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Generic polynomial vector fields are not integrable

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It is a great pleasure for us to dedicate this paper to our friend Professor Jean-Marie Strelcyn.

ABSTRACT

We study some generic aspects of polynomial vector fields or polynomial derivations with respect to their integration. In particular, using a well-suited presentation of *Darboux polynomials* at some *Darboux point* as power series in local *Darboux coordinates*, it is possible to show, by algebraic means only, that the Jouanolou derivation in four variables has no polynomial first integral for any integer value $s \geq 2$ of the parameter.

Using direct sums of derivations together with our previous results we show that, for all $n \geq 3$ and $s \geq 2$, the absence of polynomial first integrals, or even of Darboux polynomials, is generic for homogeneous polynomial vector fields of degree s in n variables.

1. INTRODUCTION

We are interested in homogeneous derivations of polynomial rings $\mathbb{K}[X] = \mathbb{K}[x_1, \cdots, x_n]$, where $\mathbb{K}$ is a field of characteristic 0.

A derivation $d = \sum P_i \partial_i$ of $\mathbb{K}[X]$ is said to be homogeneous of degree s if all polynomials P_i are homogeneous of the same degree $s + 1$. In this case, the

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image $d(F)$ of a homogeneous polynomial F of degree m is homogeneous of degree $m+s$.

A non-trivial *first integral of degree* 0 of a homogeneous derivation d is a common constant F of δ and of the *Euler derivation* $E = \sum x_i\partial_i$, which is not a common constant for the n partial derivatives $\partial_i, 1 \leq i \leq n$.

In this context, integration in finite form consists in the search of first integrals of degree 0 for a homogeneous derivation d in some well-defined differential extension of the field $\mathbb{K}(X)$. Usual considered extensions are algebraic and liouvillian ones.

A very important tool, now called the Darboux polynomials, has been introduced by Darboux [1] in connection with this problem.
Let d be a homogeneous derivation of $\mathbb{K}[X]$. A homogeneous polynomial $F \in \mathbb{K}[X]$ is said to be *a Darboux polynomial* of d with cofactor Λ if $d(F) = \Lambda F$, where Λ is a homogeneous polynomial of degree s. The cofactor Λ of a non-zero Darboux polynomial is well-defined. A Darboux polynomial F is said to be *non-trivial* if $F \notin \mathbb{K}$.

The absence of Darboux polynomials is typical and their existence is rare if we consider the whole set (a finite dimensional $\mathbb{K}$-vector space) of all homogeneous derivations of a given degree $s \geq 1$. To be precise, we follow the notion of the Baire category when $\mathbb{K}$ is the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers.

According to the previous studies [3,5], the set of all homogeneous derivations of a given degree s of $\mathbb{K}[X]$ without a non-trivial Darboux polynomial in a countable intersection of Zariski open algebraic sets; it is therefore sufficient to find, for every degree $s \geq 1$, one derivation d without a non-trivial Darboux polynomial. To deal with all possible $\mathbb{K}$, it is natural to look for examples of such derivations with rational coefficients. In the three-variable case, a well-known example is the Jouanolou derivation $J_{3,s} = y^s\partial_x + z^s\partial_y + x^s\partial_z$: there are many different proofs that $J_{3,s}, s \geq 2$, has no non-trivial Darboux polynomial [3,6,4]. In more variables, H. Żołądek [10] recently proposed an analytical proof that $J_{n,s}, s \geq 2, n \geq 3$, has no non-trivial Darboux polynomial; at the end of his paper, in Remark 7, Żołądek gives a special proof for the case $n = 4, s > 4$, which is quite different from ours.

In the present work, we propose another way of constructing a homogeneous derivation of degree $s-1$ of $\mathbb{Q}[x_1, \cdots, x_n]$ without any non-trivial Darboux polynomial for every $n \geq 3$ and every $s \geq 2$: direct sums of derivations. The main point is then to prove that $J_{4,s}, s \geq 2$, has no non-trivial Darboux polynomial. We do it in a purely algebraic way.

The key tool of our proof consists in the study of Darboux polynomials of a homogeneous derivation around some particular points of the projective space, called the *Darboux points*. A Darboux point of a homogeneous derivation d is a point of the projective space in $n-1$ dimensions where the vectors $[d(x_1), \cdots, d(x_n)]$ and $[x_1, \cdots, x_n]$ are collinear.

The idea of studying Darboux polynomials at Darboux points is not completely new; for instance, the Lagutinskii-Levelt procedure (LL for short) [5,6] can be considered as the *linear part* of this study.

One of the novelties of the present paper is to consider much more completely the possible structure of a Darboux polynomial as a power series in the local coordinates at a Darboux point. In the planar case, this deeper analysis leads to a branch decomposition [7].

2. DIRECT SUMS OF DERIVATIONS

2.1. Basic facts

In this subsection, we describe how to construct homogeneous derivations of polynomials rings over a field $\mathbb{K}$ by *direct sums* of previously known ones and show that $d_1 \oplus d_2$ inherits nice properties of d_1 and d_2. Some additional facts concerning such direct sums of derivations are given in [8].

Definition 1 *A derivation* $d = \sum P_i \partial_i$ *of* $\mathbb{K}[X]$ *is said to be homogeneous of degree* s *if all polynomials* P_i *are homogeneous of the same degree* $s+1$. *In this case, the image* $d(F)$ *of a homogeneous polynomial* F *of degree* m *is homogeneous of degree* $m+s$. *To stress this natural definition of the degree, let us remark that linear derivations (the* P_i *are homogeneous of degree* 1*) are homogeneous of degree* 0.

Definition 2 *Let* d_1 *and* d_2 *be homogeneous* $\mathbb{K}$*-derivations of the same degree* s *of the polynomial rings* $\mathbb{K}[X] = \mathbb{K}[x_1, \cdots, x_n]$ *and* $\mathbb{K}[Y] = \mathbb{K}[y_1, \cdots, y_p]$, *respectively. The sets* X *and* Y *of indeterminates being disjoint, there is a unique* $\mathbb{K}$*-derivation* d *on the polynomial ring* $\mathbb{K}[X \cup Y]$ *whose restrictions to* $\mathbb{K}[X]$ *and to* $\mathbb{K}[Y]$ *are respectively* d_1 *and* d_2. *This* d *is called the* direct sum *of* d_1 *and* d_2 *and it is denoted by* $d = d_1 \oplus d_2$.

Two hereditary properties of direct sums of derivations are interesting in our study of generic non-integrability:

Proposition 1 *If* d_1 *and* d_2 *have no non-trivial polynomial constant, the same is true for* $d_1 \oplus d_2$.

Proposition 2 *If* d_1 *and* d_2 *have no non-trivial Darboux polynomial, the same is true for* $d_1 \oplus d_2$.

