a power series with complex coefficients and γ a complex number, satisfying an equation*

$$s(z) = \gamma + z \cdot g(s(z)).$$

Then,

$$[z^n]u(s(z)) = u(\gamma) + \frac{1}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}\left\{\frac{\mathrm{d}u(t)}{\mathrm{d}t} g^n(t)\right\}_{t=\gamma}.$$

We are ready for the main result of this section.

Theorem 2.3. *For every $d \geqslant 1$ and $n > d$*

$$t(n,d) = \frac{2d}{3n-d-3}\binom{3n-d-3}{n-d-1}.$$

Proof. The convolution of the above lemma translates directly into the following equation:

$$T_d = \frac{1}{z^{d-1}} T^{2d} = z^{d+1} S^{2d}.$$

Since $T/z = S$ satisfies $S - 1 = zS^3$, we can apply Lagrange's inversion to the series T_d with $\gamma = 1$, $g(t) = t^3$ and $u(t) = t^{2d}$. We get

$$[z^n]S^{2d} = \frac{1}{n!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} \{2dt^{3n+2d-1}\}_{t=1},$$

and after a simple computation one arrives at the expression for $t(n,d)$ claimed above. □

We will also need a simple consequence of Theorem 2.3.

Corollary 2.4. *The number of nc-trees on n vertices containing the edge $(1,n)$ is equal to*

$$t(n,1) = \frac{2}{3n-4}\binom{3n-4}{n-2}.$$

Proof. For an nc-tree containing edge $(1,n)$ there exist j, with $1 \leqslant j \leqslant n$, such that each of the subgraphs induced on $\{1,\dots,j\}$ and $\{j+1,\dots,n\}$ is an nc-tree. Thus, the required number is $t_1 t_{n-1} + \cdots + t_{n-1} t_1$ which, by Lemma 2.1, is equal to $t(n,1)$. □

If we consider random nc-trees on n labelled points, then Theorem 2.3 gives the density of the distribution according to the degrees of the nodes. If $d(x)$ denotes the degree of node x in a random nc-tree, then

$$P(d(x) = d) = \frac{t(n,d)}{t_n} = \frac{2d}{3n-d-3}\binom{3n-d-3}{n-d-1}(2n-1)\binom{3n-3}{n-1}^{-1}.$$

We have computed the basic statistics of this distribution.

Theorem 2.5. *The mean and the variance of the distribution $d(x)$ are given by*

$$E(d(x)) = 2(1 - 1/n),$$
$$\sigma^2(d(x)) = (1 - 1/n)(1 - 2/n)(3n - 2)/(2n + 1).$$

Proof. The mean is the same as for any family of labelled trees: it is just the sum $2n-2$ of the degrees divided by n. On the other hand, since $\sigma^2(d(x)) = E(d(x)^2) - E(d(x))^2$, we only need to prove that

$$E(d(x)^2) = 4(1 - 1/n)^2 + (1 - 1/n)(1 - 2/n)(3n - 2)/(2n + 1)$$
$$= \frac{(n-1)(11n-12)}{n(2n+1)}.$$

This is equivalent to the combinatorial identity

$$\sum_d \frac{2d^3}{3n-d-3}\binom{3n-d-3}{n-d-1} = \frac{1}{2n-1}\binom{3n-3}{n-1}\frac{(n-1)(11n-12)}{n(2n+1)},$$

which can be proved automatically using Zeilberger's algorithm [13]. □

The variance of $d(x)$ for general labelled trees is equal to $(1-1/n)(1-2/n)$ (see [8] for this and related results). In the case of nc-trees the above theorem says it is about $\frac{3}{2}$ times higher. It is also interesting to compare Theorem 2.3 with the corresponding result for arbitrary labelled trees. If we let $c(n,d)$ be the number of labelled trees on n vertices in which a given node has degree d then it is known that $c(n,d) = \binom{n-2}{d-1}(n-1)^{n-d-1}$ (and, of course, the number of labelled trees is n^{n-2}). It can be of interest to compare the ratios $t(n,d)/t_n \sim 4d/3^{d+1}$ and $c(n,d)/n^{n-2} \sim (e(d-1)!)^{-1}$ for fixed d.

