Journal of MATHEMATICAL

www.elsevier.com/locate/jmaa

On discrete q-ultraspherical polynomials and their duals

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Received 30 April 2004

Available online 4 February 2005

Submitted by H.M. Srivastava

Abstract

A special case of the big q-Jacobi polynomials $P_n(x; a, b, c; q)$, which corresponds to $a = b = -c$, is shown to satisfy a discrete orthogonality relation for imaginary values of the parameter a (outside of its commonly known domain $0 < a < q^{-1}$). Since $P_n(x; q^\alpha, q^\alpha, -q^\alpha; q)$ tend to Gegenbauer (or ultraspherical) polynomials in the limit as $q \to 1$, this family represents another q-extension of these classical polynomials, different from the continuous q-ultraspherical polynomials of Rogers. For a dual family with respect to the polynomials $P_n(x; a, a, -a; q)$ (i.e., for dual discrete q-ultraspherical polynomials) we also find new orthogonality relations with extremal measures.

Keywords: Big q-Jacobi polynomials; Little q-Jacobi polynomials; Discrete q-ultraspherical polynomials; Duality; Orthogonality relation

1. Main results

It is well known that the big q-Jacobi polynomials $P_n(x; a, b, c; q)$ are orthogonal for values of the parameters in the intervals $0 < a, b < q^{-1}$, $c < 0$. We show that these poly-

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doi:10.1016/j.jmaa.2004.10.016

nomials are also orthogonal outside of these intervals in a special case when $a = b = -c$. Since the polynomials $P_n(x; q^\alpha, q^\alpha, -q^\alpha|q)$ tend to ultraspherical polynomials when $q \to 1$, it is natural to call them *discrete q-ultraspherical polynomials* (because their orthogonality measure is discrete, contrary to the orthogonality measure for continuous q-ultraspherical polynomials of Rogers). We give explicitly an orthogonality relation for $P_n(x; a, a, -a|q)$ when a becomes imaginary. Orthogonality relations for dual q-Jacobi polynomials for the same special cases are also given.

Throughout the sequel we always assume that q is a fixed positive number such that $q < 1$. We use (without additional explanation) notations of the theory of q-special functions (see, for example, [5]).

If one introduces the notation

$$C_n^{(a^2)}(x; q) := P_n(x; a, a, -a; q) = {}_3\phi_2(q^{-n}, a^2q^{n+1}, x; aq, -aq; q, q), \tag{1}$$

then an orthogonality relation for $C_n^{(a)}(x; q)$, which follows from that for the big q-Jacobi polynomials (see formula (7.3.12) in [5]), holds for positive values of a. We prove that the polynomials $C_n^{(a)}(x; q)$ are orthogonal also for imaginary values of a and x. In order to dispense with imaginary numbers in this case, let us denote

$$\tilde{C}_n^{(a)}(x; q) := (-i)^n C_n^{(-a)}(ix; q) = (-i)^n {}_3\phi_2\left(\begin{matrix} q^{-n}, -aq^{n+1}, ix \\ i\sqrt{a}\,q, -i\sqrt{a}\,q \end{matrix} \middle| q, q \right), \tag{2}$$

where x is real and $0 < a < \infty$. These polynomials satisfy the recurrence relation

$$x\tilde{C}_n^{(a)}(x; q) = a_n\tilde{C}_{n+1}^{(a)}(x; q) + c_n\tilde{C}_{n-1}^{(a)}(x; q), \tag{3}$$

where $a_n = (1 + aq^{n+1})/(1 + aq^{2n+1})$, $c_n = a_n - 1$, and $\tilde{C}_0^{(a)}(x; q) \equiv 1$. Observe that $a_n \geqslant 1$ and, hence, coefficients in (3) satisfy the conditions $a_n c_{n+1} > 0$ of Favard's characterization theorem for $n = 0, 1, 2, \ldots$ (see, for example, [5]). This means that these polynomials are orthogonal with respect to a positive measure. We prove that the orthogonality relation for them is

$$\sum_{s=0}^{\infty} \sum_{\varepsilon=\pm 1} \frac{(-aq^2; q^2)_s q^s}{(q^2; q^2)_s} \tilde{C}_n^{(a)}(\varepsilon\sqrt{a}\,q^{s+1}; q)\tilde{C}_{n'}^{(a)}(\varepsilon\sqrt{a}\,q^{s+1}; q)$$
$$= \frac{(-aq^3; q^2)_\infty}{(q; q^2)_\infty} \frac{(1 + aq)a^n}{(1 + aq^{2n+1})} \frac{(q; q)_n}{(-aq; q)_n} q^{n(n+3)/2}\delta_{nn'}. \tag{4}$$

Since in the limit as $q \to 1$ the polynomials (1) and (2) tend to Gegenbauer polynomials, they represent *discrete q-ultraspherical polynomials*.

