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# Goncharov's relations in Bloch's higher Chow group $CH^3(F,5)$ ☆

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**Abstract**

In this paper we will prove Goncharov's 22-term relations (see [A.B. Goncharov, Geometry of configurations, polylogarithms and motivic cohomology, Adv. Math. 114 (1995) 179–319. [G1]]) in the linearized version of Bloch's higher Chow group $CH^3(F,5)$ using linear fractional cycles of Bloch, Kriz and Totaro under the Beilinson–Soulé vanishing conjecture that $CH^2(F,n)=0$ for $n \geqslant 4$.


*MSC:* primary 14C25; secondary 33B30

## 1. Introduction

Around 1980 Goncharov defined his polylogarithmic (cohomological) motivic complex over an arbitrary field $F$:

$$\Gamma(F,n): \mathcal{G}_n(F) \xrightarrow{\delta_n} \mathcal{G}_{n-1}(F)\otimes F^\times \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_3} \mathcal{G}_2(F)\otimes \bigwedge^{n-2} F^\times \xrightarrow{\delta_2} \bigwedge^{n} F^\times,$$

where $\mathcal{G}_n$, denoted by $\mathcal{B}_n$ by Goncharov, is placed at degree 1. To save space we here only point out that $\mathcal{G}_n(F)$ are quotient groups of $\mathbb{Z}[\mathbb{P}^1_F]$ and refer the interested readers to [G2, p. 49] for the detailed definition of these groups.

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On the other hand, currently there are two versions of higher Chow groups available: a simplicial one and a cubical one, and they are known to be isomorphic (cf. [Le]). We will recall the cubical version in Section 2 and use it throughout this paper.

Define the $\mathbb{Z}$-linear map $\beta_2 : \mathbb{Z}[\mathbb{P}^1_F] \to \bigwedge^2 F^\times$ by $\beta_2(\{x\}) = (1-x) \wedge x$ for $x \neq 0, 1$ and $\beta_2(\{x\}) = 0$ for $x = 0, 1$. Let $B_2(F)$ be the Bloch group defined as the quotient group of $\ker(\beta_2)$ by the subgroup generated by (specializations of) the 5-term relation for the dilogarithm (see [BD]). In [GM] Gangl and Müller-Stach prove that there is a well-defined map to the higher Chow group

$$\bar{\rho}_2 : B_2(F)_\mathbb{Q} \longrightarrow CH^2(F, 3)_\mathbb{Q},$$

where we denote $G_\mathbb{Q} = G \otimes \mathbb{Q}$ for any abelian group $G$. The essential difficulty of the proof lies in showing that $\rho_2 : \mathbb{Z}[\mathbb{P}^1_F] \to CH^2(F, 3)$ sends 5-term relations to 0, where for $a \in F$ we set $\rho_2(\{a\})$ to be the linear fractional cycle $C_a^{(2)}$ of Totaro [To], generalized by Bloch and Kriz [BK]. For $m \geqslant 2$ these cycles are defined as

$$C_a^{(m)} = \left[x_1, \ldots, x_{m-1}, 1 - x_1, 1 - \frac{x_2}{x_1}, \ldots, 1 - \frac{x_{m-1}}{x_{m-2}}, 1 - \frac{a}{x_{m-1}}\right] \in CH^m(F, 2m-1).$$

It is believed that $\bar{\rho}_2$ gives rise to an isomorphism because of the following results of Suslin (cf. [S1,S2]): $B_2(F)_\mathbb{Q} \cong K_3^{\mathrm{ind}}(F)_\mathbb{Q} \cong CH^2(F, 3)_\mathbb{Q}$ where $K_3^{\mathrm{ind}}(F)$ is the indecomposable part of $K_3(F)$.

One expects that the above carries over to the higher Chow groups $CH^m(F, 2m-1)_\mathbb{Q}$ for $m \geqslant 3$. It is suggestive to define $\mathcal{B}_m(F)$ as the subgroup $\ker \delta_m$ of $\mathcal{G}_m$ because it is known that $\mathcal{B}_2(F) \cong B_2(F)$ for number fields $F$, at least modulo torsion. (There are some other ways to define these groups, see [ZG].) One then has:

**Conjecture 1.1.** *For $m \geqslant 3$,*

$$\mathcal{B}_m(F)_\mathbb{Q} \cong CH^m(F, 2m-1)_\mathbb{Q}.$$

Even for $m = 3$ the current state of knowledge requires modifications of the groups on both sides. For example, we do not yet have a very good understanding of the relation group of $\mathcal{G}_3(F)$ although we expect it is equal to $R_3(F)$ which is generated by the following relations:

(1) $\{x\} - \{x^{-1}\}$, $x \in F$;
(2) $\{x\} + \{1-x\} + \{1-x^{-1}\} - \{1\}$, $x \in F$;
(3) Goncharov's 22-term relations: for any $a, b, c \in \mathbb{P}^1_F$

$$R(a,b,c) = \{-abc\} + \bigoplus_{\mathrm{cyc}(a,b,c)} \left( \{ca - a + 1\} + \left\{\frac{ca - a + 1}{ca}\right\} - \left\{\frac{ca - a + 1}{c}\right\} \right.$$
$$\left. + \left\{\frac{a(bc - c + 1)}{-(ca - a + 1)}\right\} + \left\{\frac{bc - c + 1}{b(ca - a + 1)}\right\} + \{c\} - \left\{\frac{bc - c + 1}{bc(ca - a + 1)}\right\} - \{1\} \right).$$

We thus define $B_3(F)$ as $\ker(\beta_3 : \mathbb{Z}[\mathbb{P}^1_F] \to B_2(F) \otimes F^\times)/R_3(F)$. This is well defined by a result of Goncharov [G2]. We then replace the group $CH^3(F, 5)_\mathbb{Q}$ by $\mathcal{C}H^3(F, 5)_\mathbb{Q}$ (see Section 2) which is isomorphic to $CH^3(F,5)$ under some mild conjecture. The ultimate goal of our work is to prove:

**Conjecture 1.2.** *Let $F$ be a field. Then*

$$B_3(F)_{\mathbb{Q}} \cong \mathcal{C}H^3(F,5)_{\mathbb{Q}} \cong CH^3(F,5)_{\mathbb{Q}}.$$

Define the map

$$\rho_3\colon \quad \mathbb{Q}[\mathbb{P}^1_F] \longrightarrow \mathcal{C}H^3(F,5)_{\mathbb{Q}}, \quad \{a\} \longmapsto C_a^{(3)}.$$

Let $T(a)=\{a\}+\{1-a\}+\{1-a^{-1}\}$. By [GM, Theorem 2.9(b)] we know that for any $a,b\neq 0,1$ in $F$ we have $\rho_3(T(a))=\rho_3(T(b))$. We denote this cycle by $\eta$. The main purpose of this paper is to show that if we replace $\{1\}$ by $\eta$ in relation (3) then this relation is sent to 0 under $\rho_3$ when none of the terms is $\{0\}$ or $\{1\}$. Note that Gangl and Müller-Stach have done the same for (1) and they even prove the Kummer–Spence relations which are special cases of (3). Naturally, our work builds on theirs. The proof of relation (2) in $\mathcal{C}H^3(F,5)_{\mathbb{Q}}$ is still open as of now.

To simplify exposition we disregard torsion throughout this paper. In fact, all the results are still valid if we work modulo 4-torsion only.

## 2. The setup

Let $F$ be an arbitrary field. The algebraic $n$-cube

$$\square^n = \left(\mathbb{P}^1_F \setminus \{1\}\right)^n$$

has $2^n$ codimension one faces given by $\{t_i=0\}$ and $\{t_i=\infty\}$ for $1\leqslant i\leqslant n$. We have the boundary map

$$\partial = \sum_{i=1}^{n}(-1)^{i-1}\left(\partial_i^0-\partial_i^\infty\right),$$

where $\partial_i^a$ denotes the restriction map onto face $t_i=a$. Recall that for a field $F$ one denotes by $Z_c^p(F,n)$ (subscript $c$ for "cubical") the quotient of the group of admissible codimension $p$ cycles in $\square^n$ by the subgroup of degenerate cycles as defined in [To, p. 180]. *Admissible* means that the cycles have to intersect all the faces of any dimension properly. Levine [Le] shows that the $n$th homology group of the resulting complex $Z_c^p(F,\bullet)$ is isomorphic to Bloch's higher Chow group $CH^p(F,n)$. This establishes the isomorphism between the cubical and simplicial version of Bloch's higher Chow groups. Furthermore, Bloch [Bl] constructs a rational alternating version $\mathcal{C}^p(F,n)$ of $Z_c^p(F,n)$ (see also [GM, Section 2]) whose homological complex

$$\mathcal{C}^m(F,\bullet)\colon \cdots \longrightarrow \mathcal{C}^m(F,2m) \longrightarrow \mathcal{C}^m(F,2m-1) \longrightarrow \cdots \longrightarrow \mathcal{C}^m(F,m) \longrightarrow 0$$

still computes $CH^p(X,n)_{\mathbb{Q}}$ as proved in [Le]. The properties of the elements in $\mathcal{C}^m(F,n)$ are essentially encoded in the following equation: for any choice of $\delta_1,\dots,\delta_n=\pm 1$ and any permutation $\sigma$ of $\{1,\dots,n\}$

$$\left[f_1^{\delta_1},\dots,f_n^{\delta_n}\right] = \operatorname{sgn}(\sigma)\left(\prod_{i=1}^{n}\delta_i\right)[f_{\sigma(1)},\dots,f_{\sigma(n)}].$$

To simplify computations Gangl and Müller-Stach further modify the complex $C^3(F,\bullet)$ by taking the quotient by an acyclic subcomplex $S^3(F,\bullet)$. (See their paper for the definition. Also note that acyclicity is proved under Beilinson–Soulé conjecture $CH^2(F,n)=0$ for $n\geqslant 4$.) Following them we call cycles in $S^3(F,\bullet)$ *negligible* and denote the quotient complex by $A^3(F,\bullet)$. We further put $\mathcal{C}H^3(F,n)=H_n(A^3(F,\bullet))$ (note the different fonts). Hence

$$\mathcal{C}H^3(F,n)\cong CH^3(F,n)$$

under the conjecture $CH^2(F,n)=0$ for $n\geqslant 4$.

