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J. Math. Anal. Appl. 284 (2003) 236–249

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

The necessary and sufficient conditions for the existence of periodic orbits in a Lotka–Volterra system

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Received 6 December 2000

Submitted by J. McKenna

Abstract

By extending Darboux method to three dimension, we present necessary and sufficient conditions for the existence of periodic orbits in three species Lotka–Volterra systems with the same intrinsic growth rates. Therefore, all the published sufficient or necessary conditions for the existence of periodic orbits of the system are included in our results. Furthermore, we prove the stability of periodic orbits. Hopf bifurcation is shown for the emergence of periodic orbits and new phenomenon is presented: at critical values, each equilibrium are surrounded by either equilibria or periodic orbits.

Keywords: Lotka–Volterra model of three species; Existence of periodic orbits; Center manifold theorem for flows; Hopf bifurcation

1. Introduction

Consider a community of three interacting species modelled by a Lotka–Volterra system with the same intrinsic growth rates

$$\begin{aligned} x_1' &= x_1(b - a_{11}x_1 - a_{12}x_2 - a_{13}x_3), \\ x_2' &= x_2(b - a_{21}x_1 - a_{22}x_2 - a_{23}x_3), \\ x_3' &= x_3(b - a_{31}x_1 - a_{32}x_2 - a_{33}x_3), \end{aligned} \tag{1.1}$$

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0022-247X/$ – see front matter
doi:10.1016/S0022-247X(03)00340-8

where $x_i(t)$ is the population size of the ith species at time t, $x_i(0) > 0$, and x_i' denotes dx_i/dt, $i = 1, 2, 3$. We restrict our attention to the open positive cone int R_+^3, where $R_+^3 := \{x \in R^3,\ x \geqslant 0\}$, where $x = (x_1, x_2, x_3)$.

From now on, we use the following notations:

$$F = (F_1, F_2, F_3), \qquad F_i = x_i\left(b - \sum_{j=1}^{3} a_{ij}x_j\right), \quad i \in Z_3,$$

$$A_i := a_{ii} - a_{i-1,i}, \quad B_i := a_{i+1,i} - a_{ii}, \quad i \in Z_3,$$

$$\Delta_i := \det\begin{pmatrix} 1 & a_{1,i+1} & a_{1,i-1} \\ 1 & a_{2,i+1} & a_{2,i-1} \\ 1 & a_{3,i+1} & a_{3,i-1} \end{pmatrix} = A_{i-1}A_{i+1} + A_{i-1}B_{i+1} + B_{i-1}B_{i+1}, \quad i \in Z_3,$$

$$\Delta := \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}\Delta_1 + a_{22}\Delta_2 + a_{33}\Delta_3 + B_1B_2B_3 - A_1A_2A_3,$$

where $Z_3 := \{1, 2, 3\}$ is considered in cyclic.

We denote by ψ_1 the radial projection from $R^3 \setminus \{0\}$ to the unit sphere $S^2 := \{x \in R^3\colon |x| = 1\}$, that is, $\psi_1(x) = x/|x|$. We denote by ψ_2 the radial projection from $(S_+^2)^0 := S^2 \cap \operatorname{int} R_+^3$ to the plane $\pi_1 := \{x \in \operatorname{int} R_+^3\colon\ x_3 = 1\}$, that is, $\psi_2(x) = x/x_3$.

Lemma 1.1 [6]. *Consider the vector field F of* (1.1); *then there is a vector field G on $(S_+^2)^0$ such that every orbit of F is projected onto that of G by the projection ψ_1, and*

$$G = (G_1, G_2, G_3),$$

where

$$G_i(x) := \left(|x|^2 - x_i^2\right)\left(F_i(x) - bx_i\right) - \sum_{j\neq i,\ j=1}^{3} x_i x_j\left(F_j(x) - bx_j\right),$$
$$x \in \left(S_+^2\right)^0,\ i \in Z_3.$$

The corresponding system is

$$x' = G(x), \quad x \in \left(S_+^2\right)^0, \tag{1.2}$$

which has the following properties:

(i) *If $A_1A_2A_3 - B_1B_2B_3 \neq 0$, then it has no periodic orbits*;
(ii) *If $A_1A_2A_3 - B_1B_2B_3 = 0$, then it has a nonconstant analytic first integral.*

In this paper, by extending Darboux method in [1,4] to system (1.1), we give the first integral of (1.1) if $A_1A_2A_3 - B_1B_2B_3 = 0$, then the predicted nonconstant analytic first integral of (1.2) in Lemma 1.1 can be obtained by ψ_1. Furthermore, the first integral is used to give the Hopf bifurcations which generate an infinite family of neutrally stable (unstable) periodic orbits. Our bifurcation conditions include all those in [3,5,6]. Generally, the necessary and sufficient conditions are given for the existence of periodic orbits of (1.1).

Theorem 1.2. *The periodic orbit of* (1.1) *exists if and only if one of the following conditions is satisfied*:

(i) $A_1A_2A_3 - B_1B_2B_3 = 0$, $b = \Delta = 0$, $\Delta_i > 0$, $i \in Z_3$;
(ii) $A_1A_2A_3 - B_1B_2B_3 = 0$, $b\Delta > 0$, $\Delta_i > 0$, $i \in Z_3$.

The proof of Theorem 1.2 is in Section 4 of this paper.