Note that the situation when along with orthogonal polynomials $p_n(x)$, depending on some parameters, the set of polynomials $(-i)^n p_n(ix)$ is also orthogonal, but for other values of parameters, is known (see, for example, [1,4,6]). The detailed discussion of orthogonality property of Jacobi polynomials can be found in [4]. Contrary to the situation in [1,4,6], in our case we have orthogonality relations for the whole set of polynomials $\tilde{C}_n^{(a)}(x; q)$, $n = 0, 1, 2, \ldots$.

In [2] we have introduced the polynomials $D_n(\mu(x; a); a, b, c|q)$, $\mu(x; a) := q^{-x} + abq^{x+1}$, dual to the big q-Jacobi polynomials $P_n(x; a, b, c|q)$. If we set $a = b = -c$ in these polynomials, this leads to the polynomials

$$D_n^{(a^2)}\big(\mu(x;a^2)|q\big) := D_n\big(\mu(x;a^2);a,a,-a|q\big) = {}_3\phi_2\left(\begin{matrix} q^{-x}, a^2q^{x+1}, q^{-n} \\ aq, -aq \end{matrix}\,\middle|\, q, -q^{n+1}\right) \tag{5}$$

in $\mu(x;a^2) = q^{-x} + a^2q^{x+1}$. They correspond to indeterminate moment problem. We also proved that the polynomials (5) satisfy the following orthogonality relations:

$$\sum_{k=0}^{\infty} \frac{(1-aq^{4k+1})(aq;q)_{2k}}{(1-aq)(q;q)_{2k}} q^{k(2k-1)} D_n^{(a)}\big(\mu(2k)|q\big) D_{n'}^{(a)}\big(\mu(2k)|q\big) = \beta \frac{(q^2;q^2)_n q^{-n}}{(aq^2;q^2)_n} \delta_{nn'}, \tag{6}$$

$$\sum_{k=0}^{\infty} \frac{(1-aq^{4k+3})(aq;q)_{2k+1}}{(1-aq)(q;q)_{2k+1}} q^{k(2k+1)} D_n^{(a)}\big(\mu(2k+1)|q\big) D_{n'}^{(a)}\big(\mu(2k+1)|q\big) = \beta \frac{(q^2;q^2)_n q^{-n}}{(aq^2;q^2)_n} \delta_{nn'}, \tag{7}$$

where $\beta = (aq^3;q^2)_\infty/(q;q^2)_\infty$, $\mu(2k) \equiv \mu(2k;a)$, $\mu(2k+1) \equiv \mu(2k+1;a)$ and $0 < a < q^{-2}$. The orthogonality measures here are extremal.

For the polynomials $D_n^{(a^2)}(\mu(x;a^2)|q)$ with imaginary a it is natural to define

$$\tilde{D}_n^{(a^2)}\big(\mu(x;-a^2)|q\big) := D_n\big(\mu(x;-a^2);ia,ia,-ia|q\big) = {}_3\phi_2\left(\begin{matrix} q^{-x}, -a^2q^{x+1}, q^{-n} \\ iaq, -iaq \end{matrix}\,\middle|\, q, -q^{n+1}\right). \tag{8}$$

These polynomials satisfy the recurrence relation

$$\begin{aligned}
&(q^{-x} - aq^{x+1})\tilde{D}_n^{(a)}\big(\mu(x;-a)|q\big) \\
&\quad = -q^{-2n-1}(1+aq^{2n+2})\tilde{D}_{n+1}^{(a)}\big(\mu(x;-a)|q\big) \\
&\qquad + q^{-2n-1}(1+q)\tilde{D}_n^{(a)}\big(\mu(x;-a)|q\big) - q^{-2n}(1-q^{2n})\tilde{D}_{n-1}^{(a)}\big(\mu(x;-a)|q\big).
\end{aligned}$$

It is obvious from this relation that $\tilde{D}_n^{(a)}(\mu(x;-a)|q)$ are real for $x \in \mathbb{R}$ and $a > 0$. For $a > 0$ they satisfy the conditions of Favard's theorem and, therefore, are orthogonal with respect to a positive measure.