## 3. Some lemmas

We will mostly adopt the notation system in [GM] except that we denote

$$\{a\}_c=\left[x,y,1-x,1-\frac{y}{x},1-\frac{a}{y}\right].$$

This is denoted by $C_a$ in [GM]. The subscript $c$ here is for "cubical."

**Lemma 3.1** *(Gangl and Müller-Stach). Let $f_i$ $(i=1,2,3,5)$ be rational functions in one variable and $f_4(x,y)$ be a product of fractional linear transformations of the form $(a_1x+b_1y+c_1)/(a_2x+b_2y+c_2)$. We assume that all the cycles in the lemma are* admissible *and write*

$$Z(f_1,f_2)=[f_1,f_2,f_3,f_4,f_5]=\big[f_1(x),f_2(y),f_3(x),f_4(x,y),f_5(y)\big]$$

*if no confusion arises.*

(i) *If $f_4(x,y)=g(x,y)h(x,y)$ then*

$$[f_1,f_2,f_3,f_4,f_5]=[f_1,f_2,f_3,g,f_5]+[f_1,f_2,f_3,h,f_5].$$

(ii) *Assume that $f_1=f_2$ and that for each nonconstant solution $y=r(x)$ of $f_4(x,y)=0$ and $1/f_4(x,y)=0$ one has $f_2(r(x))=f_2(x)$.*
   (a) *If $f_3(x)=g(x)h(x)$ then*

$$[f_1,f_2,f_3,f_4,f_5]=[f_1,f_2,g,f_4,f_5]+[f_1,f_2,h,f_4,f_5].$$

   (b) *Similarly, if $f_5(y)=g(y)h(y)$ then*

$$[f_1,f_2,f_3,f_4,f_5]=[f_1,f_2,f_3,f_4,g]+[f_1,f_2,f_3,f_4,h].$$

   (c) *If $f_1=f_2=gh$ and $g(r(x))=g(x)$ or $g(r(x))=h(x)$ then*

$$2Z(f_1,f_2)=Z(g,f_2)+Z(h,f_2)+Z(f_1,g)+Z(f_1,h)\quad \textit{and} \tag{1}$$

$$Z(f_1,f_2)=Z(g,g)+Z(h,h)+Z(h,g)+Z(g,h). \tag{2}$$

**Proof.** Part (i) is contained in Lemma 2.8(b) of [GM]; (ii) can be proved using the same idea as in the proof of [GM, Lemma 2.8(a)(c)]. □

**Lemma 3.2.** *Assume that $f_i$, $i = 1, 2, 3, 5$, are rational functions of one variable and $p_4$ and $q_4$ are rational functions of two variables. Assume that the only nonconstant solution of $p_4(x, y) = 0, \infty$ is $y = x$ and the same for $q_4(x, y)$.*

(i) *If $f_3 = gh$ then*

$$[f_1, f_2, f_3, p_4, f_5] + [f_2, f_1, f_3, q_4, f_5] = [f_1, f_2, g, p_4, f_5] + [f_2, f_1, g, q_4, f_5] + [f_1, f_2, h, p_4, f_5] + [f_2, f_1, h, q_4, f_5]$$

*if all cycles are admissible. A similar result holds if $f_5 = gh$.*

(ii) *If $f_2 = gh$ then*

$$[f_1, f_2, f_3, p_4, f_5] + [f_2, f_1, f_3, q_4, f_5] = [f_1, g, f_3, p_4, f_5] + [g, f_1, f_3, q_4, f_5] + [f_1, h, f_3, p_4, f_5] + [h, f_1, f_3, q_4, f_5]$$

*if all cycles are admissible.*

**Proof.** (i) Write $[F_1, \dots, F_6] = [F_1(x), F_2(y), \dots, F_5(x, y), F_6(y)]$ and let

$$W = \left[f_1, f_2, \frac{z - g(x)h(x)}{z - g(x)}, z, p_4, f_5\right] + \left[f_2, f_1, \frac{z - g(x)h(x)}{z - g(x)}, z, q_4, f_5\right].$$

Taking the boundary we get the desired result because the cycle

$$V = \left[f_1(x), f_2(x), \frac{z - g(x)h(x)}{z - g(x)}, z, f_5(x)\right]$$

cancels with

$$-V = \left[f_2(x), f_1(x), \frac{z - g(x)h(x)}{z - g(x)}, z, f_5(x)\right]$$

by skew-symmetry.

(ii) This is similar to (i) if we set

$$W = \left[f_1(x), \frac{z - g(y)h(y)}{z - g(y)}, z, f_3, p_4, f_5\right] - \left[\frac{z - g(x)h(x)}{z - g(x)}, z, f_1(y), f_3, q_4, f_5\right]. \qquad \square$$

**Corollary 3.3.** *If the conditions in the lemma are all satisfied then for $\alpha \in F - \{0\}$*

$$[f_1, f_2, \alpha f_3, p_4, f_5] + [f_2, f_1, \alpha f_3, q_4, f_5] = [f_1, f_2, f_3, p_4, f_5] + [f_2, f_1, f_3, q_4, f_5].$$

*A similar result holds if the constant $\alpha$ is in front of $f_5$.*

The next computational lemma is easy. Here and in what follows we formally extend the definition of $\{?\}_c$ to include $\{0\}_c = \{\infty\}_c = 0$.

**Lemma 3.4.** *For all $s, t, u, v \in F$ there is an identity of admissible cycles*

$$\left[x, y, \frac{1-sx}{1-tx}, 1-\frac{y}{x}, \frac{u-y}{v-y}\right] = \{us\}_c - \{vs\}_c - \{ut\}_c + \{vt\}_c.$$

*Similarly, there is an identity of admissible cycles*

$$\left[x, y, \frac{s-x}{t-x}, 1-\frac{x}{y}, \frac{1-uy}{1-vy}\right] = \{us\}_c - \{vs\}_c - \{ut\}_c + \{vt\}_c.$$

**Proof.** By Lemma 3.1(ii) we only need to show

$$\left[x, y, 1-sx, 1-\frac{y}{x}, 1-\frac{u}{y}\right] = \{us\}_c \tag{3}$$

which follows easily from a substitution $(x, y) \mapsto (x/s, y/s)$ if $s \neq 0$. If $s = 0$ then (3) is trivial. The second equation follows from the obvious substitution $(x, y) \mapsto (y, x)$. $\square$

## 4. Goncharov's relations

Let $\mathcal{T}(a) = \{a\}_c + \{1-a\}_c + \{1-a^{-1}\}_c$. By [GM, Theorem 2.9(b)] we know that for any $a, b \neq 0, 1$ in $F$ we have $\mathcal{T}(a) = \mathcal{T}(b)$. We denote this cycle by $\eta$.

**Theorem 4.1.** *Goncharov's 22 term relations hold in $\mathcal{C}H^3(F, 5)$: for any $a, b, c \in \mathbb{P}^1_F$*

$$\begin{aligned} R(a,b,c) = \{-abc\} + \bigoplus_{\mathrm{cyc}(a,b,c)} \Bigg( &\{ca-a+1\} + \left\{\frac{ca-a+1}{ca}\right\} - \left\{\frac{ca-a+1}{c}\right\} \\ &+ \left\{\frac{a(bc-c+1)}{-(ca-a+1)}\right\} + \left\{\frac{bc-c+1}{b(ca-a+1)}\right\} + \{c\} - \left\{\frac{bc-c+1}{bc(ca-a+1)}\right\} - \eta\Bigg) = 0, \end{aligned} \tag{4}$$

*where $\mathrm{cyc}(a,b,c)$ means cyclic permutations of $a, b$ and $c$, provided that none of the terms in $R(a,b,c)$ is $\{0\}$ or $\{1\}$ except for $\eta$ (nondegeneracy condition). Here we drop the subscript $c$ for the cycle notation $\{?\}_c$.*

**Proof.** To make the proof explicit we will carry it out in a series of steps. Throughout the proof we will use $\{1/t\} = \{t\}$ repeatedly without stating it explicitly. As Gangl pointed out to the author the major difficulty is to guarantee that all the cycles we use lie in the "admissible world." Due to its length and pure computational feature we put the proof of admissibility of all the cycles appearing in this paper in the online supplement [Zh] except for one cycle in step (2) where we spell out all the details to provide the readers the procedure how we do the checking in general.