Since the restriction of ψ_2 to $(S_+^2)^0$ is one-one, every orbit of (1.2) is projected onto π_1 by the projection ψ_2, that is:

Lemma 1.3. *Consider the vector field G on $(S_+^2)^0$; there is a vector field H on the plane π_1 such that every orbit of* (1.2) *on $(S_+^2)^0$ is projected onto that of H on π_1 by the projection ψ_2, where*

$$H = \big(x_1\big[(a_{31} - a_{11})x_1 + (a_{32} - a_{12})x_2 + (a_{33} - a_{13})\big], \\ x_2\big[(a_{31} - a_{21})x_1 + (a_{32} - a_{22})x_2 + (a_{33} - a_{23})\big], 0\big),$$

the corresponding system is

$$x' = H(x), \quad x \in \pi_1. \tag{1.3}$$

Suppose ν is an orbit of (1.3). Let

$$S_\nu := \{kx\colon\ k \geqslant 0,\ x \in \nu\}.$$

Then by Lemmas 1.1 and 1.3, $S_\nu = \psi_1^{-1}(\psi_2^{-1}(\nu)) \cup \{0\}$, and S_ν is an invariant cone of (1.1). We call S_ν is closed if there is a plane π such that $\pi \cap S_\nu$ is a closed curve in int R_+^3.

Lemma 1.4. *If γ is a periodic orbit of* (1.1)*, then $\psi_1(\gamma)$ is a periodic orbit of* (1.2) *on $(S_+^2)^0$, and $\psi_2(\psi_1(\gamma))$ is a periodic orbit of* (1.3) *on π_1.*

Proof. Suppose $\psi_1(\gamma)$ is not a periodic orbit of (1.2) on $(S_+^2)^0$. By Lemma 1.1, $\psi_1(\gamma)$ is a bounded curve on $(S_+^2)^0$ with two end points. Suppose $p^0 = (p_1^0, p_2^0, p_3^0)$ is one of the two end points and let

$$L := \{tp^0\colon\ t > 0\}, \qquad \nu := \psi_2\big(\psi_1(\gamma)\big).$$

Then $\gamma, L \subset S_\nu$. On S_ν, L is tangent to γ at some point t_1p^0, where $t_1 > 0$. That is, there exists $c \neq 0$ such that

$$F(t_1p^0) = ct_1p^0,$$

then

$$\sum_{j=1}^{3} a_{ij}p_j^0 = (b - c)/t_1, \quad i \in Z_3,$$

$$F(tp^0) = \varphi(t)p^0, \quad \forall t > 0,$$

where $\varphi(t) = t(b - (b - c)t/t_1)$. Since

$$\varphi(t_1) = t_1 c \neq 0,$$

then there is $\delta > 0$ such that

$$\varphi(t) \neq 0, \quad t \in (t_1 - \delta, t_1 + \delta).$$

If $b = 0$, then $\varphi(t) = ct^2/t_1 \neq 0$, and

$$\frac{d(tp^0)}{dt} = \frac{1}{\varphi(t)} F(tp^0), \quad t > 0,$$

that is, L is an orbit of (1.1) with $t_1 p^0$ in it, this contradicts the fact that γ is a periodic orbit of (1.1) with $t_1 p^0$ in it. If $b > 0$, $c > 0$, then $\varphi(t) > 0$, $t \in (0, t_1 + \delta)$. Let $L_1 := \{y(t)\colon\ y(t) = tp^0,\ t \in (0, t_1 + \delta)\}$, then $y(t)$ satisfies

$$\frac{dy(t)}{dt} = \frac{1}{\varphi(t)} F\bigl(y(t)\bigr), \quad t \in (0, t_1 + \delta).$$

That is, L_1 is an orbit of (1.1) with $t_1 p^0$ in it, this contradicts the fact that γ is a periodic orbit of (1.1) with $t_1 p^0$ in it.

If $b > 0$, $c < 0$, the contradiction also exists while another orbit of (1.1) is constructed,

$$L_2 := \bigl\{z(t)\colon\ z(t) = tp^0,\ t \in (t_1 - \delta, +\infty)\bigr\}.$$

Similar to $b > 0$, the contradiction also exists if $b < 0$, the details are omitted.

Then $\psi_1(\gamma)$ is a periodic orbit of (1.2) on $(S_+^2)^0$. It follows from Lemma 1.3 that $\psi_2(\psi_1(\gamma))$ is a periodic orbit of (1.3) on π_1. Lemma 1.4 is proved. □

Lemma 1.5 [7]. *Consider the system*

$$x' = X(x), \quad x \in R^2, \tag{1.4}$$

where $X(x)$ is continuous in R^2. If γ is a periodic orbit of (1.4)*, and D is the bounded area surrounded by γ, there must exist an equilibrium of* (1.4) *in D. If every equilibrium of* (1.4) *is isolated, the sum of the indexes of the equilibria in D must be* 1.

While we restrict our attention to π_1, (1.3) becomes a planar system. If $\Delta_1\Delta_3 < 0$ or $\Delta_2\Delta_3 < 0$, there is no interior equilibrium of (1.3). By Lemma 1.5, there is no periodic orbit of (1.3). Suppose $\Delta_3 = 0$. If there is an equilibrium of (1.3), the set of equilibria of (1.3) must be the line

$$(a_{31} - a_{11})x_1 + (a_{32} - a_{12})x_2 + (a_{33} - a_{13}) = 0,$$

or the line

$$(a_{31} - a_{21})x_1 + (a_{32} - a_{22})x_2 + (a_{33} - a_{23}) = 0,$$

which separates π_1 into two open areas with no intersection. By Lemma 1.5, there is no periodic orbit of (1.3). It follows from Lemma 1.4 and the symmetry of Δ_1, Δ_2, Δ_3 that

Lemma 1.6. (i) *If $\Delta_1\Delta_3 < 0$ or $\Delta_2\Delta_3 < 0$, there is no periodic orbit of* (1.1).