Finally, the mean number of leaves (vertices of degree 1) in a random nc-tree is equal to $nt(n,1)/t_n \sim 4n/9 = 0.444n$ while for random labelled trees it is $c(n,1)/n^{n-2} \sim n/e = 0.367n$. Hence, a random nc-tree has asymptotically more leaves than a random labelled tree.

3. Unicyclic non-crossing graphs

A unicyclic graph is a connected graph with a unique cycle or, equivalently, a connected graph with the same number of edges and vertices. If we remove the unique cycle from a unicyclic graph what remains is a collection of trees.

A *non-crossing unicyclic graph* (*unicyclic nc-graph* for short) on n points will be simply a unicyclic graph drawn on a circle without crossings. Let u_n be the number of unicyclic nc-graphs on n points, and let w_n be the number of them containing the edge $(1,n)$. The aim of this section is to find a closed-form formula for u_n.

First a technical lemma in the spirit of Corollary 2.4.

Lemma 3.1. *The following equations hold*:

$$u_n = t(2,1)w_n + \cdots + t(n,1)w_2,$$

$$w_n = t_n - t(n,1) + 2(u_1 t_{n-1} + \cdots + u_{n-1}t_1),$$

where t_n and $t(n,d)$ are as in the former sections.

Proof. Let e be an edge belonging to a unicyclic nc-graph G (see Fig. 3). Then to the left of e we have a unicyclic nc-graph containing e and to the right an nc-tree containing e, or conversely. If we count in two ways the number of pairs (e,G) with $e \in G$ then, since G contains n edges, applying Corollary 2.4 we have

$$nu_n = n(t(2,1)w_n + \cdots + t(n,1)w_2).$$

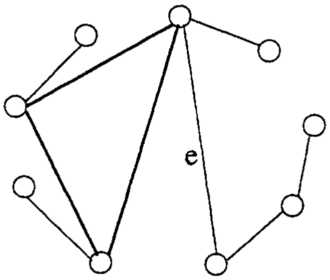


Fig. 3. Proof of Lemma 3.1.

For the second equation, let G be a unicyclic nc-graph containing $(1,n)$. If $(1,n)$ belongs to the unique cycle then $G\backslash 1n$ (deletion of an edge) is an nc-tree not containing $(1,n)$, and there are $t_n - t(n,1)$ of them by Corollary 2.4. Otherwise $G\backslash 1n$ has two connected components: one has to be a nc-tree and the other a unicyclic nc-graph. This gives $2(u_1 t_{n-1} + \cdots + |u_{n-1} t_1)$ possibilities. □

For the second equation to hold when $n=1$ we need to set $t(1,1)=1$. We take that as a convention throughout this section.

Theorem 3.2. *The number of unicyclic nc-graphs on n points is given by*

$$u_n = \binom{3n-3}{n-3}.$$

Proof. From Lemma 3.1 one obtains functional equations involving the generating functions $U = \sum u_n z^n$ and $W = \sum w_n z^n$:

$$z^2 U = (T_1 - z)W,$$

$$W = T - T_1 + 2UT,$$

where T and $T_1 = T^2 + z$ are as in Section 2 (remember that we have set $t(1,1)=1$ in this case). We can easily express U in terms of the g.f. S of the generalized Catalan numbers σ_n using the fact that $T = zS$ and $S - 1 = zS^3$, obtaining

$$U = z\,\frac{S(S-1)^2}{3-2S} = z^3\,\frac{S^7}{3-2S}.$$

One could, in principle, use again Lagrange's inversion to find the coefficients of U but the ensuing expressions become too clumsy. Instead, one can first establish the relation

$$2\,\frac{dU}{dz} = z^2\,\frac{d^2 S}{dz^2}.$$

This is done using the above expression for U and the functional equation for S. We omit the details. From this it follows that