As Proposition 1 is the particular case of Proposition 2, in which the cofactor is 0, the proof of the second proposition will include the proof of the first one.

Proof. Let $F \in \mathbb{K}[X, Y]$ $\mathbb{K}$ be a homogeneous Darboux polynomial of d of degree $m \geq 1$ with $\Lambda \in \mathbb{K}[X, Y]$ as its cofactor. Then Λ is homogeneous of degree s.

The notation $|||[1ex]|||$ standing for the sum of coordinates of a tuple of non-negative integers, the polynomials F and Λ have the following forms in $\mathbb{K}[X][Y]$:

$$(1) \qquad F = \sum_{|\alpha| \le m} F_\alpha Y^\alpha, \quad \Lambda = \sum_{|\beta| \le s} \Lambda_\beta Y^\beta,$$

where F_α, Λ_β (for any α and β) are homogeneous polynomials from $k[X]$ of degrees $m - |\alpha|$ and $s - |\beta|$, respectively.

From Equation (1), the polynomials $d(F)$ and ΛF may be developed as

$$(2) \qquad d(F) = \sum_{|\alpha| \le m} (d_1(F_\alpha) Y^\alpha + F_\alpha d_2(Y^\alpha)), \quad \Lambda F = \sum_{|\beta| \le s} \sum_{|\alpha| \le m} \Lambda_\beta F_\alpha Y^{\alpha+\beta}.$$

Since $d(F) = \Lambda F$, we have:

$$(3) \qquad H = \sum_{|\alpha| \le m} (d_1(F_\alpha) Y^\alpha + F_\alpha d_2(Y^\alpha)) - \sum_{|\beta| \le s} \sum_{|\alpha| \le m} \Lambda_\beta F_\alpha Y^{\alpha+\beta} = 0.$$

The previous difference H has the form $H = H_0 + H_1 + \cdots + H_{m+s}$, where each H_i is homogeneous of degree i in $\mathbb{K}[X][Y]$.

Since $H = 0$, we have $H_0 = H_1 = \cdots = H_{m+s} = 0$.

Using induction with respect to the total degree of the exponents, we now show that $F_\alpha = 0$ for all α such that $|\alpha| \le m - 1$.

First consider the case $j = |\alpha| = 0$.

We know that $0 = H_0 = d_1(F_0) Y^0 - \Lambda_0 F_0 Y^0$, and $d_1(F_0) = \Lambda_0 F_0$. But d_1 has only trivial Darboux polynomials, so $F_0 \in \mathbb{K}$. Moreover, $\deg(F_0) = m - |0| = m \ge 1$, so $F_0 = 0$.

Consider now the cases $0 < j \le m - 1$ and suppose that $F_\alpha = 0$ for all exponents such that $|\alpha| < j$.

We want to deduce that $F_\alpha = 0$ for all exponents such that $|\alpha| = j$. From the fact that $H_j = 0$; we have therefore to distinguish between two cases: $j < s + 1$ (no contribution from d_2) and $j \ge s + 1$ (d_2 contributes to H_j).

Case 1 [$j < s + 1$]:

$$H_j = \sum_{|\alpha| = j} d_1(F_\alpha) Y^\alpha - \sum_{|\alpha| + |\beta| = j} \Lambda_\beta F_\alpha Y^{\alpha+\beta}.$$

By induction, as $F_\alpha = 0$ for $|\alpha| < j$, H_j reduces to

$$0 = H_j = \sum_{|\alpha| = j} d_1(F_\alpha) Y^\alpha - \sum_{|\alpha| = j} \Lambda_0 F_\alpha Y^\alpha = \sum_{|\alpha| = j} (d_1(F_\alpha) - \Lambda_0 F_\alpha) Y^\alpha,$$

which implies that $d_1(F_\alpha) = \Lambda_0 F_\alpha$ for $|\alpha| = j$.

But d_1 has only trivial Darboux polynomials, so $F_\alpha \in \mathbb{K}$ for $|\alpha| = j$. Moreover, $\deg(F_\alpha) = m - |\alpha| = m - j \ge 1$, so $F_\alpha = 0$.

Case 2 [$j \ge s + 1$]:

$$H_j = \sum_{|\alpha| = j} d_1(F_\alpha) Y^\alpha + \sum_{|\alpha'| = j - s} F_{\alpha'} d_2(Y^{\alpha'}) - \sum_{|\alpha| + |\beta| = j} \Lambda_\beta F_\alpha Y^{\alpha+\beta}.$$

By induction, each $F_{\alpha'}$ is equal to 0 and $F_\alpha = 0$ for $|\alpha| < j$, so

$$0 = H_j = \sum_{|\alpha|=j} d_1(F_\alpha) Y^\alpha - \sum_{|\alpha|=j} \Lambda_0 F_\alpha Y^\alpha = \sum_{|\alpha|=j} (d_1(F_\alpha) - \Lambda_0 F_\alpha) Y^\alpha.$$

Hence, $d_1(F_\alpha) = \Lambda_0 F_\alpha$ and $F_\alpha \in \mathbb{K}$ for $|\alpha| = j$. Moreover, $\deg(F_\alpha) = m - |\alpha| = m - j \geq 1$, so $F_\alpha = 0$ for $|\alpha| = j$.

We can now conclude. F reduces to $\sum_{|\alpha|=m} F_\alpha Y^\alpha$, where the degree of each F_α is equal to $p - |\alpha| = p - p = 0$. Thus, $F \in \mathbb{K}[Y]$. Moreover, $\Lambda F = d(F) = d_2(F) \in \mathbb{K}[Y]$, that is, $\Lambda \in \mathbb{K}[Y]$. Therefore, F is a Darboux polynomial of d_2. But d_2 has no non-trivial Darboux polynomial. □

2.2. Application

Direct sums of derivations are a useful tool to deal with some generic aspects of non-integrability of polynomial derivations. According to our previous discussion, we have to show, for any number of variables $n \geq 3$, any degree $s \geq 2$ and any field $\mathbb{K}$ of characteristic 0, that there exists a $\mathbb{K}$-derivation d of $\mathbb{K}[x_1, \cdots, x_n]$ of degree $s - 1$ (the $d(x_i)$ are homogeneous polynomials of degree s) without a non-trivial Darboux polynomial. It is enough to consider the case where $\mathbb{K}$ is the field $\mathbb{Q}$ of rational numbers.

According to [5], the Jouanolou derivation $J_{n,s} = \sum x_{i+1}^s \partial_i$ has no Darboux polynomial for $s \geq 3$ when $n \geq 5$ is a prime number. Jouanolou's original result [3,5] is that the same is true for $n = 3, s \geq 2$.