(ii) *If $\Delta_1\Delta_2\Delta_3 = 0$, there is no periodic orbit of* (1.1).

Suppose $\Delta_3 \neq 0$ and let

$$q := \left(\frac{\Delta_1}{\Delta_3}, \frac{\Delta_2}{\Delta_3}, 1\right), \qquad \Gamma := \{kq\colon k > 0\}.$$

If $\Delta_1\Delta_3 > 0$, $\Delta_2\Delta_3 > 0$, q is the unique equilibrium of (1.3), and Γ is an invariant line of (1.1).

Lemma 1.7 [7]. *Consider the system*

$$\begin{aligned}\frac{dz_1}{d\tau} &= -z_2 + d_{20}z_1^2 + d_{11}z_1z_2 + d_{02}z_2^2,\\ \frac{dz_2}{d\tau} &= z_1 + e_{20}z_1^2 + e_{11}z_1z_2 + e_{02}z_2^2.\end{aligned} \tag{1.5}$$

If $d_{20} + d_{02} = 0$, $e_{20} + e_{02} = 0$, *the equilibrium* $(0,0)$ *of* (1.5) *is a center.*

Lemma 1.8. *Consider the vector field* H *on* π_1. *If*

$$A_1A_2A_3 - B_1B_2B_3 = 0, \quad \Delta_i < 0,\ i \in Z_3,$$

there is no periodic orbit of (1.3) *on* π_1. *If*

$$A_1A_2A_3 - B_1B_2B_3 = 0, \quad \Delta_i > 0,\ i \in Z_3, \tag{1.6}$$

q *is a center of* (1.3).

Proof. Since $\Delta_1\Delta_3 > 0$, $\Delta_2\Delta_3 > 0$, q is the unique equilibrium of (1.3). If $A_1A_2A_3 - B_1B_2B_3 = 0$, the eigenvalues of $dH(q)$ are given by

$$\pm\sqrt{-\Delta_1\Delta_2/\Delta_3}.$$

If $\Delta_i < 0$, $i \in Z_3$, q is a saddle of (1.3), that is, the index of q is -1. It follows from Lemma 1.5 that there is no periodic orbit of (1.3).

Suppose $\Delta_i > 0$, $i \in Z_3$. Let $y_i := x_i - q_i$, $i = 1, 2$; then (1.3) becomes

$$\begin{aligned}y_1' &= q_1(a_{31} - a_{11})y_1 + q_1(a_{32} - a_{12})y_2 + (a_{31} - a_{11})y_1^2 + (a_{32} - a_{12})y_1y_2,\\ y_2' &= q_2(a_{31} - a_{21})y_1 + q_2(a_{32} - a_{22})y_2 + (a_{31} - a_{21})y_1y_2 + (a_{32} - a_{12})y_2^2,\end{aligned} \tag{1.7}$$

where $y_i > -q_i$, $i = 1, 2$.

Let

$$\begin{aligned}&A := q_1(a_{31} - a_{11}), \qquad B := q_1(a_{32} - a_{12}), \qquad C := q_2(a_{31} - a_{21}),\\ &\sigma := \sqrt{-A^2 - BC} = \sqrt{\Delta_1\Delta_2/\Delta_3}.\end{aligned}$$

Let

$$y_1 = -\frac{1}{C}z_1 - \frac{A}{C\sigma}z_2, \qquad y_2 = -\frac{1}{\sigma}z_2, \qquad t = \frac{\tau}{\sigma}.$$

Then (1.7) becomes (1.5), where

$$d_{20} = \frac{(a_{31} - a_{11})}{C^2}, \qquad d_{11} = \frac{2(a_{31} - a_{11})A}{C^2\sigma} + \frac{(a_{32} - a_{12})}{C\sigma},$$

$$d_{02} = \frac{(a_{31} - a_{11})A^2}{C^2\sigma^2} + \frac{(a_{32} - a_{12})A}{C\sigma^2}, \qquad e_{20} = e_{02} = 0, \qquad e_{11} = \frac{1}{q_2\sigma}.$$

Since $d_{20} + d_{02} = 0$, $e_{20} + e_{02} = 0$, it follows from Lemma 1.7 that the equilibrium $(0, 0)$ of (1.5) is a center, that is, q is a center of (1.3). Lemma 1.8 is proved. □

Suppose the conditions in (1.6) are satisfied, then there is a family of periodic orbits of (1.3) which surround q. For such periodic orbit ν_0 of (1.3), S_{ν_0} is a closed, invariant cone of (1.1) which surrounds Γ. We denote by ν an orbit of (1.3); then $\forall k > 0$, there exists $\delta_k > 0$ such that if $S_\nu \cap O_{kq,\delta_k} \neq \phi$, S_ν is a closed, invariant cone of (1.1) which surrounds Γ.

It follows from Lemmas 1.4 and 1.8 that

Corollary 1.9. *If* $A_1A_2A_3 - B_1B_2B_3 = 0$, $\Delta_i < 0$, $i \in Z_3$, *there is no periodic orbit of* (1.1).

2. The case $A_1A_2A_3 - B_1B_2B_3 = 0$, $\Delta = 0$, and $\Delta_i > 0$, $i \in Z_3$

Theorem 2.1. *If* $A_1A_2A_3 - B_1B_2B_3 = 0$, $b = \Delta = 0$, $\Delta_i > 0$, $i \in Z_3$, *every equilibrium of* (1.1) *is a center in* $\operatorname{int} R^3_+$.