$$nu_n = \binom{n-1}{2} t_n$$

and, since $t_n = (1/(2n-1))\binom{3n-3}{n-1}$, the result follows. □

4. Bipartite non-crossing trees

Even if a tree is always a bipartite graph, in this section we are interested in trees with a fixed bipartition. In other words, we take r red points and s blue points distributed on a circle, where $n = r + s$, and ask about the number of bipartite nc-trees on these two sets of vertices. The answer will, of course, depend on the particular distribution of the colours, so we consider two extreme cases particularly appealing: first when the red and the blue points are consecutive, and then when the two colours alternate.

Theorem 4.1. *The number of bipartite nc-trees on a chain of r consecutive red points followed by a chain of s consecutive blue points on a circle is equal to*

$$\binom{r+s-2}{r-1} = \binom{r+s-2}{s-1}.$$

Proof. Let $\delta(1), \delta(2), \ldots, \delta(r)$ be the degrees of the red vertices in such a tree. Obviously, $\delta(1) + \cdots + \delta(r) = r + s - 1$, the size of the tree, and it is clear that, in the non-crossing case, this ordered partition completely determines the bipartite tree. But the number of such partitions is precisely $\binom{r+s-2}{r-1}$. □

The case of alternating colours presents more difficulties. We need a result for obtaining asymptotic estimates from functional equations satisfied by generating functions, which we quote directly from Bender [1].

Lemma 4.2 (Bender [1]). *Assume the power series $w(z) = \sum a_n z^n$ with nonnegative coefficients satisfies $\phi(z, w) \equiv 0$. Suppose there exist real numbers $r > 0$ and $s > a_0$ such that*

(i) *for some $\delta > 0, \phi(z, w)$ is analytic whenever $|z| < r + \delta$ and $|q| < s + \delta$;*
(ii) *$\phi(r, s) = \phi_w(r, s) = 0$;*
(iii) *$\phi_z(r, s) \neq 0$, and $\phi_{ww}(r, s) \neq 0$; and*
(iv) *if $|z| \leqslant r$, $|w| \leqslant s$, and $\phi(z, w) = \phi_w(z, w) = 0$, then $z = r$ and $w = s$.*

Then

$$a_n \sim ((r\phi_z)/(2\pi\phi_{ww}))^{1/2} n^{-3/2} r^{-n},$$

where the partial derivatives ϕ_z and ϕ_{ww} are evaluated at $z = r$ and $w = s$.

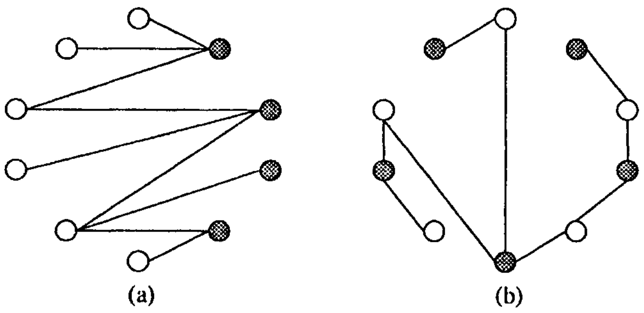


Fig. 4. Bipartite nc-trees. Left: consecutive colours. Right: alternating colours.

Theorem 4.3. *The number of bipartite nc-trees on n alternating red–blue vertices on a circle is, asymptotically, equal to*

$$a_n \sim Kn^{-3/2}\left(\frac{135+78\sqrt{3}}{16}\right)^{n/2}$$

for some constant K.

Proof. First note that, when n is odd, there will be necessarily two consecutive points with the same colour, which will be assumed to be vertices 1 and n. However, we do not need to distinguish between the even and the odd case. Our first claim is that the a_n satisfy the following recurrence:

$$a_n = \sum_{\substack{i,j,k\geqslant 1\\ i+j+k=n+1\\ i+j \text{ even}}} a_i a_j a_k,$$

starting with $a_1 = a_2 = a_3 = 1$ and $a_4 = 4$. To prove it, consider a bipartite nc-tree T on n vertices with alternating colours and let β be the neighbour of 1 with largest index, which is necessarily even. Then on the vertex set $\{i, i+1, \ldots, n\}$ we have induced a bipartite tree of the same kind.

