Hamilton decompositions of directed cubes and products

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Abstract

Call a directed graph \( \overset{\rightarrow}{G} \) symmetric if it is obtained from an undirected graph \( G \) by replacing each edge of \( G \) by two directed edges, one in each direction. We will show that if \( G \) has a Hamilton decomposition with certain additional structure, then \( \overset{\rightarrow}{G} \times \overset{\rightarrow}{C_2} \times \overset{\rightarrow}{K}_2 \) has a directed Hamilton decomposition. In particular, it will follow that the bidirected cubes \( \overset{\rightarrow}{Q}_{2m+1} \) for \( m \geq 2 \) are decomposable into \( 2m + 1 \) directed Hamilton cycles and that a product of cycles \( \overset{\rightarrow}{C}_{n_1} \times \cdots \times \overset{\rightarrow}{C}_{n_m} \times \overset{\rightarrow}{K}_2 \) is decomposable into \( 2m + 1 \) directed Hamilton cycles if \( n_i \geq 3 \) and \( m \geq 2 \).

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1. Introduction

Hamilton decompositions of cartesian products of undirected graphs have been the subject of a number of papers. Kotzig [3] showed that a product of two cycles \( C_m \times C_n \) has a Hamilton decomposition. This was generalized to products of three cycles \( C_\ell \times C_m \times C_n \) by Foregger [2]. Aubert and Schneider [1] proved that if \( G \) is decomposable into two Hamilton cycles, then \( G \times C_n \) decomposes into three cycles. One corollary of this theorem is that a product of an arbitrary number of cycles \( C_{n_1} \times \cdots \times C_{n_m} \) is decomposable into \( m \) Hamilton cycles, and in particular the \( 2n \)-dimensional cube \( Q_{2n} = (C_4)^n \) is decomposable into \( n \) Hamilton cycles. This work was generalized by Stong [4] to show that if \( G \) is decomposable into \( m \) Hamilton cycles and either \( n \) is even or \( G \) has at least \( 6m - 3 \) vertices, then \( G \times C_n \) is decomposable into \( m + 1 \) Hamilton cycles.

These results are of interest as examples of graph decompositions, but they are also of practical interest. Multiprocessor computers are often built with the processors arranged in a high dimensional cube with vertices corresponding to processors and edges to communications channels. Decompositions of the edge set of the cube can then be used to give algorithms which use each edge equally and therefore make more efficient use of the architecture. However, communications channels are not undirected, they are in fact directed with one channel in each direction. It would therefore be nice to know that directed cubes are also decomposable. The goal of this paper is to provide such a decomposition. In fact we will show that a large number of cartesian products of graphs have directed Hamilton decompositions.

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We will first set up a little notation and a few conventions. All of our graphs will be assumed to be simple graphs without loops (though many of our results can be made to go through for graphs with multiple edges provided one adds some technical hypotheses). When we say a graph is Hamilton decomposable, we mean that its edge set has a partition into Hamilton cycles. (Some authors extend the notion of Hamilton decomposability to undirected graphs of odd degree by allowing a single 1-factor in the decomposition. We will not adopt this convention.) Let $C_n$, $n \geq 3$, denote a cycle of length $n$. Let $Q_n$ denote an $n$-dimensional cube. We will generally use $Q_1$ rather than $K_2$ to denote the graph with two vertices and one edge. Let $G \times H$ denote cartesian product of the graphs $G$ and $H$. Note that since $Q_2 = C_4$, we have $Q_{2m+1} = (C_4)^m \times Q_1$. For a directed or undirected graph $\Gamma$, let $|V(\Gamma)|$ denote the number of vertices of $\Gamma$.

If $G$ is an undirected graph, let $\vec{G}$ denote the directed graph one obtains by replacing each edge of $G$ by two directed edges, one in each direction. We will call such a graph $\vec{G}$ a symmetric directed graph. If $G$ is regular of even degree and decomposes into Hamilton cycles, then of course $\vec{G}$ decomposes into directed Hamilton cycles. Therefore, this paper will be exclusively concerned with graphs $G$ which are regular of odd degree.