Proof. Since $b = \Delta = 0$, there are α_i, $i \in Z_3$, which satisfy

$$\sum_{i=1}^{3} \alpha_i \frac{x_i'}{x_i} = 0, \qquad \sum_{i=1}^{3} \alpha_i^2 \neq 0.$$

We obtain the first integral of (1.1),

$$I(x) = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}.$$

Without loss of generality, we suppose $\alpha_3 \neq 0$. Since

$$\Delta_1 = \frac{\alpha_1 + \alpha_2 + \alpha_3}{\alpha_3} \det \begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix} > 0,$$

then $\alpha_1 + \alpha_2 + \alpha_3 \neq 0$. It follows from $\Delta = 0$, $\Delta_i > 0$, $i \in Z_3$, that every equilibrium of (1.1) must be on Γ. Let $I_k := \{x \in \operatorname{int} R^3_+ \colon I(x) = I(kq)\}$, where $k > 0$, the normal vector of I_k at kq is $(\alpha_1\Delta_1^{-1}, \alpha_2\Delta_2^{-1}, \alpha_3\Delta_3^{-1})\Delta_3 k^{-1} I(kq)$.

We denote by π_{kq} the plane

$$\left(\alpha_1\Delta_1^{-1}, \alpha_2\Delta_2^{-1}, \alpha_3\Delta_3^{-1}\right) \cdot \left(x_1 - k\frac{\Delta_1}{\Delta_3}, x_2 - k\frac{\Delta_2}{\Delta_3}, x_3 - k\right) = 0.$$

I_k is tangent to π_{kq} at kq.

Since

$$\left(\alpha_1\Delta_1^{-1},\alpha_2\Delta_2^{-1},\alpha_3\Delta_3^{-1}\right)\cdot\left(l\frac{\Delta_1}{\Delta_3},l\frac{\Delta_2}{\Delta_3},l\right)=l\sum_{i=1}^{3}\alpha_i\neq 0,$$

where $l>0$, then $\Gamma\notin\pi_{kq}$. Let ν be an orbit of (1.3). By Lemma 1.8, there is $\delta_1>0$ such that if $S_\nu\cap O(kq,\delta_1)\neq\phi$, S_ν is a closed cone and $S_\nu\cap\pi_{kq}$ is a closed curve in $\operatorname{int}R_+^3$.

Since I_k is tangent to π_{kq} at kq, then there is $\delta_2>0$ ($\delta_2<\delta_1$) such that if $S_\nu\cap O(kq,\delta_2)\neq\phi$, $S_\nu\cap I_k$ is a closed curve in $\operatorname{int}R_+^3$, that is, a periodic orbit of (1.1). It follows from the arbitrariness of S_ν and k that Theorem 2.1 is proved. □

For example, every orbit of the following system is either a periodic orbit or an equilibrium:

$$x_1'=-\beta x_1x_2+\beta x_1x_3,\qquad x_2'=+\beta x_1x_2-\beta x_2x_3,\qquad x_3'=-\beta x_1x_3+\beta x_2x_3,$$

where $\beta>0$.

Theorem 2.2. *If $b\neq 0$, $\Delta=0$, $\Delta_i>0$, $i\in Z_3$, there is no periodic orbit of* (1.1).

Proof. Since $\Delta=0$, there are α_i, $i\in Z_3$, which satisfy

$$\sum_{i=1}^{3}\alpha_i\frac{x_i'}{x_i}=\mu,\qquad \sum_{i=1}^{3}\alpha_i^2\neq 0,$$

where $\mu=b\sum_{i=1}^{3}\alpha_i$. Then $x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}=c\exp(\mu t)$, where $c=x_1^{\alpha_1}(0)x_2^{\alpha_2}(0)x_3^{\alpha_3}(0)>0$. Without loss of generality, we suppose $\alpha_3\neq 0$. Since

$$\Delta_1=\frac{\alpha_1+\alpha_2+\alpha_3}{\alpha_3}\det\begin{pmatrix}a_{12}&a_{13}\\a_{22}&a_{23}\end{pmatrix}>0,$$

then $\alpha_1+\alpha_2+\alpha_3\neq 0$, $\mu\neq 0$.

If $\mu>0$, then $\lim_{t\to+\infty}c\exp(\mu t)=+\infty$. There is i, $i\in Z_3$, such that $\lim_{t\to+\infty}x_i^{\alpha_i}(t)=+\infty$. Then $\lim_{t\to+\infty}x_i(t)=+\infty$ or $\lim_{t\to+\infty}x_i(t)=0$, which means $x_i(t)$ is not periodic.

If $\mu<0$, then $\lim_{t\to+\infty}c\exp(\mu t)=0$. There is i, $i\in Z_3$, such that $\lim_{t\to+\infty}x_i^{\alpha_i}(t)=0$. Then $\lim_{t\to+\infty}x_i(t)=+\infty$ or $\lim_{t\to+\infty}x_i(t)=0$, which means $x_i(t)$ is not periodic. Theorem 2.2 is proved. □

3. The case $A_1A_2A_3-B_1B_2B_3=0$, $\Delta\neq 0$, and $\Delta_i>0$, $i\in Z_3$

Let $p:=(p_1,p_2,p_3)$, where $p_i:=b\Delta_i/\Delta$, $i\in Z_3$.

Theorem 3.1. *If*

$$A_1A_2A_3-B_1B_2B_3=0,\quad b\Delta\leqslant 0,\ \Delta\neq 0,\ \Delta_i>0,\ i\in Z_3,$$

there is no periodic orbit of (1.1).