The case of a prime number $n \geq 5$ with $s = 2$ has to be dealt with in a special way; we leave to the reader to prove that there is no Darboux polynomial in this case. This can be done as in the case of $J_{4,s}$ but with much simpler details.
Using direct sums of Jouanolou derivations of the same degree with various numbers of variables, one can show the existence of a $\mathbb{Q}$-derivation of $\mathbb{Q}[x_1, \cdots, x_n]$ of degree $s - 1$ without a Darboux polynomial for any $s \geq 2$ and any $n \geq 3$ provided that n can be written as a sum of positive odd primes.

Every $n \geq 3$ but $n = 4$ has the last property. Thus, we have to prove that $J_{4,s}$ has no Darboux polynomial for any $s \geq 2$ to achieve our task. This is in fact the main theorem of the present paper whose proof is the purpose of Section 3:

Theorem 1. *$J_{4,s}$ has no Darboux polynomial for any $s \geq 2$.*

On the other hand, as every integer $n \geq 3$ but 5 is a positive combination of 3 and 4, it suffices to prove that $J_{5,2}$ has no Darboux polynomial to receive the announced genericity result for all $n \geq 3$ and all $s \geq 2$. Some remarks will be given about this fact along with the proof of the main theorem.

3. $J_{4,s}$ HAS NO DARBOUX POLYNOMIAL

Some time ago, Henryk Żołądek [10] gave a complete but difficult proof that $J_{n,s}$ has no Darboux polynomial for any $s \geq 2$ by analytical means together with a remark on the case $n = 4, s > 4$. We will restrict our proof to the case

$n = 4, s \geq 2$, which is enough (together with previous results and the case $n = 5, s = 2$) to receive the *generic* conclusion we look for.

Let us remark that $J_{4,s}$ has the invariant algebraic set $\{x_1 = x_3, x_2 = x_4\}$ whose codimension is 2 in the projective space. Thus, $J_{4,s}$ is not the example to show the generic absence of invariant algebraic sets (not only of those of codimension 1); see the work of M. G. Soares [9].

Let us put some emphasis on the fact that our proof is purely algebraic and takes into account a more complete study of Darboux polynomials around Darboux points that the usual Lagutinskii-Levelt procedure (LL for short) in which we can take into account the fact that a Darboux polynomial is irreducible.

Despite the fact that we are mainly interested in the case $n = 4$, some intermediate results are valid for all Jouanolou derivations and we will present them in a general framework. We first recall some useful reductions.

3.1. Darboux polynomials and polynomials constants of $J_{n,s}$

Let $s \geq 2$ and $n \geq 3$ be integers. Then denote by d the usual Jouanolou derivation $J_{n,s}$

$$d(x_i) = x_{i+1}^s, i = 1, \cdots, n.$$

According to [5] (Lemma 2.2), $d = J_{n,s}$ has Darboux polynomials if and only if it has a non-trivial homogeneous polynomial first integral (a polynomial constant).

3.2. $J_{n,s}$ and $FJ_{n,s}$

Let $s \geq 2$ and $n \geq 3$ be integers. Then denote by δ the factored Jouanolou derivation $FJ_{n,s}$

$$\delta(x_i) = x_i(sx_{i+1} - x_i), i = 1, \cdots, n.$$

According to [5] (Corollary 3.2), if the factored derivation $\delta = FJ_{n,s}$ has no polynomial first integral, the same is true for the original Jouanolou derivation $d = FJ_{n,s}$.

3.3. Polynomials constants of $FJ_{n,s}$

The coordinates are evident Darboux polynomials for $FJ_{n,s}$. Let us call *strict* Darboux polynomial a Darboux polynomial which is not divisible by any of the coordinates.

A general Darboux polynomial is the product of a strict one by a monomial. It is easy to show that a non-trivial monomial cannot be a polynomial constant for $FJ_{n,s}$.

Thus, a non-trivial polynomial constant has some strict irreducible factor. Then, in order to conclude that $FJ_{n,s}$ has no polynomial constant and hence that $J_{n,s}$ has no Darboux polynomial, it is enough to show that $FJ_{n,s}$ has no strict (irreducible) Darboux polynomial.

The rest of this section is devoted to the proof of this sufficient condition for $n = 4$: $FJ_{n,s}$ has no strict Darboux polynomial.

3.4. Cofactors of strict Darboux polynomials of $FJ_{n,s}$

The following proposition gives strong restrictions on the cofactors of a supposed strict Darboux polynomial of $FJ_{n,s}$.

Proposition 3. *Let F be a non-trivial strict homogeneous Darboux polynomial of degree m of $FJ_{n,s}$ and let $\Lambda = \sum \lambda_i x_i$ be its cofactor. Then all λ_i are integers in the range $-m \leq \lambda_i \leq 0$. Moreover, two of the λ_i at least are different from* 0.

Proof. As F is strict, for any i, the polynomial $F_i = F_{|x_i=0}$ that we get by evaluating F in $x_i = 0$ is a non-zero homogeneous polynomial in $n-1$ variables (all but x_i) with the same degree m.

Evaluating the Darboux equation $\delta(F) = \Lambda F$ at $x_n = 0$ we obtain

$$(5) \qquad \sum_{i=1}^{n-2} x_i(sx_{i+1} - x_i)\partial_i(F_n) + x_{n-1}(-x_{n-1})\partial_{n-1}(F_n) = (\sum_{i=1}^{n-1} \lambda_i x_i)F_n,$$

Let $m_1, 0 \leq m_1 \leq m$, be the partial degree of F_n with respect to x_1.

Consider now F_n as a polynomial in $\mathbb{K}[x_2, \cdots, x_{n-1}][x_1]$. Balancing monomials of degree $m_1 + 1$ in Equation (4) gives $\lambda_1 = -m_1$.

Same results hold for all coefficients of the cofactor Λ.

As $|\lambda_i|$ is the partial degree of F_{i-1} with respect to x_i, $\lambda_i = 0$ means that the variable x_{i-1} appears in every monomial of F in which x_i appears.

Then, if all λ_i vanish, the product of all variables divides the non-trivial polynomial F, a contradiction with the fact that F is strict.
In the same way, if all λ_i but one vanish, the variable corresponding to the non-zero coefficient divides F, once again a contradiction. □

3.5. The Lagutinskii-Levelt procedure

In [5,6], we described completely a nice combinatorial tool to find necessary conditions on Darboux polynomials of some vector field and their cofactors by looking at them around one or several Darboux points of the vector field. Following Jean-Marie Strelcyn [2], we call this tool the Lagutinskii-Levelt procedure.

We will now describe the LL procedure in the only case that we are interested in: strict Darboux polynomials of $FJ_{n,s}$ at the Darboux point $U = [1, \cdots, 1]$ of the projective space.