*Step* 1. Construction of $\{k(c)\}$. Let $f(x)=x$, $A(x)=(ax-a+1)/a$ and $B(x)=bx-x+1$. Let $k(x)=B(x)/abxA(x)$ and $l(y)=1-(k(c)/k(y))$. Then taking $\mu=-(ab-b+1)/a$ we can write

$$\{k(c)\}=\left[x,y,1-x,1-\frac{y}{x},1-\frac{k(c)}{y}\right]=\left[\frac{abx}{\mu},\frac{aby}{\mu},1-x,1-\frac{y}{x},1-\frac{k(c)}{y}\right] \tag{5}$$

by Lemma 3.1(ii) because all of the following cycles are admissible and negligible

$$\left[\frac{ab}{\mu},y,f_3,f_4,f_5\right],\quad \left[x,\frac{ab}{\mu},f_3,f_4,f_5\right],\quad \left[\frac{ab}{\mu},\frac{ab}{\mu},f_3,f_4,f_5\right],$$

where $(f_3,f_4,f_5)=(1-x,1-y/x,1-k(c)/y)$. Here for the last cycle we need to use the fact that

$$1-k(c)=\frac{(c-1)(1+abc)}{abcA(c)}\neq 0. \tag{6}$$

Next by using the transformation $(x,y)\mapsto(k(x),k(y))$ we get

$$4\{k(c)\}=\left[\frac{B(x)}{\mu xA(x)},\frac{B(y)}{\mu yA(y)},1-k(x),1-\frac{k(y)}{k(x)},l(y)\right]=Z\left(\frac{B}{\mu fA},\frac{B}{\mu fA}\right).$$

Here for any two rational functions $f_1$ and $f_2$ of one variable we set

$$Z(f_1,f_2)=\left[f_1(x),f_2(y),1-k(x),1-\frac{k(y)}{k(x)},l(y)\right].$$

*Step* 2. The key reparametrization and a simple expression of $\{k(c)\}$. We first observe that under the involution $x\overset{\rho_x}{\longleftrightarrow}-A(x)/B(x)$ we have

$$k(x)\overset{\rho_x}{\longleftrightarrow}k(x),\qquad \frac{x-1}{x}\overset{\rho_x}{\longleftrightarrow}\frac{abx+1}{aA(x)},\qquad 1-\frac{\mu x}{A(y)B(x)}\overset{\rho_x}{\longleftrightarrow}\frac{y-x}{A(y)},$$
$$B(x)\overset{\rho_x}{\longleftrightarrow}\frac{-\mu}{B(x)},\qquad \frac{B(x)}{x}\overset{\rho_x}{\longleftrightarrow}\frac{\mu}{A(x)},\qquad \frac{A(x)}{x}\overset{\rho_x}{\longleftrightarrow}\frac{-\mu x}{A(x)}.$$

Next if we apply both $\rho_x$ and $\rho_y$ (denoted by $\rho_{x,y}$) then we get

$$1-\frac{x}{y}\overset{\rho_{x,y}}{\longleftrightarrow}\frac{\mu(x-y)}{A(y)B(x)},\qquad \frac{y-x}{yB(x)}\overset{\rho_{x,y}}{\longleftrightarrow}\frac{y-x}{A(y)},$$
$$\frac{A(y)}{y}\left(1-\frac{\mu x}{A(y)B(x)}\right)\overset{\rho_{x,y}}{\longleftrightarrow}B(x)\left(1-\frac{\mu x}{A(y)B(x)}\right). \tag{7}$$

By Lemma 3.1(ii)

$$\begin{aligned}4\{k(c)\}&=Z\left(\frac{\mu fA}{B},\frac{\mu fA}{B}\right)=Z(A,A)+Z\left(\frac{\mu f}{B},A\right)+Z\left(A,\frac{\mu f}{B}\right)+Z\left(\frac{\mu f}{B},\frac{\mu f}{B}\right)\\&=Z(A,A)+\rho_xZ(A,A)+\rho_yZ(A,A)+\rho_{x,y}Z(A,A)=4Z(A,A).\end{aligned} \tag{8}$$

We end this step by showing that $Z_A = Z(A, A)$ is admissible. Note that

$$1 - k(x) = \frac{(x-1)(1+abx)}{abxA(x)}, \tag{9}$$

$$1 - \frac{k(y)}{k(x)} = \frac{(y-x)(yB(x)+A(x))}{yA(y)B(x)} = \frac{(y-x)(xB(y)+A(y))}{yA(y)B(x)}. \tag{10}$$

We have

$$\begin{aligned}
&\partial_1^0(Z_A) \subset \{t_4 = 1\}, \qquad \partial_1^\infty(Z_A) \subset \{t_3 = 1\},\\
&\partial_2^0(Z_A) \subset \{t_5 = 1\}, \qquad \partial_2^\infty(Z_A) \subset \{t_4 = 1\}, \qquad \partial_3^\infty(Z_A) \subset \{t_4 = 1\},\\
&\partial_4^\infty(Z_A) \subset \{t_3 = 1\} \cup \{t_5 = 1\}, \qquad \partial_5^\infty(Z_A) \subset \{t_4 = 1\},\\
&\partial_3^0(Z_A) = \left[\frac{1}{a}, A(y), 1-k(y), l(y)\right] + \left[A\left(\frac{-1}{ab}\right), A(y), 1-k(y), l(y)\right],\\
&\partial_4^0(Z_A) = \bigl[A(y), A(y), 1-k(y), l(y)\bigr] + \left[\frac{\mu y}{B(y)}, A(y), 1-k(y), l(y)\right],\\
&\partial_5^0(Z_A) = \bigl[A(x), A(c), 1-k(x), l(x)\bigr] + \bigl[A(x), A(y_2), 1-k(x), l(x)\bigr],
\end{aligned}$$

where the last equation comes from the two solutions of $l(y) = 0$:

$$y_1 = c \quad \text{and} \quad y_2 = -\frac{ac-a+1}{a(bc-c+1)} = -\frac{A(c)}{B(c)} = \rho_c(c).$$

By the nondegeneracy assumption and the conditions

$$\begin{aligned}
A(y_2) &= \rho_c\bigl(A(c)\bigr) = c\mu/B(c) \neq 0, \infty,\\
B(y_2) &= \rho_c\bigl(B(c)\bigr) = -\mu/B(c) \neq 0, \infty,
\end{aligned} \tag{11}$$

it suffices to show that the following cycles are admissible:

$$L := \bigl[A(y), 1-k(y), l(y)\bigr], \qquad L' := \bigl[A(y), A(y), 1-k(y), l(y)\bigr],$$
$$L'' := \left[\frac{\mu y}{B(y)}, A(y), 1-k(y), l(y)\right].$$

- $L$ is admissible. Because $l(y) = 1 - yB(c)A(y)/cA(c)B(y)$ we have

$$\partial_1^0(L) \subset \{t_3 = 1\}, \quad \partial_1^\infty(L) \subset \{t_2 = 1\}, \quad \partial_2^\infty(L) \subset \{t_3 = 1\}, \quad \partial_3^\infty(L) \subset \{t_2 = 1\}.$$

Moreover, by the nondegeneracy assumption we see that by (6) and (9)

$$A(1) = \frac{1}{a} \neq 0, \qquad k(1) = 1, \qquad l(1) = 1 - k(c) \neq 0,$$

$$A\left(\frac{-1}{ab}\right)=\frac{\mu}{b}\neq 0,\qquad k(-1/ab)=1,\qquad l\left(\frac{-1}{ab}\right)=1-k(c)\neq 0,$$
$$aby_2+1=\frac{(1-c)(ab-b+1)}{bc-c+1}\neq 0.$$

Thus both $\partial_2^0(L)=[A(1),l(1)]+[A(-1/ab),l(-1/ab)]$ and $\partial_3^0(L)=[A(c),1-k(c)]+[A(y_2),1-k(c)]$ are clearly admissible by the nondegeneracy assumption.

- $L'$ is admissible. This follows from the above proof for $L$.
- $L''$ is admissible. This also follows from the proof for $L$ because $\mu y/B(y)\neq 0,\infty$ when $y=1,-1/ab,c,y_2$ by (11).