Proof. We give the proof of the case $b \geqslant 0$, $\Delta < 0$. The proof of the case $b \leqslant 0$, $\Delta > 0$ can be obtained if t is replaced with $-t$ in the proof of the case $b \geqslant 0$, $\Delta < 0$.

Let $x_i = y_i + p_i$, where $y_i > -p_i \geqslant 0$. Since $\Delta_i > 0$, $i \in Z_3$, and $\Delta = a_{11}\Delta_1 + a_{22}\Delta_2 + a_{33}\Delta_3 < 0$, then there exists i, $i \in Z_3$, such that $a_{ii} < 0$. Without loss of generality, we suppose $a_{11} < 0$. Since $\Delta \neq 0$, then there is a matrix $M = (m_{ij})_{3\times 3}$, $\det M = 1$ such that

$$M \begin{pmatrix} \frac{y_1'}{(y_1+p_1)} \\ \frac{y_2'}{(y_2+p_2)} \\ \frac{y_3'}{(y_3+p_3)} \end{pmatrix} = \begin{pmatrix} -a_{11}y_1 \\ -a_{22}^0 y_2 \\ -a_{33}^0 y_3 \end{pmatrix}, \quad a_{22}^0 a_{33}^0 \neq 0,$$

$$\sum_{j=1}^{3} m_{1j} \frac{y_j'}{(y_j + p_j)} = -a_{11}y_1 > a_{11}p_1,$$

$$\prod_{j=1}^{3} x_j^{m_{1j}}(t) = c \exp\left(-a_{11} \int_0^t y_1(s)\, ds\right), \quad c = \prod_{j=1}^{3} x_j^{m_{1j}}(0) > 0.$$

Suppose $b > 0$. Then $\lim_{t\to+\infty} c \exp(a_{11}p_1 t) = +\infty$. There is j, $j \in Z_3$, such that $\lim_{t\to+\infty} x_j^{m_{1j}}(t) = +\infty$. Then $\lim_{t\to+\infty} x_j(t) = +\infty$ or $\lim_{t\to+\infty} x_j(t) = 0$, which means $x_j(t)$ is not periodic.

Suppose $b = 0$. If $\liminf_{t\to+\infty} y_1(t) = 0$, then $\liminf_{t\to+\infty} x_1(t) = 0$, which means $x_1(t)$ is not periodic. If $\liminf_{t\to+\infty} y_1(t) = m_1 > 0$, then $\lim_{t\to+\infty} -a_{11}\int_0^t y_1(s)\, ds = +\infty$. There is j, $j \in Z_3$, such that $\lim_{t\to+\infty} x_j^{m_{1j}}(t) = +\infty$. Then $\lim_{t\to+\infty} x_j(t) = +\infty$ or $\lim_{t\to+\infty} x_j(t) = 0$, which means $x_j(t)$ is not periodic. Theorem 3.1 is proved. □

Let $w(x) := b - \sum_{i=1}^{3} a_{ii}x_i$.

Lemma 3.2. *If* $A_1A_2A_3 - B_1B_2B_3 = 0$, $\Delta \neq 0$, *there is a first integral* $U(x)$ *of* (1.1),

$$U(x) = \prod_{i=1}^{3} x_i^{\alpha_i} v(x), \tag{3.1}$$

where

$$v(x) = B_1A_1x_1 + A_1A_2x_2 + B_1B_3x_3,$$
$$\sum_{i=1}^{3} a_{ij}\alpha_i = -a_{jj}, \quad j \in Z_3.$$

α_i *also satisfy the system*

$$1 + \sum_{i=1}^{3} \alpha_i = 0. \tag{3.2}$$

Proof. We extend Darboux method in [1,4] to system (1.1) for searching a first integral. The Darboux method is based on determining pairs of polynomials $(f_i(x), K_i(x))$ such that

$$\sum_{j=1}^{3} F_j \frac{\partial f_i}{\partial x_j} = f_i K_i. \tag{3.3}$$

We look for an invariant of the form

$$U(x,t) = \prod_{i=1}^{3} f_i^{\alpha_i} \exp(st). \tag{3.4}$$

We obtain

$$\frac{dU}{dt} = \frac{\partial U}{\partial t} + \sum_{j=1}^{3} F_j \frac{\partial U}{\partial x_j} = \left(s + \sum_{i=1}^{N} \frac{\alpha_i}{f_i} \sum_{j=1}^{3} F_j \frac{\partial f_i}{\partial x_j}\right) U.$$

Taking into account (3.3) and imposing that U is an invariant, we obtain

$$s + \sum_{i=1}^{N} \alpha_i K_i = 0, \tag{3.5}$$

where α_i, s can be determined.

The following three pairs of polynomials satisfy (3.3):

$$\begin{aligned}
\big(f_1(x), K_1(x)\big) &:= \big(x_1, (b - a_{11}x_1 - a_{12}x_2 - a_{13}x_3)\big),\\
\big(f_2(x), K_2(x)\big) &:= \big(x_2, (b - a_{21}x_1 - a_{22}x_2 - a_{23}x_3)\big),\\
\big(f_3(x), K_3(x)\big) &:= \big(x_3, (b - a_{31}x_1 - a_{32}x_2 - a_{33}x_3)\big).
\end{aligned}$$

If $A_1A_2A_3 - B_1B_2B_3 = 0$, we have another pair (v, w) which satisfies (3.3). Then there is an invariant of the form

$$U(x,t) = \prod_{i=1}^{3} f_i^{\alpha_i} v \exp(st),$$

where

$$s + b\left(1 + \sum_{i=1}^{3} \alpha_i\right) = 0, \qquad \sum_{i=1}^{3} \alpha_i a_{ij} = -a_{jj}, \quad i \in Z_3. \tag{3.6}$$

