JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 52, 660–668 (1975)

Some Exponentially Decreasing Error Bounds for a Numerical Inversion of the Laplace Transform

M. Z. NASHED

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332

AND

GRACE WAHBA

Department of Statistics, University of Wisconsin, Madison, Wisconsin 53706

Submitted by G. Dahlquist

Convergence properties of a class of least-squares methods for finding approximate inverses of the Laplace transform are obtained by using reproducing kernel Hilbert space techniques (or, alternatively, related minimization techniques) and some classical interpolation results.

1. INTRODUCTION AND PRELIMINARIES

We obtain error bounds for certain approximations to the inverse Laplace transform. Suppose

$$\int_0^\infty e^{-st} f(t)\, dt = F(s), \tag{1.1}$$

where f is assumed to have a representation of the form (2.1), which follows. We wish to construct an approximation $f_n(t)$ to the inverse transform $f(t)$, using $n+1$ values $F(s_i)$, $i = 0, 1,..., n$ of F. The problem of inversion of the Laplace transform, which is an ill-posed problem, gives rise to many interesting and challenging numerical and analytic investigations. The monographs of Bellman, Kalaba, and Lockett [2] and Krylov and Skoblya [4] are devoted to this important problem, where a number of methods are developed. A synopsis of the difficulties and the rationale of various approaches to the numerical inversion of the Laplace transform are given in Bellman [1, Chap. 19]. In the present note we consider only a very simple method used in [2, Chap. 2] and more recently by Schoenberg [9].

We suppose that $f \in \mathscr{L}_2(\alpha)$, where $\mathscr{L}_2(\alpha)$ is the Hilbert space of real-valued functions on $[0, \infty)$, square integrable with respect to the weight function $w_\alpha(t) = e^{2\alpha t}$ (α is a fixed constant). Denote the inner product and norm in $\mathscr{L}_2(\alpha)$ by

$$(f, g)_\alpha = \int_0^\infty f(t)\, g(t)\, e^{2\alpha t}\, dt, \qquad \| f \|_\alpha = (f, f)_\alpha^{1/2}.$$

Let s_i, $i = 0, 1,..., n$ be $n + 1$ distinct points in $[0, \infty)$. Let the approximate solution f_n to (1.1) be the solution to the minimization problem:

Find $f \in \mathscr{L}_2(\alpha)$ to minimize $\| f \|_\alpha$, subject to

$$\int_0^\infty e^{-s_i t} f(t)\, dt = F(s_i), \qquad i = 0, 1,..., n. \tag{1.2}$$

Let the functions ψ_{s_i} be defined by

$$\psi_{s_i}(t) = \exp[-(s_i + 2\alpha)\, t], \qquad i = 0, 1,..., n.$$

If $s_0 + \alpha > 0$, then $\psi_{s_i} \in \mathscr{L}_2(\alpha)$ and the conditions (1.2) may be rewritten

$$(\psi_{s_i}, f)_\alpha = F(s_i), \qquad i = 0, 1,..., n. \tag{1.3}$$

Thus, the problem of finding the approximate solution f_n is naturally formulated as finding the function of minimum norm in the linear variety (of codimension $n + 1$) defined by the finite number of linear constraints (1.3). It is easy to show (and is well-known, see, e.g., [5, p. 65]) that f_n is unique and is in the span of ψ_{s_i}, $i = 0, 1,..., n$. To write the solution explicitly, let Γ_n be the Gram matrix of ψ_{s_i}, $i = 0, 1,..., n$. The ijth entry of Γ_n is given by

$$(\psi_{s_i}, \psi_{s_j})_\alpha = \int_0^\infty \exp[-(s_i + s_j + 2\alpha)\, t]\, dt = (s_i + s_j + 2\alpha)^{-1}.$$

Thus, Γ_n is a generalization of a section of a Hilbert matrix; hence, Γ_n is nonsingular (see [3, p. 217]). It is easy to show that

$$f_n = (\psi_{s_0}, \psi_{s_1},..., \psi_{s_n})\, \Gamma_n^{-1}(F(s_0), F(s_1),..., F(s_n))'. \tag{1.4}$$

The ijth entry γ^{ij} of Γ_n^{-1} is given by the formula

$$\gamma^{ij} = (s_i + s_j + 2\alpha)\, A_j(-(s_i + \alpha))\, A_j(-(s_j + \alpha)), \qquad i, j = 0, 1,..., n,$$

where

$$A_i(x) = \prod_{k \neq i} \frac{s_k + \alpha - x}{s_k - s_i}$$

(see [3, p. 218]).