Consider a strict Darboux polynomial F of degree m and cofactor Λ for $FJ_{n,s}$ and write the corresponding Darboux relations for $FJ_{n,s}$ and the Euler vector field:

$$(5)\qquad \begin{aligned}\sum x_i(sx_{i+1}-x_i)\partial_i F &= \Lambda F,\\ \sum x_i\partial_i F &= mF.\end{aligned}$$

A linear combination cancels the coefficient before $\partial_n F$:

$$(6)\qquad \sum_{i=1}^{n-1} x_i(sx_{i+1}-x_i-sx_1+x_n)\partial_i F \;=\; (\Lambda - msx_1 + mx_n)F.$$

Now choose local coordinates around the point $U=[1,\cdots,1]$:

$$x_i = 1+y_i, 1\le i\le n-1, y_n = 0.$$

In the new coordinates, Equation (6) becomes

$$(7)\qquad \sum_{i=1}^{n-1}(1+y_i)(sy_{i+1}-y_i-sy_1)\partial_i F \;=\; (\Lambda - ms(1+y_1)+m)F,$$

which can be developed as

$$(8)\qquad \begin{aligned}&\sum_{i=1}^{n-2}(1+y_i)(sy_{i+1}-y_i-sy_1)\partial_i F + (1+y_{n-1})(-y_{n-1}-sy_1)\partial_{n-1}F\\ &= \;(m(1-s)+\sum_{i=1}^{n}\lambda_i - msy_1 + \sum_{i=1}^{n-1}\lambda_i y_i)F.\end{aligned}$$

As U is a Darboux point, all coefficients before the partial derivatives vanish at $[0,\cdots,0]$.

We pass now to the heart of the LL method.

Let then H be the homogeneous component of the lowest degree $\mu\le m$ of F in the y_i. Consider the homogeneous component of degree μ of Equation (8):

$$(9)\quad \sum_{i=1}^{n-2}(sy_{i+1}-y_i-sy_1)\,\partial_i H + (-y_{n-1}-sy_1)\,\partial_{n-1}H \;=\; \left(m(1-s)+\sum_{i=1}^{n}\lambda_i\right)H.$$

It is convenient to change the sign on both sides. This means that H is a Darboux polynomial with the prescribed cofactor $\left(m(s-1)-\sum_{i=1}^{n}\lambda_i\right)$ for the linear derivation

$$(10)\qquad D_0 = \sum_{i=1}^{n-2}(-sy_{i+1}+y_i+sy_1)\,\partial_i + (y_{n-1}+sy_1)\,\partial_{n-1}.$$

The corresponding square matrix is conjugate to a diagonal one. Indeed, its eigenvalues are different: they are all the $1-s\omega$ where ω is a n-th root of unity except 1 itself.

Thus, after a linear change of coordinates, D_0 can be written

$$(11)\qquad D_0 = \sum_{i=1}^{n-1}(1-s\,\omega^i)\partial_i,$$

for some primitive n-th root of unity whereas, by a scalar multiplication, H is a monomial $\prod u_i^{\alpha_i}$ in the new coordinates u_i.

Thus there exist nonnegative integers α_i, $1 \leq i \leq n-1$, such that

$$(12) \qquad \sum_{i=1}^{n-1} \alpha_i = \mu, \quad \sum_{i=1}^{n-1} \alpha_i (1 - s\,\omega^i) = m(s-1) - \sum_{i=1}^{n} \lambda_i.$$

In the case where $n \geq 3$ is a prime number and $s \geq 3$, this analysis is sufficient to give a contradiction and $FJ_{n,s}$ has no strict Darboux polynomial [5]. For $n = 4$ in particular, we need to go further in the local analysis of Darboux strict polynomials of $FJ_{n,s}$ at U.

In the case of a prime $n \geq 5$ with $s = 2$, all α_i are equal to the same α. Then either $\mu = (n-1)\alpha < m$, which implies $\alpha \geq 2$ or $\alpha = 1$ and $m = \mu = (n-1)$ and the Darboux polynomial would factor in linear forms.

3.6. Beyond the Lagutinskii-Levelt procedure

The Darboux equation (7) for F may be now written in the new coordinates u_i:

$$(13) \qquad (D_0 + D_1)(F) = (\gamma + \Gamma)F,$$

where D_0 is the previously defined linear derivation $D_0 = \sum_{i=1}^{n-1}(1 - s\,\omega^i)\partial_i$, where $D_1 = \sum_{i=1}^{n-1} U_i\,\partial_i$ with homogeneous U_i of degree 2, where $\gamma = m(s-1) - \sum_{i=1}^{n} \lambda_i$ and where Γ is some homogeneous polynomial of degree 1 whose value is not important.

Moreover, the nonzero Darboux polynomial F is defined up to a nonzero multiplicative factor. We normalize it by giving the coefficient 1 to its term $\prod u_i^{\alpha_i}$ of the lowest total degree.

Let us call the set of all solutions $\alpha = [\alpha_1, \cdots, \alpha_{n-1}] \in \mathbb{N}^{n-1}$ of

$$\sum_{i=1}^{n-1} \rho_i \alpha_i = \sum_{i=1}^{n-1} (1 - s\omega^i)\alpha_i = \gamma$$

the *exposed face* for γ and denote it by $\mathcal{E}$. The *exposed support* of F is the subset $\mathcal{S} = \mathcal{E} \cap Supp(F)$ of $\mathcal{E}$, and by $\mathcal{H}$ we denote the convex hull of $\mathcal{S}$ in $\mathbb{N}^{n-1}$.

We will say that an irreducible F *satisfies the* 0–1 *constraint* if for every i there exists an exponent α in $\mathcal{S}$ such that α_i is either 0 or 1. This is the way in which we are able to take into account the irreducibility of F.

We will explain later how the 0-1 constraint comes from the study of the local Darboux problem in the ring $\mathbb{K}[[u]]$ of formal power series in $n-1$ variables. Let us now pass to the conclusion: the 0–1 constraint gives an upper bound on the degree of irreducible strict Darboux polynomials for δ and allows us to show their absence.

3.7. Under the 0–1 constraint

In the case $n = 4$, the study of a Diophantine system is a useful tool to prove that there is no strict Darboux polynomial of $FJ_{4,s}$ that satisfies the 0–1 constraint.

Lemma 1. *Let $m \geq 1$, $s \geq 2$ and $L \geq 2$ be integers. Consider the following system in the unknowns $\alpha_1, \alpha_2, \alpha_3$*

$$\begin{cases} (1-s\,i)\,\alpha_1 \;+\; (1+s)\,\alpha_2 \;+\; (1+s\,i)\,\alpha_3 \;=\; (s-1)m+L, \\ \alpha_1+\alpha_2+\alpha_3 \leq m, \\ \alpha_2^2-\alpha_2 \;=\; 0, \end{cases}$$

where i stands for the square root of -1. The only solutions of (14) in $\mathbb{N}^3$ are

$$\text{(15)} \qquad \begin{cases} [\alpha_1,\, \alpha_2,\, \alpha_3\,] \;=\; [k,1,k], \textit{with } s=2, L=2, m=2k+1, \\ [\alpha_1,\, \alpha_2,\, \alpha_3\,] \;=\; [0,1,0], \textit{with } s \geq 3, L=2, m=1. \end{cases}$$

Clearly, for every solution of this system, we would have

$$\text{(16)} \quad \alpha_1=\alpha_3, \quad (1+s)\,\alpha_2+2\,\alpha_1 \;=\; m(s-1)+L, \quad \alpha_2+2\alpha_1 \leq m, \quad \alpha_2 \in \{0,1\}$$

If $\alpha_2=0$, then $m(s-1)+L=2\alpha_1 \leq m$ which implies $m(s-2)+L \leq 0$, hence $s=2$ and $L=0$, which is excluded.