Since $\Delta \neq 0$, we can obtain the unique solution of (3.6), and find that α_i satisfy

$$\sum_{i=1}^{3} \alpha_i = -(a_{11}\Delta_1 + a_{22}\Delta_2 + a_{33}\Delta_3)/\Delta.$$

Then $1 + \sum_{i=1}^{3} \alpha_i = 0$, $s = 0$, $U(x,t) = U(x)$. Lemma 3.2 is proved. □

Let $\pi_2 := \{x \in R^3\colon\ v(x) = 0\}$, π_2 is an invariant plane of (1.1) if $\pi_2 \cap \operatorname{int} R^3_+ \neq \phi$. Under the conditions of Lemma 3.2, if $(B_1A_1)^2 + (A_1A_2)^2 + (B_1B_3)^2 \neq 0$, it follows from (3.2) that every surface $S_c := \{x\colon\ U(x) = c\}$ is an invariant cone of (1.1), where $c = U(x^0)$, $x^0 \in \operatorname{int} R^3_+$. Let $\nu = \psi_2(\psi_1(S_c))$; then ν is an orbit of (1.3) and $S_c = \psi_1^{-1}(\psi_2^{-1}(\nu)) \cup \{0\} = S_\nu$.

Lemma 3.3. *Suppose that the following conditions are satisfied*:

$$A_1A_2A_3 - B_1B_2B_3 = 0, \quad b > 0,\ \Delta > 0,\ \Delta_i > 0,\ i \in Z_3. \tag{3.7}$$

If there exists a periodic orbit of (1.1), *there is* $\delta_0 > 0$ $(\delta_0 < p_i,\ i \in Z_3)$ *such that* $\operatorname{div} F(x) \neq 0$ *for* $x \in O(p, \delta_0)$.

Proof. Let γ be a periodic orbit of (1.1). Suppose $U(p) = 0$; then $p \in \pi_2$, $q \in \psi_2(\psi_1(\pi_2))$. Since π_2 is an invariant plane of (1.1), then by Lemmas 1.1 and 1.3, $\psi_2(\psi_1(\pi_2))$ is an invariant line of (1.3).

It follows from Lemmas 1.3 and 1.4 that $\psi_2(\psi_1(\gamma))$ is a periodic orbit of (1.3). By Lemma 1.5, the unique equilibrium q of (1.3) must be surrounded by $\psi_2(\psi_1(\gamma))$. This contradicts the fact that $\psi_2(\psi_1(\pi_2))$ is an invariant line of (1.3) with q in it. Then $U(p) \neq 0$, that is, $v(p) \neq 0$.

It follows from (3.3) that $w(p) = 0$. Since

$$\operatorname{div} F(x) = \sum_{i=1}^{3}\left(\frac{F_i(x)}{x_i} - a_{ii}x_i\right), \qquad F_i(p) = 0, \quad i \in Z_3,$$
$$\operatorname{div} F(p) = -b + w(p) = -b \neq 0,$$

then Lemma 3.3 is proved. □

Lemma 3.4. *If the conditions in* (3.7) *are satisfied, then*

(i) *There exists* $\eta_0 > 0$ *such that if* $S_c \cap O(p, \eta_0) \neq \phi$, S_c *is a closed cone which surrounds* Γ;
(ii) *There exists an infinite family of periodic orbits of* (1.1) *in* $O(p, \delta_0)$ *and* p *is locally stable.*

Proof. Since the conditions in (3.7) are satisfied, then $(B_1A_1)^2 + (A_1A_2)^2 + (B_1B_3)^2 \neq 0$.

It follows from Lemmas 1.3 and 1.8 that q is a center of (1.3); then the result in (i) is proved.

Since the eigenvalues of $dF(p)$ are given by

$$-b, \quad \frac{\pm\sqrt{-\Delta_1\Delta_2\Delta_3}}{\Delta},$$

let E_0^c be the linear spanning by the generalized eigenvectors of $\pm\sqrt{-\Delta_1\Delta_2\Delta_3}/\Delta$, and let E^s be the linear spanning by the eigenvectors of $-b$. Then $E^s = \{tp\colon\ t \in R\}$. Since $R^3 = E_0^c \oplus E^s$, then $E^s \notin E_0^c$, $-p \notin E_0^c$.

Let

$$E^c := \{p\} + E_0^c = \big\{p + y\colon\ y \in E_0^c\big\}, \qquad \pi := E^c \cap R_+^3.$$

Then $0 \notin \pi$.

Let $V := \{tx\colon\ t \geqslant 0,\ x \in \pi\}$. Since p is an interior point of V, then there is $\eta_1 > 0$ such that $O(p, \eta_1) \subset V$. If $x \in O(p, \eta_1)$, then $\{tx\colon\ t > 0\} \cap \pi \neq \phi$. So there is $\eta_2 > 0$ ($\eta_2 < \min\{\eta_0, \eta_1\}$) such that $S_c \cap \pi$ is a closed curve if $S_c \cap O(p, \eta_2) \neq \phi$.

It follows from the center manifold theorem for flows [2] that there is a two-dimensional center manifold W^2 of (1.1) which is tangent to π at p. Since $S_c \cap \pi$ is a closed curve if $S_c \cap O(p, \eta_2) \neq \phi$, there exists $\eta_3 > 0$ ($\eta_3 < \eta_2$) such that every curve $L := S_c \cap W^2$ is a closed curve if $S_c \cap O(p, \eta_3) \neq \phi$.