*Step* 3. Some admissible cycles for the decomposition of $\{k(c)\}$. In order to decompose $Z(A,A)$ we define the following admissible cycles

$$Z_1(A,A)=\left[\frac{(b-1)A(x)}{\mu},\frac{(b-1)A(y)}{\mu},\frac{x-1}{x},\frac{y-x}{A(y)},l(y)\right],$$
$$Z_2(A,A)=\left[\frac{(b-1)A(x)}{\mu},\frac{(b-1)A(y)}{\mu},\frac{x-1}{x},\left(\frac{A(y)}{y}\right)\left(1-\frac{\mu x}{A(y)B(x)}\right),l(y)\right],$$
$$Z_3(A,A)=\left[\frac{(b-1)A(x)}{\mu},\frac{(b-1)A(y)}{\mu},\frac{abx+1}{abA(x)},\frac{y-x}{A(y)},l(y)\right],$$
$$Z_4(A,A)=\left[\frac{(b-1)A(x)}{\mu},\frac{(b-1)A(y)}{\mu},\frac{abx+1}{abA(x)},\left(\frac{A(y)}{y}\right)\left(1-\frac{\mu x}{A(y)B(x)}\right),l(y)\right].$$

We now use Lemma 3.1(ii(c)) to remove the coefficients in front of $A(x)$ and $A(y)$ in $Z_1(A,A)$ and $Z_3(A,A)$, Lemma 3.1(i) to remove the factor $A(y)/y$ from the fourth coordinate of $Z_2(A,A)$, and Lemma 3.1(ii(a)) to remove the coefficient $1/b$ in front of the third coordinate of $Z_4(A,A)$:

$$Z_1(A,A)=\left[A(x),A(y),\frac{x-1}{x},\frac{y-x}{A(y)},l(y)\right],$$
$$Z_2(A,A)=\left[\frac{(b-1)A(x)}{\mu},\frac{(b-1)A(y)}{\mu},\frac{x-1}{x},1-\frac{\mu x}{A(y)B(x)},l(y)\right],$$
$$Z_3(A,A)=\left[A(x),A(y),\frac{abx+1}{abA(x)},\frac{y-x}{A(y)},l(y)\right],$$
$$Z_4(A,A)=\left[\frac{(b-1)A(x)}{\mu},\frac{(b-1)A(y)}{\mu},\frac{abx+1}{aA(x)},\left(\frac{A(y)}{y}\right)\left(1-\frac{\mu x}{A(y)B(x)}\right),l(y)\right].$$

It is not too hard to verify that all the cycles appearing in the above are admissible.

Now we can break up the fourth coordinate of $Z(A,A)$ according to Lemma 3.1(i)

$$Z(A,A)=Z'(A,A)+Z''(A,A)$$

$$
\begin{aligned}
&= \left[\frac{(b-1)A(x)}{\mu}, \frac{(b-1)A(y)}{\mu}, 1-k(x), \frac{y-x}{A(y)}, l(y)\right] \\
&\quad + \left[\frac{(b-1)A(x)}{\mu}, \frac{(b-1)A(y)}{\mu}, 1-k(x), \frac{A(y)}{y}\left(1-\frac{\mu x}{A(y)B(x)}\right), l(y)\right].
\end{aligned}
$$

Also by Lemma 3.1(ii) we find that

$$Z'(A,A) = Z_1(A,A) + Z_3(A,A).$$

However, the conditions in Lemma 3.1(i) are not all satisfied by $Z''(A,A)$. To decompose the third coordinate of $Z''(A,A)$ we combine

$$
\begin{aligned}
Z''(A,A) = \rho_y Z''(A,A) &= \left[\frac{(b-1)A(x)}{\mu}, \frac{(b-1)y}{B(y)}, 1-k(x), \frac{\mu(x-y)}{A(y)B(x)}, l(y)\right] \quad \text{and} \\
Z''(A,A) = \rho_x Z''(A,A) &= \left[\frac{(b-1)x}{B(x)}, \frac{(b-1)A(y)}{\mu}, 1-k(x), 1-\frac{x}{y}, l(y)\right]
\end{aligned}
$$

and use Lemma 3.2(i) to get

$$2Z''(A,A) = (\rho_x+\rho_y)\big(Z_2(A,A)+Z_4(A,A)\big) = 2Z_2(A,A) + 2Z_4(A,A).$$

Here the properties of the substitutions $\rho_x$ and $\rho_y$ play important roles. Another important thing is that we can write $Z_4(A,A)$ in two ways such that the third coordinate of one of these (i.e. $(abx+1)/aA(x)$) is mapped to the third coordinate of $Z_2(A,A)$ (i.e. $(x-1)/x$) under $\rho_x$ and vice versa. Hence

$$Z(A,A) = \sum_{i=1}^{4} Z_i(A,A).$$

On the other hand, we can easily see that

$$\rho_{x,y} Z_1(A,A) = Z_3\left(\frac{f}{B}, \frac{f}{B}\right) := \left[\frac{(b-1)x}{B(x)}, \frac{(1-b)y}{B(y)}, \frac{abx+1}{aA(x)}, \frac{y-x}{yB(x)}, l(y)\right]. \tag{12}$$

Therefore we have the following simple expression of $\{k(c)\}$ by (8):

$$\big\{k(c)\big\} = \sum_{i=1}^{4} Z_i(A,A) = Z_3(A,A) + Z_3\left(\frac{f}{B}, \frac{f}{B}\right) + \rho_x Z_2(A,A) + \rho_y Z_4(A,A), \tag{13}$$

where

$$\rho_x Z_2(A,A) = \left[\frac{(b-1)x}{B(x)}, \frac{(b-1)A(y)}{\mu}, \frac{abx+1}{aA(x)}, \frac{y-x}{A(y)}, l(y)\right], \tag{14}$$

$$\rho_y Z_4(A,A) = \left[\frac{(b-1)A(x)}{\mu}, \frac{(b-1)y}{B(y)}, \frac{abx+1}{aA(x)}, \frac{\mu(x-y)}{A(y)B(x)}, l(y)\right]. \tag{15}$$

*Step* 4. Decomposition of $\rho_x Z_2(A,A)+\rho_y Z_4(A,A)$ into $X_1-X_2$. Let $f_1(x)=(b-1)x/B(x)$, $f_3(x)=(abx+1)/aA(x)$, $f_2=gh$ where $g(x)=A(x)/(-\mu x)$ and $h(x)=(1-b)x$. Then we can apply Lemma 3.2(ii) and easily get

$$\rho_x Z_2(A,A)+\rho_x Z_4(A,A)=X_1-X_2, \tag{16}$$

where

$$\begin{aligned} X_1 &= \left[\frac{(b-1)x}{B(x)}, \frac{A(y)}{-\mu y}, \frac{abx+1}{aA(x)}, \frac{y-x}{A(y)}, l(y)\right] \\ &\quad + \left[\frac{A(y)}{-\mu x}, \frac{(b-1)y}{B(y)}, \frac{abx+1}{aA(x)}, \frac{\mu(x-y)}{A(y)B(x)}, l(y)\right], \\ X_2 &= \left[\frac{B(x)}{(b-1)x}, (1-b)y, \frac{abx+1}{aA(x)}, \frac{y-x}{A(y)}, l(y)\right] \\ &\quad + \left[(1-b)x, \frac{B(y)}{(b-1)y}, \frac{abx+1}{aA(x)}, \frac{\mu(x-y)}{A(y)B(x)}, l(y)\right]. \end{aligned}$$

We now apply Lemma 3.2(ii) to $X_2$ with $g=-1$ and $h(x)=(b-1)x$ to get

$$\begin{aligned} X_2 &= \left[\frac{B(x)}{(b-1)x}, (b-1)y, \frac{abx+1}{aA(x)}, \frac{y-x}{A(y)}, l(y)\right] \\ &\quad + \left[(b-1)x, \frac{B(y)}{(b-1)y}, \frac{abx+1}{aA(x)}, \frac{\mu(x-y)}{A(y)B(x)}, l(y)\right]. \end{aligned}$$

*Step* 5. Computation of $X_1$. Set

$$\tilde{Z}(f_1,f_2)=\left[f_1, f_2, \frac{abx+1}{aA(x)}, \frac{\mu(x-y)}{A(y)B(x)}, l(y)\right].$$

Throwing away the appropriate admissible and negligible cycle we have

$$X_1=\tilde{Z}\left(\frac{(b-1)f}{B}, \frac{A}{-\mu f}\right)+\tilde{Z}\left(\frac{A}{-\mu f}, \frac{(b-1)f}{B}\right).$$

Then further disregarding some admissible and negligible cycles we get

$$Z_3(F,F)=\tilde{Z}(F,F) \quad \text{for } F=\frac{A}{f}, \frac{f}{B}, \frac{A}{B},$$

where $Z_3(f/B, f/B)$ is defined by (12),

$$Z_3\left(\frac{A}{f}, \frac{A}{f}\right) := \left[\frac{A(x)}{-\mu x}, \frac{A(y)}{-\mu y}, \frac{abx+1}{aA(x)}, \frac{y-x}{A(y)}, l(y)\right] \quad \text{and} \tag{17}$$

$$Z_3\left(\frac{A}{B}, \frac{A}{B}\right) := \left[\frac{A(x)}{B(x)}, \frac{A(y)}{B(y)}, \frac{abx+1}{aA(x)}, \frac{-\mu y}{A(y)B(x)}\left(1-\frac{x}{y}\right), l(y)\right]. \tag{18}$$

Here we removed the coefficient $b-1$ in front of the $A/B$ and $-1/\mu$ in front of $A/f$ in $Z_3$ by using Lemma 3.1(ii). Then we can take $f_1=f_2=(b-1)A/B$, $g=A/f$ and $h=(b-1)f/B$ in Lemma 3.1(ii)(c) and get

$$\begin{aligned}X_1&=\tilde{Z}\left(\frac{A}{B},\frac{A}{B}\right)-\tilde{Z}\left(\frac{A}{f},\frac{A}{f}\right)-\tilde{Z}\left(\frac{f}{B},\frac{f}{B}\right)\\&=Z_3\left(\frac{A}{B},\frac{A}{B}\right)-Z_3\left(\frac{A}{f},\frac{A}{f}\right)-Z_3\left(\frac{f}{B},\frac{f}{B}\right).\end{aligned}\tag{19}$$