For graphs $G$ which are regular of degree 3, there is a somewhat surprising parity obstruction to the existence of a directed Hamilton decomposition of $\vec{G}$. Specifically, we will show in Section 2 that if $G$ is regular of degree 3

![Fig. 1. Tiles for building hamiltonian cycles: (a) a hamiltonian cycle in $\vec{C}_4 \times \vec{C}_4 \times \vec{Q}_1$ drawn as a tile; (b) a vertical extension of part (a); (c) a second vertical extension; (d) a horizontal extension (and vertical extensions for it).](image-url)
and $|V(G)|$ is a multiple of 4, then $\vec{G}$ is not decomposable into three directed Hamilton cycles. This unfortunate fact will force us to work with graphs of degree at least 5. Specifically we will be looking at graphs of the form $\vec{G} \times \overrightarrow{C_n} \times \overrightarrow{Q_1}$, where $G$ is Hamilton decomposable.

Suppose $G$ is an undirected graph which is decomposable into $m$ Hamilton cycles $H_1, \ldots, H_m$. Regard $H_1$ as the “base” Hamilton cycle in this decomposition and the remaining $H_i$ as “extra” Hamilton cycles. Suppose $e$ is an edge in $H_i$, $i \geq 2$. To $e$ we can associate a directed path $\alpha_e$ in $H_1$ which is just one of the two possible paths in $H_1$ with the same endpoints as $e$. (Generally one of the two paths will be “short” and we will use the short path.) Call a directed path $\alpha_e$ even, if it contains an even number of vertices (including the endpoints). Suppose we choose edges $e_i \in H_i$ for all $2 \leq i \leq m$ so that the resulting directed paths $\alpha_{e_i}$ are all even and are pairwise vertex-disjoint. We will say these edges constitute even data if we can order them around $H_1$ so that (strictly) between the first and second directed paths, (strictly) between the second and third directed paths, \ldots, and (strictly) between the next to last and last directed paths there are an even number of vertices of $H_1$ and (strictly) between the last directed path and the first directed path there are at least four vertices of $H_1$.

In Section 3, we will prove the main theorem, namely that if $G$ has a Hamilton decomposition with even data, then $\vec{G} \times \overrightarrow{C_n} \times \overrightarrow{Q_1}$ has a directed Hamilton decomposition. The method of proof mimics the techniques of Stong [4]. Suppose $G$ decomposes into $m$ Hamilton cycles $H_1, \ldots, H_m$, with $H_1$ the base cycle. We will show that for each $H_i$, $2 \leq i \leq m$
Fig. 2. (a) and (b) A second, (c) fourth and (d) fifth hamiltonian cycle in tile form.
Fig. 2. (continued)
we can find two directed Hamilton cycles in the product which use only a few edges in the $Q_1$ and $C_n$ directions (and further the few edges they do use will be at the endpoints of $e_i$). After removing these cycles, the remaining graph $\tilde{G}$ will be a modified version of $\tilde{C}_s \times \tilde{C}_n \times \tilde{Q}_1$ for $s = |V(G)|$. We will show that any such modification can be decomposed into 5 directed Hamilton cycles by giving specific “tiles” which can be assembled into such a decomposition. There will unfortunately be four cases depending on the parities of $n$ and $|V(G)|$. The existence of even data for the decomposition of $G$ is a technical assumption. It might be possible to weaken this assumption if one could find a much larger compatible collection of tiles.

In Section 4, we prove that this technical requirement of even data is not too restrictive by showing that a product of cycles always has a Hamilton decomposition with even data. Moreover, by adapting the techniques of Stong [4] we will show that if $G$ has degree $2m$, a Hamilton decomposition with even data, and $|V(G)| \geq 9m - 6$, then $G \times C_n$ has a Hamilton decomposition with even data. As a corollary we will obtain the facts that $\tilde{Q}_{2m+1}$ and $\tilde{C}_n \times \cdots \times \tilde{C}_n \times \tilde{Q}_1$ have directed Hamilton decompositions for $m \geq 2$.