The polynomials $\tilde{D}_n^{(a)}(\mu(x;a)|q)$ correspond to indeterminate moment problem and, therefore, they have infinitely many positive orthogonality measures. We prove in the next section that they satisfy the orthogonality relations

$$\sum_{k=0}^{\infty} \frac{(1+aq^{4k+1})(-aq;q)_{2k}}{(1+aq)(q;q)_{2k}} q^{k(2k-1)} \tilde{D}_n^{(a)}\big(\mu(2k)|q\big) \tilde{D}_{n'}^{(a)}\big(\mu(2k)|q\big) = \gamma \frac{(q^2;q^2)_n q^{-n}}{(-aq^2;q^2)_n} \delta_{nn'}, \tag{9}$$

$$\sum_{k=0}^{\infty} \frac{(1+aq^{4k+3})(-aq;q)_{2k+1}}{(1+aq)(q;q)_{2k+1}} q^{k(2k+1)} \tilde{D}_n^{(a)}\big(\mu(2k+1)|q\big) \tilde{D}_{n'}^{(a)}\big(\mu(2k+1)|q\big)$$
$$= \gamma \frac{(q^2;q^2)_n q^{-n}}{(-aq^2;q^2)_n} \delta_{nn'}, \tag{10}$$

where $\gamma = (-aq^3;q^2)_\infty/(q;q^2)_\infty$, $\mu(2k) \equiv \mu(2k;-a)$, $\mu(2k+1) \equiv \mu(2k+1;-a)$ and $a > 0$. The corresponding orthogonality measures are extremal.

The polynomials (5) and (8) are dual to the polynomials (1) and (2). For this reason, we call them *dual discrete q-ultraspherical polynomials*.

2. Proofs

The main idea of proving the orthogonality relations (4), (6), (7), (9) and (10) is to establish the connection between the polynomials (2) and the little q-Jacobi polynomials

$$p_n(x;a,b|q) := {}_2\phi_1(q^{-n}, abq^{n+1}; aq; q, qx), \tag{11}$$

as well as between the polynomials (5), (8) and the dual little q-Jacobi polynomials

$$d_n\big(\mu(x;ab);a,b|q\big) := {}_3\phi_1(q^{-x}, abq^{x+1}, q^{-n}; bq; q, q^n/a), \tag{12}$$

considered in [2].

Proposition 1. *The following expressions for the discrete q-ultraspherical polynomials* (2) *hold*:

$$\tilde{C}_{2k}^{(a)}(x;q) = \frac{(q;q^2)_k(-a)^k}{(-aq^2;q^2)_k} q^{k(k+1)} p_k(x^2/aq^2; q^{-1}, -a|q^2), \tag{13}$$

$$\tilde{C}_{2k+1}^{(a)}(x;q) = \frac{(q^3;q^2)_k(-a)^k}{(-aq^2;q^2)_k} q^{k(k+1)} x p_k(x^2/aq^2; q, -a|q^2), \tag{14}$$

where $p_k(y;a,b|q)$ are the little q-Jacobi polynomials (11).

Proof. We apply Singh's quadratic transformation (3.10.13) from [5] for a terminating ${}_3\phi_2$ series to the expression in (2) for polynomials $\tilde{C}_{2k}^{(a)}(x;q)$. This yields

$$\tilde{C}_{2k}^{(a)}(x;q) = (-1)^k {}_3\phi_2(q^{-2k}, -aq^{2k+1}, -x^2; -aq^2, 0; q^2, q^2).$$

Now apply to this basic hypergeometric series ${}_3\phi_2$ the transformation formula (III.7) from Appendix III in [5] in order to get

$$\tilde{C}_{2k}^{(a)}(x;q) = \frac{(q;q^2)_k(-a)^k}{(-aq^2;q^2)_k} q^{k(k+1)} {}_2\phi_1(q^{-2k}, -aq^{2k+1}; q; q^2, x^2/a).$$

Comparing this formula with the expression for the polynomials (11), one arrives at (13).