*Step* 6. Decomposition of $X_2$ into $Y_1+Y_2+Y_3+Y_4$. Put

$$v(x)=\frac{abx+1}{aA(x)},\qquad l_1(y)=1-\frac{y}{c},\qquad l_2(y)=\frac{y_2-y}{y_2B(y)},$$

which satisfies

$$l_1(y)l_2(y)=l(y)=1-\frac{k(c)}{k(y)},\quad l_1(0)=l_2(0)=1.$$

Then it follows from Lemma 3.2(i) that

$$X_2=Y_1+Y_2+Y_3+Y_4,\tag{20}$$

where all of the cycles

$$\begin{aligned}Y_1&=\left[\frac{B(x)}{(b-1)x},(b-1)y,\frac{abx+1}{aA(x)},\frac{y-x}{A(y)},l_1(y)\right],\\Y_2&=\left[(b-1)x,\frac{B(y)}{(b-1)y},\frac{abx+1}{aA(x)},\frac{\mu(x-y)}{A(y)B(x)},l_1(y)\right],\\Y_3&=\left[\frac{B(x)}{(b-1)x},(b-1)y,\frac{abx+1}{aA(x)},\frac{y-x}{A(y)},l_2(y)\right],\\Y_4&=\left[(b-1)x,\frac{B(y)}{(b-1)y},\frac{abx+1}{aA(x)},\frac{\mu(x-y)}{A(y)B(x)},l_2(y)\right]\end{aligned}$$

are admissible. This breakup is the key step in the whole paper.

*Step* 7. Computation of $Y_1+Y_2$. To ease the reading of the proof in this step we first summarize our approach here. We would very much like to be in a position to use Lemma 3.1(ii) but unfortunately the terms $f_3$, $f_4$ and $f_5$ cannot be fixed for all the terms simultaneously because we need to stay inside the "admissible world." Nevertheless, luckily enough for us, most of the cycles we are going to use have more than one "realization" so that we can apply Lemmas 3.1 and 3.2 to obtain the desired results. Corollary 3.3 will be crucial to us.

We begin by setting

$$\alpha=\frac{bc-c}{bc-c+1},\qquad \delta=\frac{1}{b},\quad \text{and}$$

$$v(x)=\frac{abx+1}{aA(x)},\qquad g(x)=\frac{B(x)}{(b-1)x},\qquad h(x)=(b-1)x,$$
$$p_4(x,y)=\frac{\mu(x-y)}{A(y)B(x)},\qquad q_4(x,y)=\frac{y-x}{A(y)},\qquad s_4(x,y)=\frac{(b-1)(y-x)}{B(y)},$$
$$r_4(x,y)=\frac{(b-1)(y-x)}{xB(y)},\qquad w_4(x,y)=\frac{y-x}{B(x)(y-1)},$$

such that $\alpha l_1(1/(1-b))=\delta v(\infty)=1$. By Lemma 3.1(ii)(1) we get

$$\begin{aligned}2[gh,gh,\delta v,q_4,\alpha l_1]&=[gh,gh,\delta v,q_4,\alpha l_1]+[gh,gh,\delta v,s_4,\alpha l_1]\\&=[g,gh,\delta v,q_4,\alpha l_1]+[h,gh,\delta v,q_4,\alpha l_1]\\&\quad+[gh,g,\delta v,s_4,\alpha l_1]+[gh,h,\delta v,s_4,\alpha l_1]\end{aligned}$$

are all admissible. Then repeatedly applying Lemmas 3.1 and 3.2 we have

$$\begin{aligned}&[g,gh,\delta v,q_4,\alpha l_1]+[gh,g,\delta v,s_4,\alpha l_1]\\&\quad=[g,gh,\delta v,q_4,\alpha l_1]+[gh,g,\delta v,r_4,\alpha l_1]\\&\quad=[g,gh,v,q_4,\alpha l_1]+[gh,g,v,r_4,\alpha l_1]\\&\quad=[g,gh,v,q_4,\alpha l_1]+[gh,g,v,w_4,\alpha l_1]\\&\quad=[g,gh,v,q_4,l_1]+[gh,g,v,w_4,l_1]\\&\quad=[g,gh,v,q_4,l_1]+[gh,g,v,p_4,l_1]\\&\quad=[g,h,v,q_4,l_1]+[h,g,v,p_4,l_1]+[g,g,v,q_4,l_1]+[g,g,v,p_4,l_1]\\&\quad=[g,h,v,q_4,l_1]+[h,g,v,p_4,l_1]+2[g,g,v,p_4,l_1].\end{aligned}$$

Again by applying Lemmas 3.1 and 3.2 we get

$$\begin{aligned}&[h,gh,\delta v,q_4,\alpha l_1]+[gh,h,\delta v,s_4,\alpha l_1]\\&\quad=[h,gh,\delta v,q_4,l_1]+[gh,h,\delta v,s_4,l_1]\\&\quad=[h,gh,\delta v,q_4,l_1]+[gh,h,\delta v,q_4,l_1]\\&\quad=[h,g,\delta v,q_4,l_1]+[g,h,\delta v,q_4,l_1]+2[h,h,\delta v,q_4,l_1]\\&\quad=[h,g,v,q_4,l_1]+[g,h,v,q_4,l_1]+2[h,h,\delta v,q_4,l_1]\\&\quad=[h,g,v,p_4,l_1]+[g,h,v,q_4,l_1]+2[h,h,\delta v,q_4,l_1].\end{aligned}$$

Therefore

$$\begin{aligned}Y_1+Y_2&=[h,g,v,p_4,l_1]+[g,h,v,q_4,l_1]\\&=[gh,gh,\delta v,q_4,\alpha l_1]-[g,g,v,p_4,l_1]-[h,h,\delta v,q_4,l_1].\end{aligned}\tag{21}$$

*Step* 8. Computation of $Y_3 + Y_4$. We could use a similar process as in Step (7) to do the computation. But we can get around this by the following argument. Define the substitutions

$$\sigma_{x,y} : (x, y) \longmapsto \left(\frac{-x}{B(x)}, \frac{-y}{B(y)}\right), \qquad \tau_{a,c} : (a, c) \longmapsto \left(\frac{ab-b+1}{b(a-1)}, \frac{ca-a+1}{ab-b+1}\right).$$

Let

$$Y_3' = \left[\frac{(1-b)x}{B(x)}, (1-b)y, \frac{abx+1}{abA(x)}, \frac{y-x}{A(y)}, l_2(y)\right],$$
$$Y_4' = \left[(1-b)x, \frac{(1-b)y}{B(y)}, \frac{abx+1}{abA(x)}, \frac{\mu(x-y)}{A(y)B(x)}, l_2(y)\right].$$

Then an easy computation shows that

$$-Y_4' = \tau_{a,c}\sigma_{x,y}Y_1 = \left[\frac{1}{(1-b)x}, \frac{(1-b)y}{B(y)}, \frac{abA(x)}{abx+1}, \frac{(ab-b+1)(y-x)}{(aby+1)B(x)}, l_2(y)\right],$$
$$-Y_3' = \tau_{a,c}\sigma_{x,y}Y_2 = \left[\frac{(1-b)x}{B(x)}, \frac{1}{(1-b)y}, \frac{abA(x)}{abx+1}, \frac{ab(y-x)}{aby+1}, l_2(y)\right].$$

Hence by first splitting off the $-1$ in front of the first two coordinates of $Y_3'$ and $Y_4'$ respectively, then removing some other admissible and negligible cycles we find

$$Y_3 + Y_4 = Y_3' + Y_4' = -\tau_{a,c}(Y_1 + Y_2). \tag{22}$$

Notice that $\tau_{a,c}$ is only used here to pass from $l_1(y)$ to $l_2(y)$.

*Step* 9. Final decomposition of $\{k(c)\}$ into $T_i(F)$'s. Combining all the results from (13) to (22) we see that

$$\begin{aligned}\{k(c)\} &= Z_3(A, A) + Z_3\left(\frac{A}{B}, \frac{A}{B}\right) - Z_3\left(\frac{A}{f}, \frac{A}{f}\right)\\ &\quad + (1-\tau_{a,c})\big([g, g, v, p_4, l_1] + [h, h, \delta v, q_4, l_1] - [gh, gh, \delta v, q_4, \alpha l_1]\big).\end{aligned} \tag{23}$$

We will first simplify the terms in the above expression. Set

$$\varepsilon_1(f) = \varepsilon_2(f) = 1, \qquad \varepsilon_1(A) = \frac{ca}{ca-a+1}, \qquad \varepsilon_2(A) = \frac{ca-a+1}{ca}.$$

Define the admissible cycles

$$T_i(F) = \begin{cases} [\frac{A(x)}{x}, \frac{A(y)}{y}, \frac{(1-a)(x-1)}{x}, 1-\frac{x}{y}, l_i(y)] & \text{if } F = \frac{A}{f},\ i = 1, 2,\\ [F(x), F(y), \frac{x-1}{x}, \frac{y-x}{F(y)}, \varepsilon_i(F) l_i(y)] & \text{if } F = f, A,\ i = 1, 2,\\ [F(x), F(y), \frac{abx+1}{abA(x)}, \frac{y-x}{F(y)}, \varepsilon_{i-2}(F) l_{i-2}(y)] & \begin{cases}\text{if } F = A,\ i = 3, 4,\\ \text{if } F = f,\ i = 3,\end{cases}\\ [B(x), B(y), \frac{abx+1}{abA(x)}, \frac{y-x}{A(y)}, \alpha l_1(y)] & \text{if } F = B, i = 1.\end{cases}$$