Now let $\alpha_2=1$. Then $m(s-1)+L=2\alpha_1+1+s \leq m+s$ whence $m(s-2)+L \leq s$. As $L \leq 1$ is excluded, we are left with the two announced possibilities: $[s=2, L=2, m=2\alpha_1+1]$ and $[s \geq 3, m=1, L=2]$, with $\alpha_1=0$. □

Proposition 4. *There is no non-trivial strict Darboux polynomial of $FJ_{4,s}$ that satisfies the 0–1 constraint.*

Proof. Such a Darboux polynomial F would have a cofactor $\Lambda=\lambda_1\,x_1\;+\lambda_2\,x_2\;+\lambda_3\,x_3\;+\lambda_4\,x_4$.

According to Proposition 3, two λ_i at least do not vanish and $|\Lambda|=\sum|\lambda_i| \geq 2$.

Application of Equation (12) to the case $n=4$ together with the 0–1 constraint gives the system (14) with $L=|\Lambda|$.

From Lemma 1, $L=|\Lambda|=2$. Thus, there are two 0 and two -1 among the values of λ_i. Moreover, either $s=2$ or, if $s \geq 3$, the degree m of F is 1.

It is an exercise to conclude there is no strict Darboux polynomial for $FJ_{4,s}$ when $s=2$ or when $s \geq 3, m=1$ with a cofactor $\lambda_1\,x_1\;+\lambda_2\,x_2\;+\lambda_3\,x_3\;+\lambda_4\,x_4$ with two 0 and two -1 among λ_i; in the case analysis, patterns $[1,1,0,0]$ and $[1,0,1,0]$ for the λ_i have to distinguished. We leave it to the reader. □

In the case of $F_{5,2}$ or more generally for $F_{n,2}$, where $n \geq 5$ is prime, the strict Darboux polynomials that obey the 0–1 constraint would be linear. It is an easy exercise to check that this is impossible.

We have now to establish the 0–1 constraint, i. e. to state that a strict irreducible Darboux polynomial of $FJ_{4,s}$ satisfies the 0–1 constraint at the Darboux point U. We have therefore to study the analogous of the Darboux problem (13) for power series in $n-1$ variables instead of polynomials in n variables.

The same arguments, with much simpler details, will prove the 0–1 constraint for a prime number $n \geq 5$ and $s = 2$.

3.8. Square-free polynomials and power series

The following lemma is the only way we found to use the fact that a Darboux polynomial is irreducible.

Lemma 2. *Let f be a square-free polynomial in $\mathbb{K}[x_1, \cdots, x_n]$ vanishing at $[0, \cdots, 0]$. Then f is not a unit in the ring $\mathbb{K}[[x_1, \cdots, x_n]]$ of power series and it is square-free in $\mathbb{K}[[x_1, \cdots, x_n]]$.*

Before proving the lemma, let us remark that, if f does not vanish at $[0, \cdots, 0]$, then f is a unit in the ring of power series and asking if it is square-free is an empty question. Indeed, the units of $\mathbb{K}[[x_1, \cdots, x_n]]$ are the power series with a non-zero constant term.

We can now pass to the proof, in which the partial derivatives are a good tool to study multiple factors of polynomials and power series when $\mathbb{K}$ has the characteristic 0.

Proof. First, a non-constant polynomial f is square-free in $\mathbb{K}[x_1, \cdots, x_n]$ if and only if the greatest common divisor of f and all its partial derivatives $\partial_i(f)$, $1 \leq i \leq n$, is 1.

This result is easy to prove in one direction: a irreducible multiple factor of f would be a factor of f and of all its partial derivatives.

To prove the assertion in the other direction, first consider the case of an irreducible f: f cannot divide all its partial derivatives; for degree reasons, if f divides $\partial_i(f)$ then $\partial_i(f) = 0$ and f does not depend on x_i.

Let now f be a product of different irreducible f_i. A common irreducible factor of f and all its partial derivatives has to be chosen among the f_i. But f_i does not divide all its partial derivatives and there is some partial derivative of f which is not divisible by f_i.

The same result holds in $\mathbb{K}[[x_1, \cdots, x_n]]$, which is also a unique factorization domain: a non-invertible f is square-free if and only if the greatest common divisor of f and all its partial derivatives is 1.
The proof of this result is quite similar to the proof of the previous assertion on $\mathbb{K}[x_1, \cdots, x_n]$.

The only change appears in the proof that an irreducible non-invertible f cannot divide all its partial derivatives; instead of degree arguments, we need valuation arguments.

Choose some lexicographical order on exponents; the minimal degree of all monomials appearing in f is not 0; one of the variables at least in involved in this term and the partial derivative with respect to it has thus a lowest degree which is smaller than the lowest degree of f and cannot be a multiple of f.

To achieve the proof of the lemma with the help of the previous characterization that f is square-free in terms of partial derivatives, it remains to prove that the greatest common divisor of a finite set of polynomials in $\mathbb{K}[x_1, \cdots, x_n]$ is also their greatest common divisor in $\mathbb{K}[[x_1, \cdots, x_n]]$. Of course, it suffices to prove it for two polynomials. Equivalently, if P and Q are relatively prime in $\mathbb{K}[x_1, \cdots, x_n]$, then a common divisor ϕ of them in $\mathbb{K}[[x_1, \cdots, x_n]]$ has to be invertible.

As P and Q are relatively prime, for any i, there exists polynomials U_i, V_i and R_i such that $U_iP + V_iQ = R_i$, where the non-zero R_i is a polynomial in all variables but x_i.

For every i, ϕ divides R_i and thus its lowest exponent must have 0 as its i-coordinate. Therefore, this lowest exponent is 0 and ϕ is invertible. $\square$

Remark 1. When partial derivatives cannot be used to characterize square-free elements of unique factorization domains, the general statement that "an element a of a unique factorization domain A which is square-free in A is also square-free in the unique factorization domain B" is false. Consider for instance 2, which is square-free in $\mathbb{Z}$ and associate to the square of $(1 + i)$ in the unique factorization domain $\mathbb{Z}[i]$.