For δ_0 in Lemma 3.3, it follows from the analyticity of $U(x)$ that there is $\delta > 0$ ($\delta < \eta_3$) such that $L \subset O(p, \delta_0)$ if $S_c \cap O(p, \delta) \neq \phi$. Since S_c, W^2 are invariant manifolds of (1.1), L is a periodic orbit of (1.1). It follows from the arbitrariness of S_c (with the condition $S_c \cap O(p, \delta) \neq \phi$) that p is a center on W^2. Since p is asymptotically stable on E^s, p is locally stable. Lemma 3.4 is proved. □

Lemma 3.5. *Suppose that the conditions in* (3.7) *are satisfied. If there exists a periodic orbit of* (1.1) *on* $S_c \cap O(p, \delta_0)$ *for some* c, *it is the unique periodic orbit of* (1.1) *on* $S_c \cap O(p, \delta_0)$.

Proof. Suppose there are two different periodic orbits $\gamma_1, \gamma_2 \subset S_c \cap O(p, \delta_0)$ for some c. Let D be the area which is surrounded by γ_1, γ_2 on S_c and let $n := (n_1, n_2, n_3)$ be the unit normal vector of D on S_c, then $D \subset S_c \cap O(p, \delta_0)$, and it follows from Stokes theorem that

$$\oint_{\partial D} (n \times F)\, dr = \iint_D n \cdot \mathrm{curl}(n \times F)\, d\sigma,$$

but

$$(n \times F)\, dr = (n \times F) F\, dt \equiv 0,$$

$$\mathrm{curl}(n \times F) = n\, \mathrm{div}\, F - F\, \mathrm{div}\, n + (F, \mathrm{grad}) n - (n, \mathrm{grad}) F,$$

where

$$(g, \mathrm{grad}) h := \sum_{i=1}^{3} g_i \frac{\partial h}{\partial x_i}$$

for

$$g = \big(g_1(x), g_2(x), g_3(x)\big), \qquad h = \big(h_1(x), h_2(x), h_3(x)\big).$$

Since

$$n \cdot \mathrm{curl}(n \times F) = \mathrm{div}\, F - (n \cdot F)\, \mathrm{div}\, n + n \cdot (F, \mathrm{grad}) n - n \cdot (n, \mathrm{grad}) F,$$

$$n \cdot (F, \mathrm{grad}) n = n \cdot \sum_{i=1}^{3} F_i \frac{\partial n}{\partial x_i} = \frac{1}{2} F \cdot \mathrm{grad}(n \cdot n) = 0,$$

$$n \cdot (n, \mathrm{grad})F = n \cdot \sum_{i=1}^{3} n_i \frac{\partial F}{\partial x_i} = n \cdot \mathrm{grad}(n \cdot F) - \frac{1}{2} F \cdot \mathrm{grad}(n \cdot n)$$
$$= n \cdot \mathrm{grad}(n \cdot F) = 0,$$

then

$$n \cdot \mathrm{curl}(n \times F) = \mathrm{div}\, F - (n \cdot F)\,\mathrm{div}\, n + n \cdot (F, \mathrm{grad})n - n \cdot (n, \mathrm{grad})F = \mathrm{div}\, F.$$

It follows from Lemma 3.3 and $b > 0$ that $\mathrm{div}\, F(x) < 0$ if $x \in O(p, \delta_0)$, which contradicts the fact that $\oint_{\partial D}(n \times F)\, dr \equiv 0$. Lemma 3.5 is proved. □

It follows from Lemma 3.5 that under the conditions in (3.7), the center manifold at p is unique. In fact, if there are two different center manifolds w_0^2 and w^2 at p, then similar to the proof of Lemma 3.4, there is $\eta > 0$ such that $S_c \cap W_0^2$ and $S_c \cap W^2$ are two different periodic orbits of (1.1) in $O(p, \delta_0)$ if $S_c \cap O(p, \eta) \neq \phi$. This contradicts Lemma 3.5.

Theorem 3.6. *If the conditions in* (3.7) *are satisfied, there exists an infinite family of neutrally stable periodic orbits of* (1.1).

Proof. We denote by γ the orbit $\{x(t)\colon\ t \geqslant 0\}$ of (1.1). It follows from Lemma 3.4 that for δ_0 in Lemma 3.3 and η_0 in Lemma 3.4, there is $\delta_1 > 0$ ($\delta_1 < \min\{\delta_0 \eta_0\}$) such that if $x(0) \in O(p, \delta_1) \setminus \Gamma$, $S_{U(x(0))}$ is a closed cone and $\gamma \subset S(p, \delta_0/2)$, that is, $\gamma \subset S_{U(x(0))} \cap S(p, \delta_0/2)$. Since there is no equilibrium on $S_{U(x(0))} \cap S(p, \delta_0/2)$, it follows from Poincare–Bendixson theorem that the ω-limit set of γ is a periodic orbit γ_1^0 of (1.1). By Lemma 3.5, γ_1^0 is the unique periodic orbit of (1.1) on $S_{U(x(0))} \cap S(p, \delta_0/2)$.

Since p is locally stable, then for $\delta_1 > 0$, there is $\delta_2 > 0$ ($\delta_2 < \delta_1$) such that if $x(0) \in O(p, \delta_2) \setminus \Gamma$, then $\gamma \subset S(p, \delta_1/2)$. The ω-limit set of γ is a periodic orbit γ^0 of (1.1), $\gamma^0 \subset S(p, \delta_1/2)$, and $S_{U(x(0))} = \{tx\colon\ t \geqslant 0,\ x \in \gamma^0\}$.