2. Bidirected graphs of degree 3

Suppose $\tilde{G}$ is a directed graph with indegree $(v) = \text{outdegree}(v) = m$ for all vertices $v$ of $\tilde{G}$. Suppose $F = (\tilde{F}_1, \ldots, \tilde{F}_m)$ is a 1-factorization of $\tilde{G}$. Each $\tilde{F}_i$ can be viewed as a permutation $\phi_i$ of the vertex set $V(\tilde{G})$, i.e., for all $v \in V(\tilde{G})$ the 1-factor $\tilde{F}_i$ contains the edge $(v, \phi_i(v))$ from $v$ to $\phi_i(v)$. Define the sign $(-1)^{\tilde{F}_i}$ of the 1-factor $\tilde{F}_i$ to be the sign of $\phi_i$ and let the sign of $F$ be defined as $(-1)^F = \prod_i (-1)^{\tilde{F}_i}$.

Say two 1-factorizations of $\tilde{G}$ are elementary exchange equivalent if all but two of their 1-factors agree, i.e., if they differ by exchanging edges between a pair of 1-factors. Say two 1-factorizations are exchange equivalent if they differ by a sequence of elementary exchange equivalences.

**Lemma 2.1.** Exchange equivalent 1-factorizations of $\tilde{G}$ have the same sign.

**Proof.** It suffices to show that elementary exchanges do not change the sign. Viewing the 1-factors as permutations, this case of the lemma reduces to the following fact about permutations.

**Claim.** If $\pi_1, \pi_2, \sigma_1$ and $\sigma_2$ are permutations of $\{1, 2, \ldots, n\}$ with $\{\pi_1(i), \pi_2(i)\} = \{\sigma_1(i), \sigma_2(i)\}$ for $1 \leq i \leq n$, then the sign of $\pi_1\pi_2$ agrees with the sign of $\sigma_1\sigma_2$.

To see that the claim holds, note that we can form an undirected graph $\Gamma$ with vertex set $\{1, \ldots, n\}$ by taking edges $\{\pi_1(i), \pi_2(i)\}$ for $1 \leq i \leq n$. The graph $\Gamma$ is regular of degree 2 and hence is a union of cycles, which are just the cycles of $\pi_2\pi_1^{-1}$ (but undirected). Hence, the sign of $\pi_1\pi_2$ is just $-1$ raised to the number of even cycles in $\Gamma$. Since $\sigma_1$ and $\sigma_2$ give the same graph $\Gamma$ the claim follows. $\square$

**Theorem 2.2.** Suppose $\hat{G}$ is a symmetric directed graph of degree 3 and $|V(\hat{G})|$ is a multiple of 4. Then $\hat{G}$ is not decomposable into 3 directed Hamilton cycles.

**Proof.** Suppose $(\hat{H}_1, \hat{H}_2, \hat{H}_3)$ were a directed Hamilton decomposition of $\hat{G}$. Let $-\hat{H}_1$ denote $\hat{H}_1$ with the reversed orientation and let $\tilde{F}$ consist of the edges of $\hat{G}$ not in $\hat{H}_1$ or $-\hat{H}_1$. Note that $\tilde{F}$ consists of $|V(\hat{G})|/2$ disjoint copies of $\tilde{Q}_1$. The 1-factorizations $(\hat{H}_1, \hat{H}_2, \hat{H}_3)$ and $(\hat{H}_1, -\hat{H}_1, \tilde{F})$ are exchange equivalent but the former has 3 cycles of even length, so has odd sign, and the latter has 2 + $|V(\hat{G})|/2$, an even number, so has even sign. This contradicts Lemma 2.1. $\square$

**Corollary 2.2.1.** $\hat{Q}_3$ and $\hat{C}_{2n} \times \hat{Q}_1$, $n \geq 2$, are not Hamilton decomposable.

While Corollary 2.2.1 is easy to prove by direct means (as is the fact that $\hat{C}_{2n+1} \times \hat{Q}_1$ is not Hamilton decomposable), the above parity argument further shows that the minimal odd degree that we can possibly hope to look at with the techniques of this paper is 5.
Fig. 3. The first cycle (a) the second cycle (b), (c) the third cycle, (d) the fourth cycle and (e) the fifth cycle for $|G|$ odd and $n$ even.
Fig. 3. (continued)
3. Decompositions of products

The goal of this section is to prove the following theorem.