3.9. The Darboux problem in $\mathbb{K}[[u_1, \cdots, u_{n-1}]]$

In the local coordinates $u_i, 1 \leq i \leq n-1$, for which D_0 is diagonal, a strict Darboux polynomial F of degree m and cofactor Λ for $FJ_{n,s}$ Darboux satisfies Equation (13) which takes the following form

$$\sum_{i=1}^{n-1}[(1 - s\omega^i)u_i + U_i]\partial_i F \;=\; \gamma(1 + T)F,$$

where ω is some chosen primitive n-th root of 1, the U_i are homogeneous polynomials of degree 2, T is a homogeneous polynomial of degree 1 and $\gamma \in \mathbb{K}$.

Moreover, the component of the lowest degree of F can be normalized such that

$$F \equiv u^\alpha = \prod u_i^{\alpha_i} \quad (\mathcal{M}^{|\alpha|+1}),$$

where $\mathcal{M}$ is the maximal ideal of $\mathbb{K}[[u]]$.

We will say that such a Darboux polynomial F has a *(local) good presentation* if it satisfies the following three conditions.

- There exists an invertible power series κ starting with 1 such that the power series $g = \kappa^{-1}F$ satisfies $D(g) = \gamma(1 + T')g$ with a simpler cofactor $\gamma(1 + T')$ that belongs to the kernel of the initial derivation $D_0 = \sum_{i=1}^{n-1} \rho_i u_i \partial_i = \sum_{i=1}^{n-1}[(1 - s\omega^i)u_i]\partial_i$.

• For every index i such that $\alpha_i \neq 0$, the *elementary* problem, in which γ is replaced by the eigenvalue $\rho_i = (1 - s\omega^i)$ of D_0 corresponding to u_i,

$$D(\phi) = (1 - s\omega^i)(1 + T')\phi,$$

has a (maybe non-unique) solution ϕ_i in $\mathbb{K}[[u]]$ such that $\phi_i \equiv u_i \quad (\mathcal{M}^2)$. (in this case we call the ϕ_i *Darboux coordinates*).

• g is equal to a power series in ϕ_i with support in $\mathcal{H}$: there exists a unique family $\{g_\alpha, \alpha \in \mathcal{H}\}$ such that g is equal to the (infinite and convergent) sum

$$g = \sum_{\alpha \in \mathcal{H}} g_\alpha \prod \phi_i^{\alpha_i}.$$

Suppose that every Darboux polynomial F has a good presentation at U. As F can be supposed square-free as a polynomial, F is square-free as a power series, according to Lemma 2.

Then, no ϕ_i^2 divides F and, for every index i, there exists an $\alpha \in \mathcal{H}$ with $g_\alpha \neq 0$. The corresponding u^α is one of the monomials appearing in F, a polynomial of total degree m.

This means that F would satisfy the 0–1 constraint.

Thus, it remains to show that strict Darboux polynomials of $FJ_{4,s}$ have a good presentation at $[1, 1, 1, 1]$. The last subsection 3.10 is devoted to this technical result.

3.10. Strict Darboux polynomials of $FJ_{4,s}$ at $[1, 1, 1, 1]$

To simplify matters, let us change the notations in this three-variable case. We thus consider the local Darboux problem in $\mathbb{K}[u, v, w]$

$$[(1 - si)u + U]\partial_u F + [(1 + s)v + V]\partial_v F + [(1 + si)w + W]\partial_w F = \gamma(1 + T)F,$$

where U, V, W are homogeneous polynomials of degree 2 in $\mathbb{K}[u, v, w]$, where T is a homogeneous polynomial of degree 1 in $\mathbb{K}[u, v, w]$ and where γ is in $\mathbb{K}$.

Suppose that this problem has a solution $F \in \mathbb{K}[u, v, w]$ whose lowest degree term is $u^I v^J w^K$. We would like to prove that F has a good presentation at $U = [1, 1, 1, 1]$. This is a specialization of the following proposition.

Proposition 5. *Let $s \geq 2$ be an integer. Let U, V, W be power series in $\mathbb{K}[[u, v, w]]$ of valuation 2 and let D be the derivation $[(1 - si)u + U]\partial_u + [(1 + s)v + V]\partial_v + [(1 + si)w + W]\partial_w$. Let T be a power series in $\mathbb{K}[[u, v, w]]$ of valuation 1. Suppose that a non-zero element f is a* Darboux power series *with cofactor $\gamma(1 + T)$ of D, which means*

$$[(1 - si)u + U]\partial_u f + [(1 + s)v + V]\partial_v f + [(1 + si)w + W]\partial_w f = \gamma(1 + T)f$$

Then, f has a good presentation.

Proof. Normalization of the cofactor
In the present case, finding a suitable κ is not very difficult. Indeed, for any

candidate cofactor $\lambda = \sum \lambda_n$ without constant term ($\lambda_0 = 0$), there exists one and only one power series κ such that

$$(17) \qquad \kappa \equiv 1 \quad (\mathcal{M}), \qquad D(\kappa) = \lambda\kappa,$$

where $\mathcal{M}$ is the maximal ideal of $\mathbb{K}[[u, v, w]]$.

To check this fact, remark that the initial diagonal derivation D_0, which is homogeneous of degree 0, acts on every finite-dimensional $\mathbb{K}$-vector space $\mathbb{K}_n[u, v, w]$ of homogeneous polynomials of degree n as a linear map. The monomials are eigenvectors

$$D_0(u_1^{\alpha_1} u_2^{\alpha_2} u_3^{\alpha_3}) = ((1 - si)\alpha_1 + (1 + s)\alpha_2 + (1 + si)\alpha_3) u_1^{\alpha_1} u_2^{\alpha_2} u_3^{\alpha_3},$$

and it is simple to check that no eigenvalue $(1 - si)\alpha_1 + (1 + s)\alpha_2 + (1 + si)\alpha_3$ is 0 when $n = |\alpha| \geq 1$ (the exposed face for 0 reduces to $\{[0, 0, 0]\}$); thus D_0 is one-to-one on every $\mathbb{K}_n[u, v, w], n \geq 1$.

The sought power series κ can be written as an infinite sum of homogeneous polynomials

$$\kappa = \sum_{n=0}^{\infty} \kappa_n = 1 + \sum_{n=1}^{\infty} \kappa_n$$

and equation $D(\kappa) = \lambda\kappa$ can be developed as

$$(18) \qquad \sum_{n=0}^{\infty} D_0(\kappa_n) = \lambda.\kappa - \sum_{n=0}^{\infty} D_{>0}(\kappa_n),$$

where the derivation $D_{>0} = D - D_0$ strictly increases degrees as well as the multiplication by λ ($\lambda_0 = 0$).

Equating terms of degree 0 in Equation (18) gives $D_0(\kappa_0) = 0$ and we can then fix $\kappa_0 = 1$.