One can prove (14) by induction with the aid of formula (III.7) from Appendix III in [5] and the recurrence relation (3). Let us show first that

$$\tilde{C}_{2k-1}^{(a)}(x;q) = (-1)^{k-1} x {}_3\phi_2(q^{-2(k-1)}, -aq^{2k+1}, -x^2; -aq^2, 0; q^2, q^2) \tag{15}$$

for $k = 1, 2, 3, \ldots$. For $k = 1, 2$ this formula is an immediate consequence of the recurrence relation (3). As the next step we evaluate a sum $a_{2k}^{-1} x \tilde{C}_{2k}^{(a)}(x; q) - (1 - a_{2k}^{-1}) \tilde{C}_{2k-1}^{(a)}(x; q)$. By the recurrence relation (3) this sum should be equal to $\tilde{C}_{2k+1}^{(a)}(x; q)$. This is the case because it is equal to

$$\begin{aligned} & x\left\{a_{2k}^{-1}\,{}_3\phi_2\left(\left.\begin{matrix} q^{-2k}, -aq^{2k+1}, -x^2 \\ -aq^2, 0 \end{matrix}\right| q^2, q^2\right)\right. \\ & \quad \left. + \left(1 - a_{2k}^{-1}\right){}_3\phi_2\left(\left.\begin{matrix} q^{-2(k-1)}, -aq^{2k+1}, -x^2 \\ -aq^2, 0 \end{matrix}\right| q^2, q^2\right)\right\} \\ & \quad = x\,{}_3\phi_2\left(\left.\begin{matrix} q^{-2k}, -aq^{2k+3}, -x^2 \\ -aq^2, 0 \end{matrix}\right| q^2, q^2\right), \end{aligned} \tag{16}$$

multiplied by $(-1)^k$. The second line in (16) follows from the readily verified identity

$$a_{2k}^{-1}(q^{-2k}; q^2)_m + \left(1 - a_{2k}^{-1}\right)(q^{-2(k-1)}; q^2)_m = \frac{1 + aq^{2(k+m)+1}}{1 + aq^{2k+1}}(q^{-2k}; q^2)_m.$$

The right side of (16) does coincide with $\tilde{C}_{2k+1}^{(a)}(x; q)$, defined by the same expression (15) with $k \to k + 1$. Thus, it remains only to apply formula (III.7) from Appendix III in [5] in order to arrive at (14). Proposition is proved. □

Remark 1. Observe that in the course of proving formula (14), we established the quadratic transformation

$${}_3\phi_2\left(\left.\begin{matrix} q^{-2k-1}, \alpha q^{2k+2}, y \\ \sqrt{\alpha}\, q, -\sqrt{\alpha}\, q \end{matrix}\right| q, q\right) = y\,{}_3\phi_2\left(\left.\begin{matrix} q^{-2k}, \alpha q^{2k+3}, y^2 \\ \alpha q^2, 0 \end{matrix}\right| q^2, q^2\right) \tag{17}$$

for the terminating basic hypergeometric polynomials ${}_3\phi_2$ with $k = 0, 1, 2, \ldots$. The left side in (17) defines (up to a simple multiplicative factor) the polynomials $\tilde{C}_{2k+1}^{(a)}(x; q)$ by (2) (when $\alpha = -a$ and $y = ix$), whereas the right side follows from the expression (16) for the same polynomials.

Remark 2. By using the interrelation (2) between the polynomials $\tilde{C}_n^{(a)}(x; q)$ and $C_n^{(a)}(x; q)$, it is easy to write down an analogue of the relations (13) and (14) for the latter polynomials. In the limit as $q \to 1$ these relations go over to the well-known transformations

$$\begin{aligned} C_{2n}^{(\lambda)}(y) &= \frac{(\lambda)_n}{(\frac{1}{2})_n} P_n^{(\lambda-1/2,-1/2)}(2y^2 - 1), \\ C_{2n+1}^{(\lambda)}(y) &= \frac{(\lambda)_{n+1}}{(\frac{1}{2})_{n+1}} y P_n^{(\lambda-1/2,1/2)}(2y^2 - 1). \end{aligned}$$