**Claim.** *We have*

$$\bigl\{k(c)\bigr\} = \sum_{i=1}^{3} T_i(f) + \sum_{i=2}^{4} T_i(A) - \sum_{i=1,2} T_i\left(\frac{A}{f}\right) - T_1(B) - \tau_{a,c}\bigl(T_3(f) + T_2(A) - T_1(B)\bigr).$$

**Proof.** Using the involution $\rho_{x,y}$ we quickly find

$$Z_3(A, A) = T_3(A) + T_4(A), \qquad Z_3\left(\frac{A}{B}, \frac{A}{B}\right) = T_1(f) + T_2(f).$$

For $Z_3(A/f, A/f)$ defined by (17) we can remove the coefficient $-1/\mu$ from the first two coordinates and then apply $\rho_{x,y}$ to get

$$\begin{aligned} Z_3\left(\frac{A}{f}, \frac{A}{f}\right) &= \left[\frac{A(x)}{x}, \frac{A(y)}{y}, \frac{x-1}{x}, \frac{y-x}{yB(x)}, l(y)\right] = \left[\frac{A(x)}{x}, \frac{A(y)}{y}, \frac{x-1}{x}, 1-\frac{x}{y}, l(y)\right] \\ &= \left[\frac{A(x)}{x}, \frac{A(y)}{y}, \frac{(1-a)(x-1)}{x}, \frac{y-x}{yB(x)}, l(y)\right] = T_1\left(\frac{A}{f}\right) + T_2\left(\frac{A}{f}\right). \end{aligned}$$

Here we have sequentially added two admissible and negligible cycles.

For the first term in (23), from (7) and $l_1(y) \overset{\rho_{x,y}}{\longleftrightarrow} \varepsilon_2(A)l_2(y)$ we find

$$\rho_{x,y}[g, g, v, p_4, l_1] = \left[A(x), A(y), \frac{x-1}{x}, 1-\frac{x}{y}, \varepsilon_2(A)l_2(y)\right] = T_2(A).$$

For the second term in (23), we can first remove the coefficient $b-1$ in $h$ and then delete one admissible and negligible cycle to get $[h, h, \delta v, q_4, l_1] = T_3(f)$. This completes the proof of our claim. $\square$

*Step* 10. Final computation of $\{k(c)\}$. Let us compute each $T_i(F)$ separately. Throughout this computation we will repeatedly invoke Lemma 3.4 without explicitly stating it.

$\boxed{F = f}$

By definition

$$\begin{aligned} T_1(f) &= \{c\}, \qquad T_2(f) = \left\{\frac{-a(bc-c+1)}{ca-a+1}\right\} - \{1-b\}, \\ T_3(f) &= \{-abc\} - \left\{1 - \frac{ca-a+1}{ca}\right\}, \\ \tau_{a,c}T_3(f) &= \left\{1 - \frac{ca}{ca-a+1}\right\} - \{ca-a+1\}. \end{aligned}$$

$\boxed{F = A}$

Using $(x, y) \mapsto (x + (a-1)/a, y + (a-1)/a)$ we find

$$\begin{aligned}
T_2(A) &= \left\{\frac{-c(ab-b+1)}{bc-c+1}\right\} - \left\{1-\frac{ab}{ab-b+1}\right\}\\
&\quad - \left\{1-\frac{ca-a+1}{c(ab-b+1)}\right\} + \left\{1-\frac{a}{ab-b+1}\right\},\\
\tau_{a,c}T_2(A) &= \left\{1-\frac{c(ab-b+1)}{ca-a+1}\right\} - \left\{1-\frac{ab-b+1}{a}\right\}\\
&\quad - \left\{\frac{bc-c+1}{b(ca-a+1)}\right\} + \left\{1-\frac{ab-b+1}{ab}\right\},\\
T_3(A) &= \left\{\frac{-b(ca-a+1)}{ab-b+1}\right\}, \qquad T_4(A) = \left\{\frac{bc-c+1}{bc}\right\} - \left\{1-\frac{1}{b}\right\}.
\end{aligned}$$

$\boxed{F = A/f}$

Using the substitution $(x, y) \mapsto ((1-a)/(ax-a), (1-a)/(ay-a))$ we get

$$T_1\left(\frac{A}{f}\right) = \left\{\frac{ca-a+1}{c}\right\} - \{a\}, \qquad T_2\left(\frac{A}{f}\right) = \left\{\frac{ca(ab-b+1)}{ca-a+1}\right\} - \{ab-b+1\}.$$

$\boxed{F = B}$

By definition and using the substitution $(x, y) \mapsto ((x-1)/(b-1), (y-1)/(b-1))$ we get

$$\begin{aligned}
T_1(B) &= \left\{\frac{ab-b+1}{ab(bc-c+1)}\right\} - \left\{\frac{ab-b+1}{a(bc-c+1)}\right\},\\
\tau_{a,c}T_1(B) &= \{bc-c+1\} - \left\{\frac{bc-c+1}{b}\right\}.
\end{aligned}$$

Putting the above together we now complete the proof the theorem in the case that none of the terms in Goncharov's relations is equal to $\{0\}$ or $\{1\}$ except for $\eta$. $\square$

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## Appendix A. Goncharov's relations in Bloch's higher Chow group $CH^3(F, 5)$, by Herbert Gangl

# Goncharov's trilogarithm relation on pictures


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## 1. Introduction

The main question left open in [4] was to prove Goncharov's 22-term relation for the trilogarithm [5] in the cubical version of the higher Chow group $CH^3(F, 5)$, at least if the field $F$ satisfies the Beilinson–Soulé vanishing conjecture. It has been settled recently by Zhao in [7], where he has used a lot of cycles each of which needs to be tested for admissibility. This latter test is very tedious to verify and since a single instance of failure of admissibility in any one of the many cycles occurring would jeopardize his argument, we propose a graphical notation for the cycles in question, from which admissibility (and strategy of proof) can be read off rather conveniently.


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## 2. A graphical notation for cycles

### 2.1. *Pictures for curves in 3-space*

We use the notations and conventions from [7] (and [4]) and replace each (parametrized) coordinate by its zeros, poles and preimages of 1; for example $f(t) = \frac{a_1 t - a_2}{b_1 t - b_2}$ will be encoded as

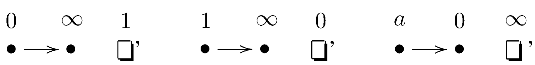
$$\frac{\{\text{zeros}\} \mid \{\text{poles}\}}{f_i^{-1}(1)} = \frac{\frac{a_2}{a_1} \mid \frac{b_2}{b_1}}{\frac{a_2 - b_2}{a_1 - b_1}},$$

and a Totaro cycle $C_a^{(2)}$ which is given as the alternation of $[t, 1-t, 1-\frac{a}{t}] \subset \square_F^3$ (cf. [6]) can be encoded as

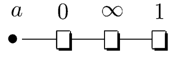
$$\left[\frac{0 \mid \infty}{1}, \frac{1 \mid \infty}{0}, \frac{a \mid 0}{\infty}\right]. \tag{1}$$

Here we use the "cube" $\square_F^n = (\square_F^1)^n$, where $\square_F^1 = \mathbb{P}_F^1 \setminus \{1\} \simeq \mathbb{A}_F^1$ with two distinguished points given by $\{x = 0\}$ and $\{x = \infty\}$.

In this cubical framework, the main difficulty is to make sure that each generator which occurs must be admissible, and this condition for a fractional linear cycle translates as follows to the encoding above: each time a number (in $\mathbb{P}_F^1$) appears in the "upper row" for two of the coordinates (i.e., either as a zero or pole) of some generator, this same number has to occur in the "lower row" (i.e., as one of the preimages of 1) of the same generator as well. This suggests the following graphical notation: for any critical point $x$ of a given coordinate we draw a bullet, decorated by $x$, as well as an arrow from each zero to each pole of this coordinate, and we cover a bullet by a square ❑ if its decoration occurs in the preimage of 1 for one of the remaining coordinates of the given generator (sometimes it will be convenient to keep the information about the index $i$). The admissibility condition then translates for the pictures as follows: each time a bullet is incident with more than one line, it must be marked by a square. Thus, (1) yields for the three respective coordinates




and we combine these three into a single picture (by "overlaying" them) to the picture of the above cycle $C_a^{(2)}$ in $\square_F^3$ (here we understand that —❑ represents —⊡, i.e., we drop the bullet inside the box):



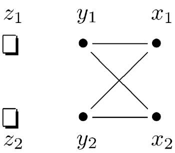

Note that the squares attached to the points 0 and $\infty$ guarantee the admissibility of this cycle.