Equating terms of degree $n \geq 1$ in Equation (18) gives a linear equation on κ_n

$$(19) \qquad D_0(\kappa_n) = \sum_{i+j=n} \lambda_i \kappa_j - \sum_{i=0}^{n-1} [D_{>0}(\kappa_i)]_{|n},$$

where the notation $[\ ]_{|n}$ stands for the homogeneous component of total degree n.

As D_0 is one-to-one on $\mathbb{K}_n[u, v, w], n \geq 1$, Equation (19) gives κ_n in a unique way from previously known $\kappa_i, i < n$.

Then, the problem (17) can be solved by induction: κ is completely and uniquely determined from the initial value $\kappa_0 = 1$ and from the successive equations (19) for $n \geq 1$.

Now, if we choose $\lambda = \gamma T$, $g = \kappa^{-1} f$ is a Darboux power series for the derivation D, but with the cofactor $\gamma \in \mathbb{K}$ instead of $\gamma(1 + T)$ and its initial term (the one of lowest degree) is $u^I v^J w^K$, the same as the one of f.

Looking for Darboux coordinates

The *Darboux coordinates* that we look for are power series ϕ_1, ϕ_2, ϕ_3 whose initial terms are u, v, w respectively (the coordinates) and whose cofactors for D are $1 - si$, $1 + s$ and $1 + si$ respectively (the eigenvalues of D_0).

By an induction process similar to the one we used for computing κ, we can define uniquely and completely ϕ_1 and ϕ_3. In this case, the series start at the degree 1 and $D_0 - (1 \pm si)$ is a one-to-one linear mapping from every $\mathbb{K}_n[u, v, w]$ to itself, when $n \geq 2$. Indeed,

- $(1 - si)\alpha_1 + (1 + s)\alpha_2 + (1 + si)\alpha_3 = 1 - si$ has only the solution $\alpha = [1, 0, 0]$ in $\mathbb{N}^3$,
- $(1 - si)\alpha_1 + (1 + s)\alpha_2 + (1 + si)\alpha_3 = 1 + si$ has only the solution $\alpha = [0, 0, 1]$ in $\mathbb{N}^3$.

In the case of $J_{n,2}$ with a prime $n \geq 5$, all equations

$$\sum_{j=1}^{n-1}(1 - 2\omega^j)\alpha_j = (1 - 2\omega^{j_0})\alpha_{j_0},$$

where ω is a primitive n-th root of 1 and j_0 goes from 1 to $n - 1$, have only the trivial solution in nonnegative integers; thus Darboux coordinates do exist, which provides the good presentation and achieves the proof in this case without further considerations. This is not the case for $J_{4,s}$.

Critical conditions

A new fact appears in defining the second Darboux coordinate ϕ_2 from v.
It is still true that equation $(1 - si)\alpha_1 + (1 + s)\alpha_2 + (1 + si)\alpha_3 = 1 + s$ has only one solution in $\mathbb{N}^3$ when s is even, $\alpha = [0, 1, 0]$.

But, when s is odd, the equation has two solutions in $\mathbb{N}^3$, $\alpha = [0, 1, 0]$, $\alpha = [\frac{s+1}{2}, 0, \frac{s+1}{2}]$.

Thus, if s is even, ϕ_2 is defined completely and uniquely by induction and we receive the sought Darboux coordinate.

On the contrary, if s is odd, the process has to be stopped at the degree $n = s + 1$ where the corresponding equation is

$$(D_0 - (1 + s))(\phi_{2,n}) = -\sum_{j=0}^{n-1}[D_{>0}(\phi_{2,j})]_{|n}. \tag{20}$$

The linear map $D_0 - (s + 1)$ is neither injective nor surjective on $\mathbb{K}_n[u, v, w]$:

- the coefficient of $(uw)^{(s+1)/2}$ in ϕ_2 is not defined by Equation (20),
- the coefficient of $(uw)^{(s+1)/2}$ in $\sum_{j=0}^{n-1}[D_{>0}(\phi_{2,j})]_{|n}$ has to be 0.

Thus we have a freedom to define a coefficient and a *necessary condition* in order to start the induction process again; let us call this necessary condition *critical*. If the critical condition is fulfilled, we give an arbitrary value to the free coefficient of ϕ_2 and ϕ_2 is completely (but not uniquely) defined.

It is possible to deduce this critical condition from the existence of g as a Darboux power series for D with cofactor γ and initial term $u^I v^J w^K$, provided that $J > 0$.

But, if $J = 0$, we do not need ϕ_2 to give a good presentation of g.

Cancellation of critical conditions

Let $s \geq 3$ be an odd natural number. Two Darboux coordinates are known for D, ϕ_1 and ϕ_3; moreover, we can start the same induction process to define ϕ_2 up to degree s, or modulo $\mathcal{M}^{s+1}$, which can be written as

$$(21) \qquad \phi_2 \equiv v \quad (\mathcal{M}^2), \qquad D(\phi_2) \equiv (1+s)\phi_2 \quad (\mathcal{M}^{s+1}).$$

Moreover, all coefficients of $\phi_{2,s+1}$ are also well-defined by induction, except of course the critical one before $(uw)^{(s+1)/2}$. Thus ϕ_2 is defined modulo the larger ideal $\left(\mathcal{M}^{s+2}, (uw)^{(s+1)/2}\right)$ and satisfies

$$(22) \qquad D(\phi_2) \;\equiv\; (1+s)\phi_2 \quad \left(\mathcal{M}^{s+2}, (uw)^{(s+1)/2}\right).$$

Recall that we also assume the existence of g such that

$$(23) \;\; D(g) = ((1-si)I + (1+s)J + (1+si)K)\, g, \quad g \equiv u^I v^J w^K \quad (\mathcal{M}^{I+J+K+1}), \quad J > 0.$$

The triple $\alpha = [I, J, K]$ is an obvious solution in N^3 of the equation

$$(24) \;\; (1-si)\alpha_1 + (1+s)\alpha_2 + (1+si)\alpha_3) = \gamma = (1-si)I + (1+s)J + (1+si)K.$$

This solution is the only one of degree $|\alpha| = I + J + K$.

For any n, $1 \leq n < s$, there is no solution to (24) with $|\alpha| = n + I + J + K$.

For $n = s$, there is one solution to (24) with $|\alpha| = n + I + J + K$, $\alpha = [I + \frac{s+1}{2}, J-1, K + \frac{s+1}{2}]$.

Thus, modulo the ideal $\left(\mathcal{M}^{I+J+K+s+1}, u^I v^{J-1} w^K (uw)^{(s+1)/2}\right)$, g and the product $P = \phi_1^I \phi_2^J \phi_3^K$ are well defined by induction from their common initial monomial of the lowest degree, $u^I v^J w^K$.