Writing down the orthogonality relation (7.3.3) in [5] for the little q-Jacobi polynomials $p_k(x^2/aq^2; q^{-1}, -a|q^2)$ and using the relation (13), one finds an orthogonality relation for the set of polynomials $\tilde{C}_{2k}^{(a)}(x; q)$, $k = 0, 1, 2, \ldots,$ with $a > 0$:

$$\sum_{s=0}^{\infty} d_s \tilde{C}_{2k}^{(a)}(\sqrt{a}\, q^{s+1}; q) \tilde{C}_{2k'}^{(a)}(\sqrt{a}\, q^{s+1}; q)$$

$$= \frac{(-aq^3;q^2)_\infty}{(q;q^2)_\infty} \frac{(1+aq)a^{2k}}{(1+aq^{4k+1})} \frac{(q;q)_{2k}}{(-aq;q)_{2k}} q^{k(2k+3)} \delta_{kk'},$$

where $d_s = (-aq^2;q^2)_s q^s/(q^2;q^2)_s$. Consequently, *the family of polynomials* $\tilde{C}^{(a)}_{2k}(x;q)$, $k = 0,1,2,\dots,$ *is orthogonal on the set of points* $\sqrt{a}\,q^{s+1}$, $s = 0,1,2,\dots.$

Similarly, using formula (14) and the orthogonality relation for the little q-Jacobi polynomials $p_k(x^2/aq^2;q,-a|q^2)$, we find an orthogonality relation

$$\sum_{s=0}^{\infty} d_s \tilde{C}^{(a)}_{2k+1}(\sqrt{a}\,q^{s+1};q)\tilde{C}^{(a)}_{2k'+1}(\sqrt{a}\,q^{s+1};q)$$
$$= \frac{(-aq^3;q^2)_\infty}{(q;q^2)_\infty} \frac{(1+aq)a^{2k+1}}{(1+aq^{4k+3})} \frac{(q;q)_{2k+1}}{(-aq;q)_{2k+1}} q^{(k+2)(2k+1)} \delta_{kk'}$$

for the set of polynomials $\tilde{C}^{(a)}_{2k+1}(x;q)$, $k = 0,1,2,\dots,$ with $a > 0$, where d_s is the same as before. We see from this relation that *the polynomials* $\tilde{C}^{(a)}_{2k+1}(x;q)$, $k = 0,1,2,\dots,$ *are orthogonal on the same set of points* $\sqrt{a}\,q^{s+1}$, $s = 0,1,2,\dots.$

However, the polynomials $\tilde{C}^{(a)}_{2k}(x;q)$, $k = 0,1,2,\dots,$ are not orthogonal to the polynomials $\tilde{C}^{(a)}_{2k+1}(x;q)$, $k = 0,1,2,\dots,$ on this set of points $\sqrt{a}\,q^{s+1}$, $s = 0,1,2,\dots.$ In order to prove that the polynomials $\tilde{C}^{(a)}_{2k}(x;q)$, $k = 0,1,2,\dots,$ are orthogonal to the polynomials $\tilde{C}^{(a)}_{2k+1}(x;q)$, $k = 0,1,2,\dots,$ one has to consider them on the set of points $\pm\sqrt{a}\,q^{s+1}$, $s = 0,1,2,\dots.$ Since the polynomials from the first set are even and the polynomials from the second set are odd, then

$$\sum_{s=0}^{\infty} d_s \tilde{C}^{(a)}_{2k}(\sqrt{a}\,q^{s+1};q)\tilde{C}^{(a)}_{2k'+1}(\sqrt{a}\,q^{s+1};q)$$
$$+ \sum_{s=0}^{\infty} d_s \tilde{C}^{(a)}_{2k}(-\sqrt{a}\,q^{s+1};q)\tilde{C}^{(a)}_{2k'+1}(-\sqrt{a}\,q^{s+1};q) = 0.$$

This gives the mutual orthogonality of the polynomials $\tilde{C}^{(a)}_{2k}(x;q)$, $k = 0,1,2,\dots,$ to the polynomials $\tilde{C}^{(a)}_{2k+1}(x;q)$, $k = 0,1,2,\dots.$ Then the orthogonality relation for the whole set of polynomials $\tilde{C}^{(a)}_{n}(x;q)$, $n = 0,1,2,\dots,$ can be written in the form (4). Thus, the orthogonality relation (4) is proved.