Let $x(0)$ ($x(0) \in O(p, \delta_2) \setminus \Gamma$) fixed, and let $\Lambda := S_{U(x(0))} \cap O(p, \delta_1)$. Since $\gamma^0 \subset S(p, \delta_1/2)$, then on $S_{U(x(0))}$, Λ is a connected area with γ^0 in it. Every orbit $\{y(t)\colon\ t \geqslant 0\}$ of (1.1) with $y(0) \in \Lambda$ satisfies $y(t) \in S_{U(x(0))} \cap S(p, \delta_0/2)$. Since there is no equilibrium on $S_{U(x(0))} \cap S(p, \delta_0/2)$ and γ^0 is the unique periodic orbit of (1.1) on $S_{U(x(0))} \cap S(p, \delta_0/2)$, then it follows from Poincare–Bendixson theorem that the ω-limit set of $\{y(t)\colon\ t \geqslant 0\}$ is γ^0. By the analyticity of $U(x)$ and the arbitrariness of $x(0)$ in $O(p, \delta_2) \setminus \Gamma$, Theorem 3.6 is proved. □

Since the eigenvalues of $dF(p)$ are given by

$$\lambda_1 = -b,$$
$$\lambda_{2,3} = \frac{B_1 B_2 B_3 - A_1 A_2 A_3 \pm \sqrt{(A_1 A_2 A_3 - B_1 B_2 B_3)^2 - 4\Delta_1 \Delta_2 \Delta_3}}{2\Delta},$$

let E_0^c be the linear spanning by the eigenvectors (or generalized eigenvectors if $\mathrm{Im}\, \lambda_{2,3} \neq 0$) of $\lambda_{2,3}$, $E^c := \{p\} + E_0^c = \{p + y\colon\ y \in E_0^c\}$.

Suppose $b > 0$, $\Delta > 0$, $\Delta_i > 0$, $i \in Z_3$. It follows from the center manifold theorem for flows [2] that if $A_1 A_2 A_3 - B_1 B_2 B_3 < 0$, there is a two-dimensional unstable manifold W_1^2

of (1.1) which is tangent to E^c at p; if $A_1A_2A_3 - B_1B_2B_3 = 0$, there is a two-dimensional center manifold W^2_{1c} of (1.1) which is tangent to E^c at p.

Similarly, suppose $b < 0$, $\Delta < 0$, $\Delta_i > 0$, $i \in Z_3$. It follows from the center manifold theorem for flows [2] that if $A_1A_2A_3 - B_1B_2B_3 > 0$, there is a two-dimensional stable manifold W^2_2 of (1.1) which is tangent to E^c at p; if $A_1A_2A_3 - B_1B_2B_3 = 0$, there is a two-dimensional center manifold W^2_{2c} of (1.1) which is tangent to E^c at p.

It follows from Theorem 3.6 that

Corollary 3.7. *If $b > 0$, $\Delta > 0$, $\Delta_i > 0$, $i \in Z_3$, Hopf bifurcation occurs when $A_1A_2A_3 - B_1B_2B_3 = 0$.*

(i) *If $A_1A_2A_3 - B_1B_2B_3 > 0$, p is asymptotically stable*;
(ii) *If $A_1A_2A_3 - B_1B_2B_3 = 0$, p is stable and is a center on W^2_{1c}*;
(iii) *If $-2\sqrt{\Delta_1\Delta_2\Delta_3} < A_1A_2A_3 - B_1B_2B_3 < 0$, p is unstable, and is an unstable focus on W^2_1.*

Theorem 3.8. *If the following conditions are satisfied*:

$$A_1A_2A_3 - B_1B_2B_3 = 0, \quad b < 0,\ \Delta < 0,\ \Delta_i > 0,\ i \in Z_3,$$

then there exists an infinite family of neutrally unstable periodic orbits of (1.1).

Theorem 3.8 can be proved if t is replaced with $-t$ in the proof of Theorem 3.6. The details are omitted. Similarly with Corollary 3.7, it follows from Theorem 3.8 that

Corollary 3.9. *If $b < 0$, $\Delta < 0$, $\Delta_i > 0$, $i \in Z_3$, Hopf bifurcation occurs when $A_1A_2A_3 - B_1B_2B_3 = 0$.*

(i) *If $0 < A_1A_2A_3 - B_1B_2B_3 < 2\sqrt{\Delta_1\Delta_2\Delta_3}$, p is unstable, but is a stable focus on W^2_2*;
(ii) *If $A_1A_2A_3 - B_1B_2B_3 = 0$, p is unstable and is a center on W^2_{2c}*;
(iii) *If $A_1A_2A_3 - B_1B_2B_3 < 0$, p is unstable.*

4. Proof of Theorem 1.1

It follows from Lemmas 1.1 and 1.4 that

Lemma 4.1. *If $A_1A_2A_3 - B_1B_2B_3 \neq 0$, there exists no periodic orbit of* (1.1).

Proof of Theorem 1.1. It follows from Theorems 2.1, 2.2, 3.1, 3.6, 3.8 that under the condition $A_1A_2A_3 - B_1B_2B_3 = 0$, the periodic orbit of (1.1) exists if and only if $b\Delta > 0$, $\Delta_i > 0$, $i \in Z_3$, or $b = \Delta = 0$, $\Delta_i > 0$, $i \in Z_3$. It follows from Lemma 4.1 that if $A_1A_2A_3 - B_1B_2B_3 \neq 0$, there exists no periodic orbit of (1.1). Theorem 1.1 is proved. □

Acknowledgments

The author wishes to express his sincere thanks to Professor S.M. Zhu and Professor H.H. Dai for useful comments.

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