### 2.2. *Pictures for surfaces in 5-space*

The above pictures have been used in [4] to guide a way through the "jungle of (non-)admissible cycles" in $Z^2(F,3)$. In order to treat the next higher case, i.e., $Z^3(F,5)$, where the corresponding cycles $C_a^{(3)}$ had been given by Bloch [1], and slightly modified in Bloch–Kříž [2], we need to “generalize” the above picture in two directions: on the one hand, to functions $f_i$ of higher degree, and on the other hand, to cycles in higher dimension. First, we allow the $f_i$ to have degree $> 1$, e.g. for degree 2 we have the encoding

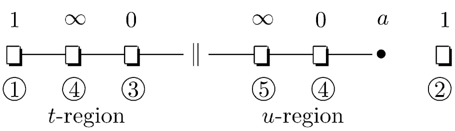
$$\frac{x_1, x_2 \mid y_1, y_2}{z_1, z_2}$$

and the corresponding picture where, for suggestive reasons, we draw the zeros, poles and preimages of 1, respectively, on the same vertical line:




Admissibility for those cycles requires two types of checks. We first indicate the simpler one which is analogous to the previous check: if at least two coordinates of some generator $[f_1, \ldots, f_5]$ are critical at some specialization $t_0$ of a given variable $t$, then there has to be a third coordinate of the same generator which is “good for $t_0$” in that it becomes equal to 1 for $t = t_0$. In terms of pictures, this amounts to the following condition: let $b$ be a bullet (corresponding to a critical value of some of the $f_i$) which is incident with (at least) two lines whose other endpoints correspond to critical points of two different coordinates. Then $b$ has to be marked by a square, otherwise it violates admissibility. Note that in the above picture (for an $f_i$ of degree 2) the two lines incident with the bullet labelled by $y_1$ have endpoint decoration in the same coordinate. Therefore the admissibility condition does not require this bullet to have a square around it.

Somewhat more delicate is the problem to picture two (or more) variables in the graph (which constitutes the second generalization of the above): we mark the “boundary” between the range of the two respective variables ($t$ and $u$, say) by a vertical double bar; it symbolizes the locus $t = u$. As an example, we picture a Bloch–Kříž–Totaro cycle $C_a := C_a^{(3)} := \mathrm{Alt}[t, u, 1-t, 1-u/t, 1-a/u]$ in $Z^3(F, 5)$ (“Alt” stands for alternation with respect to permutation and inversion of coordinates) by




where the encircled numbers indicate the coordinate of the cycle in the above presentation which is responsible for the ❑-marking (these encircled numbers will be omitted in the remaining text), while its encoding is given by

$$\left[\frac{0 \mid \infty}{1}, \frac{0 \mid \infty}{1}, \frac{1 \mid \infty}{0}, \frac{t = u \mid t = 0,\ u = \infty}{t = \infty, u = 0}, \frac{a \mid 0}{\infty}\right]. \tag{2}$$

Putting both generalizations together, we will now consider for any $c \in F^\times - \{1\}$ a Bloch–Kříž–Totaro cycle $C_{\varphi(c)}$, where $\varphi$ is a rational function of degree 2 with zeros $\{x_i\}_i$, poles $\{y_i\}_i$ and $\varphi^{-1}(1) = \{z_i\}_i$, $i = 1, 2$; we denote furthermore the zeros (in $u$) of the expression $1 - \varphi(c)/\varphi(u)$ by $\{w_i\}$.

The second check for admissibility can now be formulated as follows: whenever taking a partial differential of some generator $g$ relates two variables $t$, $u$, to each other (typically $t = u$), one has to take into account that the critical values of $t$ and $u$ get combined, and this may reveal the non-admissibility of $g$ (by a subsequent application of the first type of check above).

Therefore, in the pictures for generators below, most of the bullets associated to values which are possibly critical for both variables simultaneously (this will concern the "$x_i$" and "$y_j$" below), will have a square around them, in at least one of the regions. Since the variables $t$ and $u$ are usually related by $t = u$ (or $t = \rho(u)$ for some involution $\rho$ on the critical values $x_i$, $y_j$ below), the squares get "overlayed" in the picture for the partially derived generator, and admissibility will be apparent.

We can assume, after a reparametrization of the variable $t$ by a fractional linear transformation, that $\varphi^{-1}(1) = \{0, \infty\}$ which implies that $\varphi$ has the form

$$\varphi(t) = \frac{(t - x_1)(t - x_2)}{(t - y_1)(t - y_2)}$$

with $x_1 x_2 = y_1 y_2 (\neq 0)$. One verifies that the involutory reparametrization $\rho : t \to x_1 x_2 / t$ fixes $\varphi$ but interchanges the $x_i$ and the $y_i$.

In most of our pictures the underlying vertices and the arrows between them, going again from any zero to any pole for a given coordinate, do not change their direction, so we will usually omit this information and only draw the underlying line, as in the picture above.

### 2.3. *Reparametrization by k*

Now we relate directly to Zhao's initial cycle $\{k(c)\}$ (which in the notation above is $C_{k(c)}$).

We reparametrize both variables $t$ and $u$ in $\{k(c)\}$ using $k$. Recall [3] that the reparametrization of a parametrizing variable for a cycle by rational functions of degree $n$ gives the same cycle, except that one has to multiply its coefficient by a factor $1/n$ for each of the reparametrized variables. Therefore we get

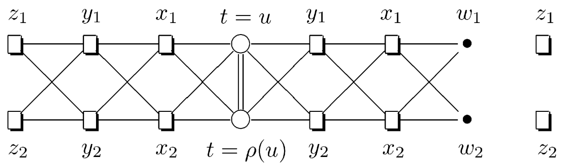
$$4\big\{k(c)\big\} = 4\left[t, u, 1 - t, 1 - \frac{u}{t}, 1 - \frac{k(c)}{u}\right] = \left[k(t), k(u), 1 - k(t), 1 - \frac{k(u)}{k(t)}, 1 - \frac{k(c)}{k(u)}\right].$$

We obtain the encoding of the latter cycle as

$$\left[\frac{x_1, x_2 \mid y_1, y_2}{z_1, z_2}, \frac{x_1, x_2 \mid y_1, y_2}{z_1, z_2}, \frac{z_1, z_2 \mid y_1, y_2}{x_1, x_2}, \frac{\begin{array}{c|c} t = u, & t = x_1,\ t = x_2, \\ t = \rho(u) & u = y_1,\ u = y_2 \end{array}}{\begin{array}{c} t = y_1,\ t = y_2, \\ u = x_1,\ u = x_2 \end{array}}, \frac{w_1, w_2 \mid x_1, x_2}{y_1, y_2}\right]. \tag{3}$$

Here, as in all the cycles in the following, the first and third coordinates depend on $t$ only, while the second and fifth one depend on $u$ only. (It would be more fitting to have the coordinates in the order $(3, 1, 4, 2, 5)$ to match the pictures better, but the above notation is compatible with the one used in [2,4] and [7].)

The above encoding finally leads to its picture




*2.4. Decompose in the coordinates* $(1,2)$

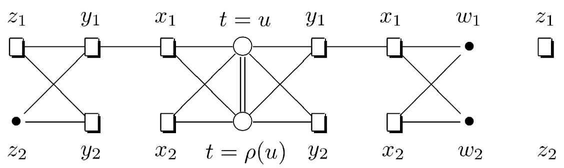
This refers to Step (2) in [7]. We decompose the cycle $4\{k(c)\}$ encoded in (3) into four new ones by decomposing in the first two coordinates, where we use Lemma 2.8(c) of [4] with $f_1(t) = \varphi(t)^2$, $g(t) = cr(t, z_1, x_1, y_1) \cdot cr(t, z_2, x_1, y_1)$ and $h(t) = cr(t, z_1, x_2, y_2) \cdot cr(t, z_2, x_2, y_2)$ and where "$cr$" denotes the cross-ratio. (Note that $k(t)$ can be written as a product of two cross-ratios in several different ways:

$$k(t) = cr(t, z_i, x_j, y_k) \cdot cr(t, z_i, x_{3-j}, y_{3-k}), \quad i, j, k = 1, 2.) \tag{4}$$

All four of them represent the same cycle (use the involution $\rho$ above for $t$ or for $u$), and we obtain the following encoding of $\{k(c)\}$

$$\left[\frac{x_1 \mid y_1}{z_1}, \frac{x_1 \mid y_1}{z_1}, \frac{z_1, z_2 \mid y_1, y_2}{x_1, x_2}, \frac{\begin{array}{c|c} t=u, & t=x_1,\ t=x_2, \\ t=\rho(u) & u=y_1,\ u=y_2 \end{array}}{\begin{array}{c} t=y_1,\ t=y_2, \\ u=x_1,\ u=x_2 \end{array}}, \frac{w_1, w_2 \mid x_1, x_2}{y_1, y_2}\right] \tag{5}$$

which we picture as



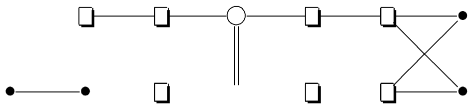

For easier comparison, we give the respective critical values in Zhao's notation (our variables $t$ and $u$ correspond to his $x$ and $y$, respectively): $x_1 = \infty$, $x_2 = (1-b)^{-1}$, $y_1 = 1 - \frac{1}{a}$, $y_2 = 0$, $z_1 = -\frac{1}{ab}$, $z_2 = 1$, $w_1 = c$, $w_2 = \rho_y(c)$.