Proposition 2. *The following expressions for the dual discrete* q*-ultraspherical polynomials* (5) *hold*:

$$D^{(a)}_n\big(\mu(2k;a)|q\big) = d_n\big(\mu(k;q^{-1}a);q^{-1},a|q^2\big)$$
$$= {}_3\phi_1\left(\begin{matrix} q^{-2k}, aq^{2k+1}, q^{-2n} \\ aq^2 \end{matrix}\,\middle|\, q^2, q^{2n+1}\right), \tag{18}$$

$$D^{(a)}_n\big(\mu(2k+1;a)|q\big) = q^n d_n\big(\mu(k;qa);q,a|q^2\big)$$
$$= q^n\,{}_3\phi_1\left(\begin{matrix} q^{-2k}, aq^{2k+3}, q^{-2n} \\ aq^2 \end{matrix}\,\middle|\, q^2, q^{2n-1}\right), \tag{19}$$

where k are nonnegative integers and $d_n(\mu(x; bc); b, c|q)$ are polynomials (12).

Proof. Applying to the right side of (5) the formula (III.13) from Appendix III in [5] and then Singh's quadratic relation (3.10.13) of [5] for terminating ${}_3\phi_2$ series, after some transformations one obtains

$$D_n^{(a^2)}\big(\mu(2k; a^2)|q\big) = a^{-2k}q^{-k(2k+1)}{}_3\phi_2\left(\begin{matrix} q^{-2k}, a^2q^{2k+1}, a^2q^{2n+2} \\ a^2q^2, 0 \end{matrix}\,\middle|\, q^2, q^2\right).$$

Now apply the relation

$${}_3\phi_2\left(\begin{matrix} q^{-n}, \alpha, \beta \\ \gamma, 0 \end{matrix}\,\middle|\, q, q\right) = \frac{(\gamma/\alpha; q)_n}{(\gamma; q)_n}{}_2\phi_1\left(\begin{matrix} q^{-n}, \alpha \\ \alpha q^{1-n}/\gamma \end{matrix}\,\middle|\, q, \beta q/\gamma\right),$$

which follows from formula (III.7) of Appendix III in [5], in order to get

$$D_n^{(a^2)}\big(\mu(2k; a^2)|q\big) = \frac{(q^{-2k+1}; q^2)_k}{(a^2q^2; q^2)_k}{}_2\phi_1\left(\begin{matrix} q^{-2k}, a^2q^{2k+1} \\ q \end{matrix}\,\middle|\, q^2, q^{2n+2}\right).$$

Using the formula (III.8) from [5], one arrives at the expression for $D_n^{(a^2)}(\mu(2k; a^2)|q)$ in terms of the basic hypergeometric function from (18), coinciding with $d_n(\mu(k; q^{-1}a^2); q^{-1}, a^2|q^2)$. The formula (19) is proved in the same way by using relation (17). Proposition is proved. □

For the polynomials $\tilde{D}_n^{(a)}(\mu(m; -a)|q)$ with nonnegative integers m, we have the expressions

$$\tilde{D}_n^{(a)}\big(\mu(2k; -a)|q\big) = d_n\big(\mu(k; -q^{-1}a); q^{-1}, -a|q^2\big)$$
$$= {}_3\phi_1\left(\begin{matrix} q^{-2k}, -aq^{2k+1}q^{-2n} \\ -aq^2 \end{matrix}\,\middle|\, q^2, q^{2n+1}\right), \tag{20}$$

$$\tilde{D}_n^{(a)}\big(\mu(2k+1; -a)|q\big) = q^n d_n\big(\mu(k; -qa); q, -a|q^2\big)$$
$$= q^n{}_3\phi_1\left(\begin{matrix} q^{-2k}, -aq^{2k+3}, q^{-2n} \\ -aq^2 \end{matrix}\,\middle|\, q^2, q^{2n-1}\right). \tag{21}$$

Now the orthogonality relations of Section 1 for the polynomials $D_n^{(a)}(\mu(x; a)|q)$ and $\tilde{D}_n^{(a)}(\mu(x; -a)|q)$ are proved by means of formulas (18)–(21) in the same way as in the case of polynomials $\tilde{C}_n^{(a)}(x; q)$. *The corresponding orthogonality measures are extremal* since they are extremal for the dual little q-Jacobi polynomials from formulas (18)–(21) (see [2]).