**Theorem 3.1.** If $G$ has a Hamilton decomposition with even data, then $\vec{G} \times \vec{C}_n \times \vec{Q}_1$ has a decomposition into directed Hamilton cycles.

First consider Fig. 1(a), which shows a Hamilton cycle in $\vec{C}_4 \times \vec{C}_4 \times \vec{Q}_1 = \vec{Q}_5$. The first factor is drawn vertically. To avoid drawing edges crossing back through the diagram we will interpret an edge leaving off the top side of the diagram as reappearing on the bottom and vice versa. Similarly the second factor is drawn horizontally with the analogous cyclic interpretation. The last factor $\vec{Q}_1$ is used to split the diagram into two halves. A $\times$ at a vertex indicates going to the analogous vertex in the other half. A $\otimes$ at a vertex indicates arriving from the analogous vertex in the other half. A convenient way to visualize the construction below is to interpret this Hamilton cycle as being drawn on a $4 \times 4$ tile, with one half on the front of the tile and the other half on the back. The ends of the tile can then be identified forming a thickened torus and the horizontal and vertical cycles reappear.

Next consider Fig. 1(b), which shows the top and bottom of a $2 \times 4$ tile. Interpret the horizontal direction cyclically, but not the vertical. We see that any path entering from the top eventually exits the tile at the matching spot on the
Fig. 4. (a) The first cycle, (b) the second cycle, (c) the third cycle, (d) the fourth cycle and (e) the fifth cycle for $|G|$ odd and $n$ odd.
Fig. 4. (continued)
bottom. Thus, if we start with the $4 \times 4$ tile above and glue on $k$ copies of this $2 \times 4$ tile and then identify the ends, we will get a Hamilton cycle in $C_{2k+4} \times C_4 \times \overrightarrow{Q_1}$.

Next consider Fig. 1(c), which appears to show the top and bottom of a $4 \times 4$ tile. Again interpret the horizontal direction cyclically. Again any path entering from the top eventually exits at the matching spot on the bottom. Thus, if we start with one copy of the original $4 \times 4$ tile and glue on any combination of the tiles from Figs. 1(b) and (c) (in any pattern) and identify ends, then we get a Hamilton cycle. This Hamilton cycle lies in a graph $\Gamma \times \overrightarrow{C_4} \times \overrightarrow{Q_1}$, where $\Gamma$ is an even cycle $C_{2r}$ with some collection of additional edges $e_1, \ldots, e_s$ added. There is one edge $e_i$ for each tile of type 1(c). The endpoints of these edges are also the endpoints of disjoint directed paths of length four in $\overrightarrow{C_{2r}}$ with even separations between any pair of consecutive directed paths and one separation of size at least four. This collection of directed paths is almost “even data”, the only problem being that all the directed paths have length exactly four instead of an arbitrary even length. However, viewing the middle two rows of Fig. 1(c) more carefully, we see that every path entering the middle two rows leaves in the analogous location on the other side. Thus we may repeat these two rows to enlarge this tile and get larger even directed paths. (Alternately, we may remove the middle two rows to get a directed path of length two, which would be useful if we allowed $G$ to have multiple edges.)
Fig. 5. (a) The first cycle, (b) the second cycle, (c) the third cycle, (d) the fourth cycle and (e) the fifth cycle for $|G|$ odd and $n$ even.
Fig. 5. (continued)
Finally, consider Fig. 1(d), which shows a $2 \times 4$ tile and two extensions of it, one $2 \times 2$ and one apparently $2 \times 4$. Interpreting the vertical direction cyclically this time, we see that any path entering this tile on the right leaves at the analogous spot on the left. Thus, it can be used to extend the Hamilton cycle horizontally. The two extensions fit together with Fig. 1(b) and (c) to extend them horizontally.