Below we will leave out the critical values attached to the squares or vertices since they have their fixed location in the pictures. Furthermore, we will omit squares which do not lie on one of these fixed locations. Note that the admissibility of $\{k(c)\}$ is now obvious from the above picture (cf. Section 2.2).

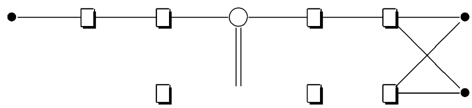
*2.5. Decompose in the third and fourth coordinate*

This refers to Step (3) in [7]. In order to prepare for the subsequent decompositions, we place the squares in the crucial locations, using the same lemma as in the previous step. In particular, we trade the two squares at $z_1$ for ones at $x_2$ (in both locations simultaneously). Subsequently we decompose in the third and fourth coordinates, the order being unimportant (for instance, in the fourth coordinate we use $1 - k(u)/k(t) = cr(t, y_1, u, x_1) \cdot cr(t, y_1, \rho(u), x_2)$). We get the following four generators, after applying $\rho$ to the $t$-region of the third one (note that the first two pictures are just mirror images of the ones for the cycle presentation that Zhao uses, e.g., $Z_1(A, A) = \rho_{x,y} Z_3(\frac{f}{B}, \frac{f}{B})$, and thus encode the same cycle)

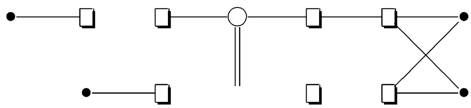
$Z_1(A, A)$:



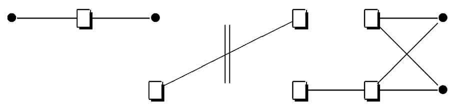
$Z_1\left(\frac{f}{B}, \frac{f}{B}\right)$:



$\rho_x Z_2(A, A)$:



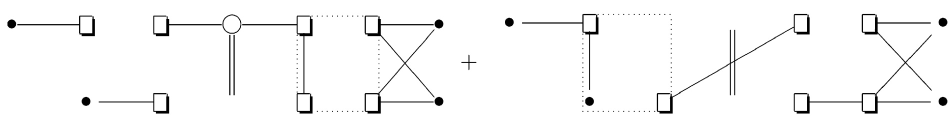
$\rho_y Z_2(A, A)$:



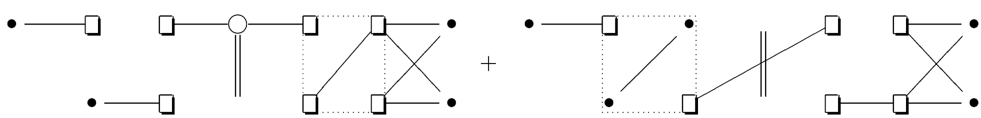
### 2.6. Step (4)

From [7], Lemma 3.2(2) with $f_1(t) = cr(t, z_1, y_1, x_1)$, $g(t) = cr(t, z_1, y_2, x_1)$ and $h(t) = cr(t, z_1, y_1, y_2)$ we conclude that we can decompose the sum of the latter two (i.e., $\rho_x Z_2(A, A) + \rho_y Z_4(A, A)$) into a sum of four cycles as follows. As a guideline for the geometrical picture, think of replacing one side of an oriented triangle (here the side $y_1, x_1$ of the triangle $y_1, x_1, y_2$ in either region) by the formal sum of the other two sides (with the right orientation). For emphasis, we have drawn dotted lines around the "crucial" parts of the pictures

$X_1$:



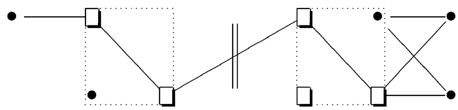
$-X_2$:



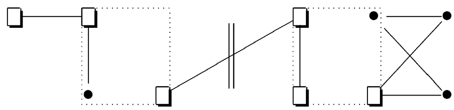
### 2.7. Step (5)

In a similar vein, we proceed for the two generators of $X_1$ by "polarizing" with respect to the two dotted regions (encircling the $x_i$ and $y_j$-locations): Zhao's lemma allows to replace $X_1$ by $Z_3(\frac{A}{B}, \frac{A}{B}) - Z_3(\frac{A}{f}, \frac{A}{f}) - Z_3(\frac{f}{B}, \frac{f}{B})$, where

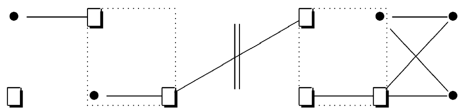
$Z_3\left(\frac{A}{B}, \frac{A}{B}\right)$:



$Z_3\left(\frac{A}{f}, \frac{A}{f}\right)$:

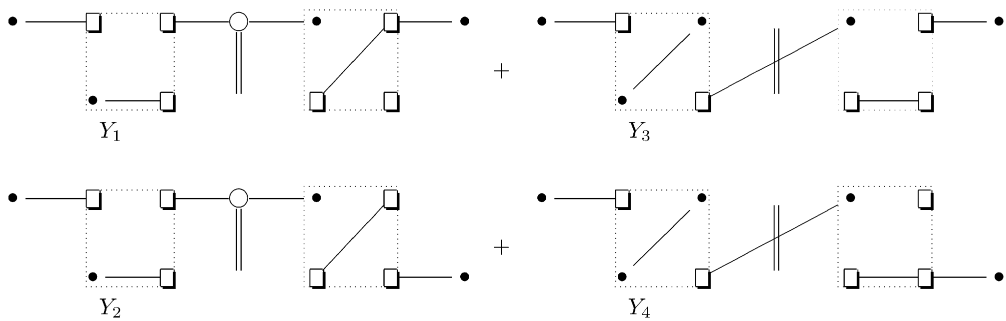


$Z_3\left(\frac{f}{B}, \frac{f}{B}\right)$:



*2.8. Step (6)*

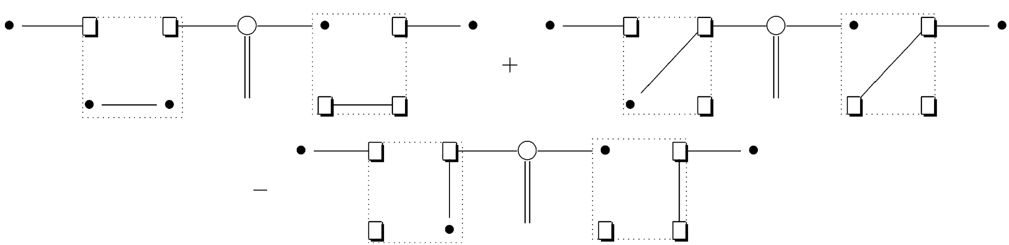
A similar "polarization" procedure is not yet applicable to $X_2$; instead, one first decomposes the fifth coordinate, giving generators $Y_i$ $(i = 1, \ldots, 4)$ as follows:




*2.9. Step (7)*

One more step of preparation is needed: the squares for $Y_3$ and $Y_4$ at the $t$-location $x_2$ are to be counted with multiplicity 2, and one of each needs to be moved to $x_1$ instead. This can be done—call the resulting generators $Y_3'$ and $Y_4'$, respectively—but with the lemmas at hand will necessarily do the same to $Y_1$ and $Y_2$, which thus lose the squares at $t$-location $x_2$.

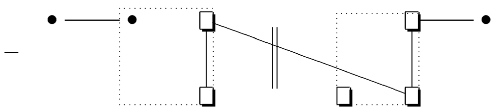
Then polarizing with respect to the dotted regions of $Y_1' + Y_3'$ gives three terms




A similar procedure can be applied to $Y_2' + Y_4'$, giving three more generators.

### 2.10. Step (10)

The final evaluation, in terms of the cycles $C_v$, now follows a similar pattern (replace an edge by the formal sum of two other edges, so that all three together bound a triangle), at least after a final preparation: according to Lemma 2.8(b) in [4], for the cycles that we now consider we can multiply (or divide) the fourth coordinate by any linear form in either of the two variables $t$ and $u$, which amounts to changing a critical point (and, simultaneously, a square for the other coordinate). The pictorial equivalent is that the central edge will change, and we do it in such a way that it ends in a point which is already critical for a different coordinate. At the same time, the square for the other region will move to that critical point; as an example, we consider the last generator in $Y_1' + Y_2'$ above, and change the critical value $y_1$ in the $u$-region to $x_2$ (by multiplying the fourth coordinate $cr(u, x_1, t, y_1)$ by $cr(u, x_1, y_1, x_2)$, giving $cr(u, x_1, t, x_2)$) which moves the square at $y_1$ in the $t$-region to $x_2$, while the squares in the $u$-region do not change location, i.e., we get



Finally, the "loose edge" in this picture (connecting $z_1$ to $y_1$ in the $t$-region) needs to be replaced by the formal sum of two edges, one connecting $x_2$ to $z_1$ and $y_1$, respectively. The resulting two generators are of Bloch–Križ–Totaro type, i.e., of the form $C_v$ for some $v$, and we can read off the invariant $v$ directly from the picture: $v$ is the cross-ratio of the four critical points, two of which are common to both $t$ and $u$-region. In the example, we obtain $C_{cr(x_1,x_2,z_1,w_1)}$ and $-C_{cr(x_1,x_2,y_1,w_1)}$.

Proceeding with this evaluation, we encounter precisely all the terms of Goncharov's 22-term relation, as in [7].

## Acknowledgments

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