3. Relation to Berg–Ismail polynomials

Since the polynomials $D_n^{(a)}(\mu(x; a)|q)$ and $\tilde{D}_n^{(a)}(\mu(x; -a)|q)$ correspond to the indeterminate moment problems, there exist infinitely many orthogonality relations for them. Let us derive some set of these relations for $\tilde{D}_n^{(a)}(\mu(x; -a)|q)$, by using their relation to the polynomials (5.18) of Berg and Ismail [3]. These polynomials are (up to a factor) of the form

$$u_n\big((e^{\xi}-e^{-\xi})/2;t_1,t_2|q\big)={}_3\phi_1\left(\begin{matrix}qe^{\xi}/t_1,-qe^{-\xi}/t_1,q^{-n}\\ -q^2/t_1t_2\end{matrix}\,\middle|\,q,q^nt_1/t_2\right) \tag{22}$$

and orthogonality relations, parametrized by a number d, $q\leqslant d<1$, are given by

$$\begin{aligned}&\sum_{n=-\infty}^{\infty}\frac{(-t_1q^{-n}/d,t_1q^nd,-t_2q^{-n}/d,t_2q^nd;q)_\infty}{(-t_1t_2/q;q)_\infty}\frac{d^{4n}q^{n(2n-1)}(1+d^2q^{2n})}{(-d^2;q)_\infty(-q/d^2;q)_\infty(q;q)_\infty}\\&\qquad\times u_r\big((d^{-1}q^{-n}-dq^n)/2;t_1,t_2\big)u_s\big((d^{-1}q^{-n}-dq^n)/2;t_1,t_2\big)\\&\quad=\frac{(q;q)_r(t_1/t_2)^r}{(-q^2/t_1t_2;q)_rq^r}\delta_{rs}.\end{aligned} \tag{23}$$

The orthogonality measure here is positive for $t_1,t_2\in\mathbb{R}$ and $t_1t_2>0$. It is not known whether these measures are extremal or not.

In order to use this orthogonality relation for the polynomials $\tilde{D}_n^{(a)}(\mu(x;-a)|q)$, let us consider the transformation formula

$$\begin{aligned}&{}_3\phi_2\left(\begin{matrix}q^{-2k},-a^2q^{2k+1},q^{-n}\\ iaq,-iaq\end{matrix}\,\middle|\,q,-q^{n+1}\right)\\&\quad={}_3\phi_1\left(\begin{matrix}q^{-2k},-a^2q^{2k+1},q^{-2n}\\ -a^2q^2\end{matrix}\,\middle|\,q^2,q^{2n+1}\right),\end{aligned} \tag{24}$$

which is obtained by equating two expressions (8) and (20) for $\tilde{D}_n^{(a)}(\mu(2k;-a)|q)$. This formula is true for any nonnegative integer values of k. *The relation* (24) *is still valid if one replaces numerator parameters q^{-2k} and $-a^2q^{2k+1}$ in both sides of it by $c^{-1}q^{-2k}$ and $-ca^2q^{2k+1}$, $c\in\mathbb{C}$, respectively.* Indeed, both sides of (24) contain the expression

$$(q^{-2k},-a^2q^{2k+1};q)_m=\prod_{j=0}^{m-1}\big[1-a^2q^{2j+1}-q^j\mu(2k;-a^2)\big],$$

where, as before, $\mu(2k;-a^2)=q^{-2k}-a^2q^{2k+1}$. The left and right sides in (24) thus represent polynomials in the $\mu(2k;-a^2)$ of degree n. Since they are equal to each other on the infinite set of distinct points $x_k=\mu(2k;-a^2)$, we may analytically continue them to any value of μ. Since

$$(c^{-1}q^{-2k},-ca^2q^{2k+1};q)_m=\prod_{j=0}^{m-1}\big[1-a^2q^{2j+1}-q^j\mu_c(2k;-a^2)\big],$$

where $\mu_c(2k;-a^2)=c^{-1}q^{-2k}-ca^2q^{2k+1}$, the replacements $q^{-2k}\to c^{-1}q^{-2k}$ and $a^2q^{2k+1}\to ca^2q^{2k+1}$ in (24) are allowed.