Since in a tree there are no cycles, there has to exist some α such that T induces respective trees in the vertex sets $\{1, \ldots, \alpha\}$ and $\{\alpha+1, \ldots, \beta\}$ (see Fig. 4). The three induced trees are bipartite with alternating colours and together they determine T. Hence, $a_n = \sum a_\alpha a_{\beta-\alpha} a_{n-\beta+1}$ and the claim is proved.

Now, let $A(z) = \sum a_{n+1} z^n$ (taking a_{n+1} instead of a_n as the coefficient of z^n is meant to simplify the computations below). The above equation translates into the functional equation

$$A(z) - 1 = zA(z)\,\frac{A(z)^2 + A(-z)^2}{2}.$$

After substituting z by $-z$ and some straightforward manipulation we arrive at

$$A_-(z)^2 = A_+(z)(A_+(z) - 1),$$
$$A_+(z) - 1 = z^2 A_+(z)^3 (2A_+(z) - 1)^2,$$

where $A_+(z) = \sum a_{2n+1} z^{2n}$ and $A_-(z) = \sum a_{2n+2} z^{2n+1}$ are the even and the odd part of $A(z)$. We can apply directly Lemma 4.2 to the second equation (since it is a polynomial, the required conditions can be checked easily) and we get the value $r^{-1} = (135 + 78\sqrt{3})/16)^{1/2}$.

Eliminating $A_+(z)$ from the above equations we get another equation of degree five in $A_-(z)^2$, namely

$$16z^4u^5 - 24z^4u^4 + (9z^4 - 8z^2)u^3 - 2z^2u^2 + (1 + 30z^2)u + 9z^2 - 2 = 0.$$

Again using Lemma 4.2 the same value of r is found and the theorem is proved. $\square$

5. Non-crossing forests

In this section, we consider the enumeration of *non-crossing forests*, i.e. acyclic graphs, on a circle. The techniques will be similar to those employed in the last section. Let f_n be the number of nc-forests on n points on a circle.

Theorem 5.1. *The generating function $F = \sum f_{n+1} z^n$ satisfies the equation*

$$zF^3 + (z^2 - z)F^2 + (2z - 1)F + 1 = 0.$$

The numbers f_n are, asymptotically,

$$f_n \sim K n^{-3/2} \alpha^n,$$

where K is a constant and $\alpha \approx 8.22469$ and α^{-1} is the smallest positive root of $4\alpha^3 - 32\alpha^2 - 8\alpha + 5 = 0$.

Proof. Let g_n be the number of nc-forests on n points in which 1 and n belong to the same component, and let $G = \sum g_{n+1} z^n$ (note that, for technical reasons, the coefficients in F and G are shifted). We derive recurrence equations involving f_n and g_n as follows. Let Γ be an nc-forest on n points and let β be the neighbour of 1 with largest index. Let $\alpha + 1$ be the point with lowest index ($\alpha > 0$) belonging to the component of β. Then we have nc-forests on $\{1, \ldots, \alpha\}$ and $\{\beta, \ldots, n\}$, and a forest on $\{\alpha + 1, \ldots, \beta\}$ in which $\alpha + 1$ and β belong to the same component. As a consequence (the term f_{n-1} arises when 1 is an isolated point).

$$f_n = f_{n-1} + \sum f_\alpha g_{\beta-\alpha} f_{n-\beta+1} = f_{n-1} + \sum_{\substack{i+j+k=n+1 \\ i,j,k \geqslant 1}} f_i f_j g_k.$$