Schoenberg [9] discusses the case $\alpha = -\frac{1}{2}$, $s_j = j + 1$, $j = 0, 1,..., n$. He gives the solution to the minimization problem (1.2) in the form $f_n(t) = S_n(e^{-t})$, where $S_n(x) = \sum_{v=0}^{n} c_v P_v(2x - 1)$, $P_v(x)$ being the classical Legendre polynomials, and

$$c_v = (2v + 1) \sum_{i=0}^{v} (-1)^{v+i} \binom{v+i}{v} \binom{v}{i} F(i + 1).$$

2. The Main Result

We now give some $\mathscr{L}_2(\alpha)$-convergence properties of this method and error bounds for $\alpha > 0$, $\alpha(s_{j+1} - s_j)^{-1}$ and $n(s_{j+1} - s_j)$ large.

Theorem. *Let f_n be given by* (1.4), *where $\alpha > 0$, $s_j = (j/n)\, T, j = 0, 1,..., n$, with T a positive number no less than 2α. Suppose $f \in \mathscr{L}_2(\alpha)$ and, furthermore, has a representation of the form*

$$f(t) = e^{-2\alpha t} \int_0^\infty e^{-tr} \rho(r)\, dr, \tag{2.1}$$

where $\int_0^\infty |\rho(r)|\, dr < \infty$. Then

$$\int_0^\infty [f(t) - f_n(t)]^2 e^{2\alpha t}\, dt$$
$$\leqslant (1 + (2)^{1/2}) \left\{ \frac{e^4}{2\alpha\pi^{1/2}} \left(\frac{n\alpha}{T}\right)^{3/2} e^{-2n\alpha/T} \left(1 + O\left(\frac{T}{n\alpha}\right)\right) \left(\int_0^T |\rho(s)|\, ds\right)^2 + \int_T^\infty \int_T^\infty \frac{\rho(s)\,\rho(t)}{s + t + 2\alpha}\, ds\, dt \right\}. \tag{2.2}$$

Proof. Let K be the operator that maps $f \in \mathscr{L}_2(\alpha)$ into its Laplace transform:

$$(Kf)(s) := \int_0^\infty e^{-st} f(t)\, dt = F(s), \qquad s \geqslant 0.$$

Using properties of reproducing kernel Hilbert spaces (RKHS) (for more details see, e.g., [8, 10]), $K(\mathscr{L}_2(\alpha))$ is the RKHS of real-valued functions on $[0, \infty)$ with inner product $\langle\, ,\, \rangle_Q$ and with the reproducing kernel $Q(s, t)$ given by

$$Q(s, t) := (\psi_s, \psi_t)_\alpha = (s + t + 2\alpha)^{-1}, \qquad 0 \leqslant s, t < \infty.$$

The condition (2.1) is equivalent to

$$F(s) = \int_0^\infty Q(s, t)\, \rho(t)\, dt = \int_0^\infty e^{-st} f(t)\, dt. \tag{2.3}$$

Denote by $Q_x(s)$ the real-valued function of s on $[0, \infty)$ defined by $Q_x(s) = Q(x, s)$. Thus, Q_x represents the evaluation functional at x in $\mathscr{H}_Q$. Let

$$F_n(s) = (Q_{s_1}(s), Q_{s_2}(s), \ldots, Q_{s_n}(s))\, \Gamma_n^{-1}(F(s_1), F(s_2), \ldots, F(s_n))'.$$

Since $Q_{s_i} = K\psi_{s_i}$, $F_n = Kf_n$, and, furthermore, F_n is the orthogonal projection in $\mathscr{H}_Q$ of F onto the subspace of $\mathscr{H}_Q$ spanned by the functions Q_{s_i}, $i = 0, 1, \ldots, n$ (Γ_n is the Gram matrix of $Q_{s_0}, \ldots, Q_{s_n}$ in $\mathscr{H}_Q$). By the properties of RKHS and the fact that

$$f \in \mathscr{L}_2(\alpha) \qquad \text{and} \qquad Kf = 0 \Rightarrow f = 0,$$

there is an isometric isomorphism between $\mathscr{L}_2(\alpha)$ and $\mathscr{H}_Q$ whereby

$$F \in \mathscr{H}_Q \sim f \in \mathscr{L}_2(\alpha) \Leftrightarrow F = Kf.$$

Thus,

$$\| F - F_n \|_Q = \| f - f_n \|_\alpha ,$$

where $\| \ \|_Q$ is the norm in $\mathscr{H}_Q$. Thus, the proof will be effected if we show that $\| F - F_n \|_Q^2$ is bounded by the right-hand side of (2.2).