We are now in a position to compare $\tilde{D}_n^{(a)}(\mu(x;-a)|q)$ with the polynomials (22). The fact is that $\tilde{D}_n^{(a)}(\mu(x;-a)|q)$ at the points $x=x_k^{(d)}:=2k-\ln(\sqrt{aq}/d)/\ln q$ are equal to

$$\tilde{D}_n^{(a)}\big(\mu\big(x_k^{(d)};-a\big)|q\big)={}_3\phi_2\left(\begin{matrix}q^{-2k}d^{-1}\sqrt{aq},-q^{2k}d\sqrt{aq},q^{-n}\\ i\sqrt{a}\,q,-i\sqrt{a}\,q\end{matrix}\,\middle|\,q,-q^{n+1}\right),$$

where $\mu(x_k^{(d)};-a) = \sqrt{aq}(d^{-1}q^{-2k} - dq^{2k})$. From (22) and (24) (with q^{-2k} and $-a^2q^{2k+1}$ replaced by $d^{-1}q^{-2k}$ and $-da^2q^{2k+1}$, respectively) it then follows that

$$\tilde{D}_n^{(a)}\big(\mu\big(x_k^{(d)};-a\big)|q\big) = u_n\big((d^{-1}q^{-2k} - dq^{2k})/2; \sqrt{q^3/a}, \sqrt{q/a}|q^2\big).$$

From the orthogonality relations (23) one obtains infinite number of orthogonality relations for the polynomials $\tilde{D}_n^{(a)}(\mu(x;-a)|q)$, which are parametrized by the same d as in (23). They are of the form

$$\sum_{n=-\infty}^{\infty} \frac{(-t_1q^{-2n}/d, t_1q^{2n}d, -t_2q^{-2n}/d, t_2q^{2n}d; q^2)_\infty}{(-t_1t_2/q^2; q^2)_\infty} \times \frac{d^{4n}q^{2n(2n-1)}(1+d^2q^{4n})}{(-d^2;q^2)_\infty(-q^2/d^2;q^2)_\infty(q^2;q^2)_\infty} \times \tilde{D}_r^{(a)}\big(\mu\big(x_n^{(d)};-a\big)|q\big)\tilde{D}_s^{(a)}\big(\mu\big(x_n^{(d)};-a\big)|q\big) = \frac{(q^2;q^2)_r}{(-q^2a;q^2)_r^2}\delta_{rs},$$

where $t_1 = \sqrt{q^3/a}$ and $t_2 = \sqrt{q/a}$. It is important to know whether the corresponding orthogonality measures here are extremal. The extremality of the measures in (25) for the polynomials $\tilde{D}_n^{(a)}(\mu_c(x;-a)|q)$ depends on the extremality of the orthogonality measures in (23) for the polynomials (22). If some of the measures in (23) are extremal, then the corresponding measures in (25) are also extremal.

References

[1] R. Askey, An integral of Ramanujan and orthogonal polynomials, J. Indian Math. Soc. (N.S.) 51 (1987) 27–36.

[2] N.M. Atakishiyev, A.U. Klimyk, On q-orthogonal polynomials, dual to little and big q-Jacobi polynomials, J. Math. Anal. Appl. 294 (2004) 246–257.

[3] Ch. Berg, M.E.H. Ismail, q-Hermite polynomials and classical orthogonal polynomials, Canad. J. Math. 48 (1996) 43–63.

[4] M.-P. Chen, H.M. Srivastava, Orthogonality relations and generating functions for Jacobi polynomials and related hypergeometric functions, Appl. Math. Comput. 68 (1995) 153–188.

[5] G. Gasper, M. Rahman, Basic Hypergeometric Functions, Cambridge Univ. Press, Cambridge, 1990.

[6] V. Romanovsky, Sur quelques classes nouvelles de polynomes orthogonaux, C. R. Acad. Sci. Paris 188 (1929) 1023–1025.