Similarly, if we start with an nc-forest in which 1 and n belong to the same component then

$$g_n = \sum_{\substack{i+j+k=n+1 \\ i,j,k \geqslant 1}} f_i g_j g_k.$$

In terms of the generating functions, we have $F - 1 = z(F^2G + F)$ and $G - 1 = zFG^2$. After elimination one gets the equation

$$zF^3 + (z^2 - z)F^2 + (2z - 1)F + 1 = 0.$$

As in the previous section we apply Lemma 4.2 to the above equation. If we let $\phi(z,F) = zF^3 + (z^2 - z)F^2 + (2z-1)F + 1$, then eliminating F from $\phi(z,F) = \phi_F(z,F) =$ one obtains

$$5z^3 - 8z^2 - 32z + 4 = 0.$$

The inverse of the smallest positive root is equal to 8.22469. □

We end this section with a remark. Let $F(n)$ be the number of (ordinary) labelled forests. There is no simple closed formula for this number (see [8,11] for more recent results) but it is known that $F(n)/n^{n-2} \sim \sqrt{e}$. However, it follows from the above theorem that there are exponentially more nc-forests than nc-trees on n points, since the number t_n of nc-trees is in the order of $(\frac{27}{4})^n n^{-3/2}$ by Corollary 1.2.

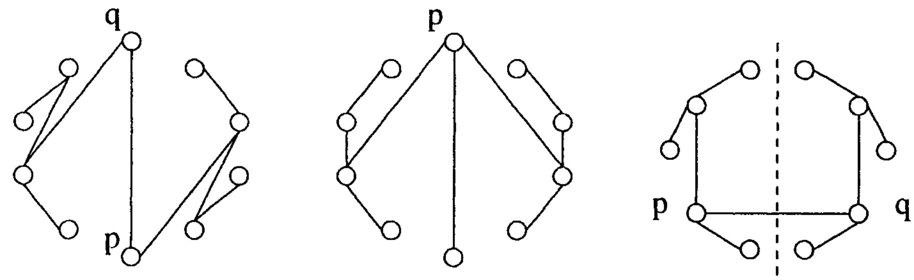
6. Unlabelled non-crossing trees

When we speak of an unlabelled structure, we mean the orbit of a set of labelled structures under the action of some group of symmetries. In the case of non-crossing trees the natural symmetries (automorphisms) to consider are given by the action of the dihedral group. Hence, we say two nc-trees on n labelled vertices are equivalent if one is obtained by a rotation and/or a reflection from the other. In other words, an unlabelled nc-tree is the *shape* of a certain nc-tree drawn on the vertices of a regular polygon.

One can count exactly the number of inequivalent nc-trees using a technique of Moon and Moser [9] in a similar problem concerning triangulations of a convex polygon.

Theorem 6.1. *The number t_n^* of unlabelled nc-trees on the vertices of a regular n-gon is equal to*

$$t_n^* = \begin{cases} \dfrac{1}{2n(n-1)}\dbinom{3n-3}{n-1} + \dfrac{3}{3n-2}\dbinom{3n/2-1}{n/2-1} & \textit{for even } n, \\[2ex] \dfrac{1}{2n(n-1)}\dbinom{3n-3}{n-1} + \dfrac{1}{2n}\dbinom{3(n-1)/2}{(n-1)/2} & \textit{for odd } n. \end{cases}$$



(a) (b) (c)

Fig. 5. Symmetries of nc-trees. (a) Rotation; (b,c) reflexions.

Proof. The proof is an application of the formula

$$N(G)=\frac{1}{|G|}\sum_{\sigma\in G}\lambda(\sigma)$$

for counting the number $N(G)$ of orbits under the action of a permutation group G, where $\lambda(\sigma)$ is the number of fixed points of σ. Hence, given a rotation or a reflection we have to count the number of nc-trees fixed by σ.