Indeed, $D_0 - \gamma$ is invertible for all monomials of total degree $I + J + K < |\alpha| < I + J + K + s$ and also for all monomials of total degree $I + J + K + s$ except $u^{I+(s+1)/2} v^{J-1} w^{K+(s+1)/2}$.

Thus, modulo this ideal, g and the P agree:

$$(25) \qquad g \equiv \phi_1^I \phi_2^J \phi_3^K \quad \left(\mathcal{M}^{I+J+K+s+1}, u^I v^{J-1} w^K (uw)^{(s+1)/2}\right).$$

As $J \neq 0$, it is possible to fix the coefficient of ϕ_2 before $u^{(s+1)/2} w^{(s+1)/2}$ in such a way that g and $\phi_1^I \phi_2^J \phi_3^K$ have the same coefficient before $u^{I+(s+1)/2} v^{J-1} w^{K+(s+1)/2}$, i. e. such that g and P agree modulo the smaller ideal $\mathcal{M}^{I+J+K+s+1}$.

By transitivity, we thus get the better congruence (modulo a smaller ideal):

$$D(\phi_1^I \phi_2^J \phi_3^K) \;\equiv\; \gamma \phi_1^I \phi_2^J \phi_3^K \quad (\mathcal{M}^{I+J+K+s+1}).$$

But $D(\phi_1^I \phi_2^J \phi_3^K) - \gamma \phi_1^I \phi_2^J \phi_3^K$ may be developed as (recall $\gamma = (1-si)I + (1+s)J + (1+si)K$):

$$\begin{aligned} D(\phi_1^I \phi_2^J \phi_3^K) - \gamma \phi_1^I \phi_2^J \phi_3^K &= I \phi_1^{I-1} \phi_2^J \phi_3^K (D(\phi_1) - (1-si)\phi_1) \\ &\quad + J \phi_1^I \phi_2^{J-1} \phi_3^K (D(\phi_2) - (1+s)\phi_2) \\ &\quad + K \phi_1^I \phi_2^J \phi_3^{K-1} (D(\phi_3) - (1+si)\phi_3). \end{aligned}$$

As $D(\phi_1) = (1 - si)\phi_1$ and $D(\phi_3) = (1 + si)\phi_3$, the first and third terms of the right-hand side of the previous equality are 0 and we get ($J > 0$):

$$(26) \qquad \phi_1^I \phi_2^{J-1} \phi_3^K (D(\phi_2) - (1+s)\phi_2) \in \mathcal{M}^{I+J+K+s+1}.$$

Using the previously known congruences

$$D(\phi_2) \equiv (1+s)\phi_2 \quad (\mathcal{M}^{s+1}) \quad \textit{and} \quad \phi_1^I \phi_2^{J-1} \phi_3^K \equiv u^I v^{J-1} w^K \quad (\mathcal{M}^{I+J+K}),$$

we get $(D(\phi_2) - (1+s)\phi_2)(\phi_1^I \phi_2^{J-1} \phi_3^K - u^I v^{J-1} w^K) \in \mathcal{M}^{I+J+K+s+1}$.

By difference with congruence (26), we get

$$(D(\phi_2) - (1+s)\phi_2) u^I v^{J-1} w^K \in \mathcal{M}^{I+J+K+s+1}.$$

By simplification, $D(\phi_2) - (1+s)\phi_2$ belongs to $\mathcal{M}^{s+2}$.

In other words, the coefficient of the monomial of exponent $[(s+1)/2, 0, (s+1)/2]$ in $D(\phi_2) - (1+s)\phi_2$ is 0, which is exactly the sought critical condition.

Good presentation

In the present case, $\mathcal{H}$ is simple to describe: this is the set of all solutions of Equation (24) with $|\alpha| \geq I + J + K$. This set is finite, but this is not important.

The coefficients g_α may be uniquely defined by induction.

Setting $g_{[I,J,K]} = 1$ is the unique way to ensure congruence for the total degree $n = n_0 = I + J + K$:

$$(27) \qquad g \equiv \sum_{\alpha \in \mathcal{H}_{n_0}} g_\alpha \prod \phi_i^{\alpha_i} \equiv g_{[I,J,K]} \phi_1^I \phi_2^J \phi_3^K \equiv \phi_1^I \phi_2^J \phi_3^K \quad (\mathcal{M}^{n+1}).$$

Let now $n \geq n_0 + 1$ and suppose that all g_α, $\alpha \in \mathcal{H}, n_0 \leq |\alpha| < n$, have been uniquely defined in such a way that

$$(28) \qquad g \equiv \sum_{\alpha \in \mathcal{H}_{<n}} g_\alpha \prod \phi_i^{\alpha_i} \quad (\mathcal{M}^n).$$

Series g and the sum in construction are both in the kernel of $D - \gamma$. Thus, their coefficients of total degree n are the same for the exponents outside $\mathcal{H}$ and the congruence is better:

$$(29) \qquad g \equiv \sum_{\alpha \in \mathcal{H}_{<n}} g_\alpha \prod \phi_i^{\alpha_i} \quad (\mathcal{M}^{n+1}, u^{\beta_1} v^{\beta_2} w^{\beta_3}, \beta \in \mathcal{H}_n),$$

where $\mathcal{H}_n$ may be empty.

Now define coefficients g_β for all (if any) $\beta \in \mathcal{H}_n$ by

$$(30) \qquad g \equiv \sum_{\alpha \in \mathcal{H}_{*<n}} g_\alpha \prod \phi_i^{\alpha_i} + \sum_{\beta \in \mathcal{H}_n} g_\beta u^{\beta_1} v^{\beta_2} w^{\beta_3} \quad (\mathcal{M}^{n+1}).$$

Remark that the only way to have

$$(31) \qquad \sum_{\alpha \in \mathcal{H}_n} \overline{g}_\alpha \prod \phi_i^{\alpha_i} \equiv \sum_{\beta \in \mathcal{H}_n} g_\beta u^{\beta_1} v^{\beta_2} w^{\beta_3} \quad (\mathcal{M}^{n+1})$$

is to choose $\overline{g}_\alpha = g_\alpha$ for all $\alpha \in \mathcal{H}_n$.

The induction step is now complete:

$$(32) \qquad g \equiv \sum_{\alpha \in \mathcal{H}_{<n+1}} g_\alpha \prod \phi_i^{\alpha_i} \quad (\mathcal{M}^{n+1}).$$

The whole process gives the sought good presentation:

$$(33) \qquad g \equiv \sum_{\alpha \in \mathcal{H}} g_\alpha \prod \phi_i^{\alpha_i} \quad (\mathcal{M}^{\infty}), \quad i.\ e. \quad g = \sum_{\alpha \in \mathcal{H}} g_\alpha \prod \phi_i^{\alpha_i}.$$

This completes the proof of Theorem 1 □

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