Now, recalling that $\langle Q_s, Q_t \rangle_Q = Q(s, t)$ from elementary properties of RKHS, it is easy to show that

$$\| F - F_n \|_Q^2 = \int_0^\infty \int_0^\infty \rho(s)\, \rho(t)\, [Q(s, t) - Q_n(s, t)]\, ds\, dt, \tag{2.4}$$

where

$$Q_n(s, t) = (Q_{s_0}(s), \ldots, Q_{s_n}(s))\, \Gamma_n^{-1}(Q_{s_0}(t), \ldots, Q_{s_n}(t))',$$

and $Q_n(s, t)$ and $E_n(s, t)$, defined by

$$E_n(s, t) = Q(s, t) - Q_n(s, t),$$

are both positive definite kernels.

The expression (2.4) also can be derived directly without the use of properties of reproducing kernels. If we put

$$x_n(t) = f_n(t)\, e^{\alpha t}, \qquad x(t) = f(t)\, e^{\alpha t}, \qquad (x, y) = \int_0^\infty x(t)\, y(t)\, dt,$$

then a formula for $\| x - x_n \|^2$, equivalent to the expression for $\| F - F_n \|_Q^2$

given in (2.4), can be derived easily by applying standard techniques to the following problem:

$$\text{Minimize } \| x_n \|^2$$

subject to the constraint

$$(x_n , G^*_{s_i}) = (x, G^*_{s_i}),$$

where

$$G^*_{s_i}(t) = G_t(s_i) = e^{-(\alpha+s_i)t}, \qquad i = 0, 1,..., n.$$

Now we consider the problem of estimating the right-hand side of (2.4). Since $E_n(s, t)$ is a positive definite kernel,

$$\int_0^\infty \int_0^\infty \rho(s)\, \rho(t)\, E_n(s, t)\, ds\, dt \geqslant 0$$

for any ρ for which the integral is defined. Therefore, replacing $\rho(s)$ by $x\rho(s)$ for $s \leqslant T$ and by $y\rho(s)$ for $s > T$ gives for all x, y

$$ax^2 + 2bx + cy^2 \geqslant 0,$$

where

$$a := \int_0^T \int_0^T \rho(s)\, \rho(t)\, E_n(s, t)\, ds\, dt,$$

$$b := \int_0^T \int_T^\infty \rho(s)\, \rho(t)\, E_n(s, t)\, ds\, dt,$$

and

$$c := \int_T^\infty \int_T^\infty \rho(s)\, \rho(t)\, E_n(s, t)\, ds\, dt.$$

So the matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

is positive definite; therefore, $a \geqslant 0$, $c \geqslant 0$ and

$$b^2 < ac < \tfrac{1}{2}(a + c)^2.$$

Hence, $2 \mid b \mid < (2)^{1/2} (a + c)$, and we get

$$\begin{aligned} \| F - F_n \|_Q^2 &= \int_0^\infty \int_0^\infty \rho(s)\, \rho(t)\, E_n(s, t)\, ds\, dt \\ &= a + 2b + c < (1 + (2)^{1/2})\,(a + c), \end{aligned}$$

or

$$\|F - F_n\|_Q^2$$
$$\leqslant (1 + (2)^{1/2}) \left\{ \sup_{0 \leqslant s \leqslant T} E_n(s, s) \left[\int_0^T |\rho(s)|\, ds \right]^2 + \int_T^\infty \int_T^\infty \frac{\rho(s)\,\rho(t)}{s + t + 2\alpha}\, ds\, dt \right\},$$

and it remains to find a bound on $\sup_{0 \leqslant s \leqslant T} E_n(s, s)$. This is done as follows. Note that

$$Q(s, t) = \int_0^\infty G(s, u)\, G(t, u)\, du,$$

where

$$G(s, u) = e^{-(\alpha+s)u}, \qquad s, u \geqslant 0,$$

and, furthermore,

$$E_n(s, s) = \inf \left\{ \int_0^\infty \left(G(s, u) - \sum_{i=0}^n c_i G(s_i, u) \right)^2 du \colon c_i \in R,\ i = 0,\dots, n \right\},$$

so that

$$E_n(s, s) \leqslant \int_0^\infty \left(G(s, u) - \sum_{i=0}^n c_i G(s_i, u) \right)^2 du \tag{2.5}$$

for any real $c_0, c_1, \dots, c_n$.

Let s be fixed, with $s_j := jT/n \leqslant s < s_{j+1} =: (j + 1)\, T/n$, and suppose $j \leqslant n - (N - 1)$, where $N - 1$ is the greatest integer in $[0, \alpha n/T]$.