Rotational symmetry: It is clear that there is no rotational symmetry if n is odd, and that for even n only when the rotation is of 180°. In the latter case there must exist an edge connecting a pair p and q of 'opposite' points (refer to Fig. 5(a)). On one side of pq we can draw any nc-tree containing the edge (p,q) and the same tree rotates on the other side. Thus, the number of invariant nc-trees is, for a fixed pair of opposite points, equal to $t(n/2+1,1)$ (by Corollary 2.4).

Axial symmetry: There are two kinds of axis of symmetry. First those containing a vertex p (Fig. 5(b)). For n even the analysis is as above for rotational symmetry and gives $t(n/2+1,1)$ invariant trees; for n odd it is clear that there is no edge joining the 'left' and the 'right' of p and the number of invariant trees is just $t_{(n+1)/2}$.

The second case involves an axis joining the middle points of opposite sides and it only occurs when n is even. In this case, there is one and only one edge pq from left to right and it must be 'horizontal' (see Fig. 5(c)). This amounts to two reflected nc-trees of sizes summing up to $n/2+1$. The number of fixed trees is then $t_1t_{n/2}+\cdots+t_{n/2}t_1$, which by Corollary 2.4 is equal to $t(n/2+1,1)$.

Applying the orbit's formula and taking into account the identity permutation, we finally get

$$t_n^*=\begin{cases}\dfrac{1}{2n}(t_n+3n/2t(n/2+1,1)) & \text{for even } n,\\[2mm] \dfrac{1}{2n}(t_n+nt_{(n+1)/2}) & \text{for odd } n.\end{cases}$$

After substituting t_n and $t(n,d)$ by the binomial expressions found before we get the result. $\square$

From the previous theorem it follows that $t_n^* \sim t_n/2n$. This means that most non-crossing trees are rigid, i.e., have no symmetry besides the identity.

We also consider the particular case of non-crossing Hamiltonian paths. The expression $n2^{n-3}$ comes from the fact that there are 2^{n-2} Hamiltonian paths starting at a given vertex. The expression for p_n^* is again an application of the orbit's formula and we omit the proof.

Theorem 6.2. *The number of labelled non-crossing Hamiltonian paths on n points on a circle is equal to* $n2^{n-3}$. *The number of unlabelled paths is*

$$p_n^* = \begin{cases} 2^{n-4} + 2^{(n-4)/2} & \textit{for even } n, \\ 2^{n-4} + 2^{(n-5)/2} & \textit{for odd } n. \end{cases}$$

7. Conclusions and open problems

We have introduced the family of non-crossing trees which, although much smaller than that of labelled trees, is also a rich source of enumeration problems. The classical recursive techniques for the enumeration of labelled trees do not apply in this case and we have developed new techniques, which have proved useful in the solution of several problems.

We would like to mention a couple of problems left open in this paper. First to find a combinatorial proof of the simple relation $n\,u_n = \binom{n-1}{2} t_n$ obtained in Section 3. And secondly, to find a closed formula (possible quite involved) for the number of nc-forests.

In [10] the problem of counting the number of nc-trees on n vertices having exactly k leaves was discussed and some partial results were obtained. At the time of this writing the author learned from Philippe Flajolet [5] a complete solution to the problem: the number of rooted nc-trees on n vertices and k leaves turns out to be $(1/(n-1))\binom{n-1}{k}\sum_{j=0}^{k-1}\binom{n-1}{j}\binom{n-1-k}{k-1-j}2^{n-2k+j}$. There is also a similar formula for unrooted trees, thus, answering in the affirmative a conjecture made in [10].

Finally, we mention a few problems for future research. Enumeration of nc-trees with a given partition: we are given a (circular) sequence $(d_1,\ldots,d_n)$ with $\sum d_i = 2n-2$ and ask how many labelled nc-trees are in which vertex i has degree d_i. Analogues to the matrix-tree theorem for nc-trees: given a labelled graph G find a way to compute the number of non-crossing trees on a circle spanning G by means of the adjacency or Laplacian matrix of G.

Added in Proof

Reference [14] contains the solution to some of the open problems mentioned in the last section, as well as additional results on the enumeration of non-crossing graphs.

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