Let $G_u(s)$ be the function of s given by

$$G_u(s) = G(s, u) := \exp[-(\alpha + s)\, u], \qquad s, u \geqslant 0,$$

and let $c_i = c_i(s)$, $i = 0, 1,\dots, n$, be defined by

$$\sum_{i=0}^n c_i G_u(s_i) = \sum_{i=0}^{N-1} \rho_{iN}(s)\, G_u(s_{j+1}),$$

where $\rho_{iN}(s)$ is the polynomial of degree $N - 1$, which takes on the value 1 at $s = s_{j+i}$ and the value 0 at $s = s_{j+k}$, $k = 0, 1,\dots, N - 1$, $k \neq i$. Thus,

$$\sum_{i=0}^{N-1} \rho_{iN}(s)\, G_u(s_{j+i})$$

is the Lagrange polynomial in s interpolating to $G_u(s)$ at the points s_j,

$s_{j+1}, \ldots, s_{j+N-1}$. By the Newton form of the remainder for Lagrange interpolation,

$$G_u(s) - \sum_{i=0}^{N-1} \rho_{iN}(s)\, G_u(s_{j+i}) = \prod_{i=0}^{N-1} (s - s_{j+i})\, G_u[s_j, s_{j+1}, \ldots, s_{j+N-1}, s], \tag{2.6}$$

where $G_u[s_j, s_{j+1}, \ldots, s_{j+N-1}, s]$ is the Nth divided difference of $G_u(x)$ at the points $x = s_j, \ldots, s_{j+N-1}, s$. Thus, there exists some $\theta \in [s_j, s_{j+N-1}]$ such that

$$\begin{aligned} G_u[s_j, s_{j+1}, \ldots, s_{j+N-1}, s] &= \frac{1}{N!} \frac{\partial^N}{\partial x^N} G_u(x) \Big|_{x=\theta} \\ &= \frac{u^N}{N!} e^{-(\theta+\alpha)u}. \end{aligned} \tag{2.7}$$

Substituting (2.7) into (2.6) and then (2.6) into (2.5) gives

$$\begin{aligned} E_n(s, s) &\leqslant \left| \prod_{i=0}^{N-1} (s - s_{j+i}) \right|^2 \int_0^\infty \frac{u^{2N}}{(N!)^2} e^{-2(\theta+\alpha)u}\, du \\ &= \left| \prod_{i=0}^{N-1} (s - s_{j+i}) \right|^2 \frac{(2N)!}{(N!)^2} [2^{2N+1}(\theta + \alpha)^{2N+1}]^{-1} \\ &\leqslant \frac{1}{2\alpha} \frac{(2N)!}{(N!)^2\, 2^{2N}} \prod_{i=1}^{N-1} \left[\frac{s_{j+i} - s_j}{\alpha}\right]^2, \qquad \text{for } s \in [s_j, s_{j+i}). \end{aligned}$$

Now, use $s_{j+i} - s_j = iT/n$, $N - 1 \leqslant \alpha n/T < N$ to obtain

$$\prod_{i=1}^{N-1} \frac{(s_{j+i} - s_j)}{\alpha} \leqslant \prod_{i=1}^{N-1} \frac{i}{N-1}.$$

Furthermore,

$$\begin{aligned} \log \prod_{i=1}^{N-1} \frac{i}{N-1} &= \sum_{i=1}^{N-1} \log \left(\frac{i}{N-1}\right) \\ &\leqslant (N-1) \int_{(N-1)^{-1}}^{1} \log u\, du = -(N-2) + \log(N-1); \end{aligned}$$

hence,

$$\prod_{i=1}^{N-1} \frac{i}{N-1} \leqslant (N-1)\, e^{-(N-2)} \leqslant e^2 \left(\frac{\alpha n}{T}\right) e^{-(\alpha n)/T}.$$

By Stirling's formula,

$$\frac{(2N)!}{(N!)^2\,2^{2N}} = \frac{1}{(\pi N)^{1/2}}\left(1 + O\left(\frac{1}{N}\right)\right)$$

$$\leqslant \frac{1}{(\pi)^{1/2}}\left(\frac{\alpha n}{T}\right)^{-1/2}\left(1 + O\left(\frac{T}{\alpha n}\right)\right).$$

Thus, for $s < s_{n-(N-1)}$,

$$E_n(s, s) \leqslant (2\alpha)^{-1}\, e^4 \pi^{-1/2}\left(\frac{\alpha n}{T}\right)^{3/2} e^{-2\alpha n/T}\left(1 + O\left(\frac{T}{\alpha n}\right)\right).$$

The same bound may be obtained for $s \geqslant s_{n-(N-1)}$, provided $n - (N-1) \geqslant N - 1$, by approximating $G_u(s)$ in (2.7) by the $G_u(s_i)$ with s_i to the left of s. The condition $T \geqslant 2\alpha$ insures that $n - (N-1) \geqslant N - 1$, and the theorem is proved.

3. Extensions

When $\alpha \leqslant 0$, a similar convergence theorem can be proved if $s_j = s_0 + (j/n)\,T$, where $s_0 + \alpha > 0$. It is necessary to assume that $\int_0^{s_0} |\,\rho(s)|\, ds = 0$. Then (2.2) can be shown to hold where in the right-hand side of (2.2) α is replaced by $\alpha + s_0$ and the lower limits on the double integral are $T + s_0$ instead of T. The left-hand side has f replaced by $f^\dagger$, where $f^\dagger$ is that element in $\mathscr{L}_2(\alpha)$ of minimal $\mathscr{L}_2(\alpha)$-norm satisfying

$$\int_0^\infty e^{-st} f(t)\, dt = F(s), \qquad s \geqslant s_0.$$

The modifications in the proof occur by noting the following facts, which can be established easily:

(1) There is an isometric isomorphism between $\mathscr{L}^\#(\alpha)$ and $\mathscr{H}_Q$, where $\mathscr{L}^\#(\alpha)$ is the quotient space $\mathscr{L}(\alpha)/\mathscr{N}(K)$,

$$\mathscr{N}(K) = \left\{ f \in \mathscr{L}_2(\alpha), \int_0^\infty e^{st} f(t)\, dt = 0, s \geqslant s_0 \right\},$$

and $\mathscr{H}_Q$ now has the reproducing kernel $Q(s, t)$, $s, t \geqslant s_0$.

(2) The condition $\int_0^{s_0} |\,\rho(s)|\, ds = 0$ insures that (2.3) holds.

(3) α is replaced by $s_0 + \alpha$ in (2.7) and the subsequent argument, and $N - 1$ is the greatest integer in $(s_0 + \alpha)n/T$.

Finally, we remark that the error bounds and convergence properties of the approximations to the inverse transform rely heavily on the particular kernel associated with the Laplace transform, and they are not a special case of other results on regularization and approximation of ill-posed linear operator Equations [6, 7] using reproducing kernel space methods.

Acknowledgment

The authors are grateful to Professors Germund Dahlquist and Ilkka Karasalo of the Royal Institute of Technology in Stockholm for their kind interest and comments. The final version of this paper was prepared while the authors were on leave in Syria and England, respectively. The authors take this opportunity to express their appreciation for the generosity of their universities and the kind hospitality of their various hosts.

References

1. R. E. Bellman, "An Introduction to Matrix Analysis," 2nd ed., McGraw-Hill, New York, 1970.
2. R. E. Bellman, R. E. Kalaba, and J. A. Lockett, "Numerical Inversion of the Laplace Transform," American Elsevier Publishing Co., Inc., New York, 1966.
3. E. Isaacson and H. B. Keller, "Analysis of Numerical Methods," John Wiley and Sons, New York, 1966.
4. V. I. Krylov and N. S. Skoblya, "Handbook of Numerical Inversion of Laplace Transforms," Nauka i tecknika, Minsk, 1968. (English translation, IPST Press, Jerusalem, 1969.)
5. D. L. Luenberger, "Optimization by Vector Space Methods," John Wiley and Sons, Inc., New York, 1969.
6. M. Z. Nashed and G. Wahba, Regularization and approximation of linear operator equations in reproducing kernel spaces, *Bull. Amer. Math. Soc.* **80** (1974), 1213–1218.
7. M. Z. Nashed and G. Wahba, Generalized inverses in reproducing kernel spaces: An approach to regularization of linear operator equations, *SIAM J. Math. Anal.* **5** (1974), 974–987.
8. M. Z. Nashed and G. Wahba, Convergence rates of approximate least squares solutions of linear integral and operator equations of the first kind, *Math. Comp.* **28** (1974), 69–80.
9. I. J. Schoenberg, Remarks concerning a numerical inversion of the Laplace transform due to Bellman, Kalaba, and Lockett, *J. Math. Anal. Appl.* **43** (1973), 823–828.
10. G. Wahba, Convergence rates for certain approximate solutions to Fredholm integral equations of the first kind, *J. Approximation Theory* **7** (1973), 167–185.