Thus, taken together we see that Fig. 1 describes a Hamilton cycle in $\overrightarrow{G} \times \overrightarrow{C_n} \times \overrightarrow{Q_1}$ for any even $n \geq 4$ and any $G$ with even data. Further this cycle only uses edges of $\overrightarrow{G}$ which are either in the base cycle of the Hamilton decomposition of $G$ or in the even data.

Continuing to use this tile method of building Hamilton cycles, we see that Fig. 2 gives four more sets of tiles which yield four more Hamilton cycles in $\overrightarrow{G} \times \overrightarrow{C_n} \times \overrightarrow{Q_1}$ of the type described above. Further these five cycles (one from Fig. 1 and four from Fig. 2) are edge disjoint. Between them, these cycles use every directed edge coming from the base Hamilton cycle of $G$ and every directed edge coming from the even data. Further they use almost every edge coming from $\overrightarrow{C_n} \times \overrightarrow{Q_1}$. These unused directed edges form a 1-factor in $\{v\} \times \overrightarrow{C_n} \times \overrightarrow{Q_1}$ where $v$ is an endpoint of an edge in the even data. The extra property that they have is slightly involved to state but fairly obvious. Suppose $v$ and $v'$ are the endpoints of an edge in the even data. There are two directed Hamilton cycles $D_1$ and $D_2$ in $\overrightarrow{C_n} \times \overrightarrow{Q_1}$ which can be divided into subsets $X_{i,j}$ ($i, j = 1$ or 2) by putting edges of $D_i$ alternately into $X_{i,1}$ or $X_{i,2}$ such that the 1-factor in $\{v\} \times \overrightarrow{C_n} \times \overrightarrow{Q_1}$ is $\{v\} \times (X_{1,1} \cup X_{2,1})$ and the 1-factor in $\{v'\} \times \overrightarrow{C_n} \times \overrightarrow{Q_1}$ is $\{v'\} \times (X_{1,2} \cup X_{2,2})$.

To see that this unwieldy condition is exactly the correct one, suppose $H_i$, $2 \leq i \leq m$, is one of the extra Hamilton cycles in $G$ and further assume that $H_i$ passes through the vertices of $G$ in the order $v_1, v_2, \ldots, v_s, v_1$ with $e_i = \{v_s, v_1\}$ the edge of $H_i$ in the even data. Let $D = (w_1, w_2, \ldots, w_{2n})$ be a directed Hamiltonian cycle in $\overrightarrow{C_n} \times \overrightarrow{Q_1}$. Note that $D$
can be divided into two subsets
\[ X_1 = \{(w_1, w_2), (w_3, w_4), \ldots, (w_{2n-1}, w_{2n})\} \]
and
\[ X_2 = \{(w_2, w_3), \ldots, (w_{2n-2}, w_{2n-1}), (w_{2n}, w_1)\} \]
by putting edges alternately into \(X_1\) or \(X_2\). Then
\[
(v_1, w_3), \ldots, (v_s, w_3), (v_s, w_4), \ldots, (v_1, w_4), \ldots, (v_1, w_{2n}), (v_1, w_1)
\]
is a directed Hamilton cycle in \((\hat{H}_{i} - e_i) \times \hat{C}_n \times \hat{Q}_1\). This cycle uses only edges \(\{v_1\} \times X_2\) and \(\{v_i\} \times X_1\) coming from \(\hat{C}_n \times \hat{Q}_1\). Thus, two such cycles use up exactly the unused edges from \(H_i\) and the unused edges in \(\{v_1\} \times C_n \times Q_1\) and \(\{v_i\} \times C_n \times Q_1\).

Thus, Figs. 1 and 2 prove Theorem 3.1 in the case where \(|V(G)|\) and \(n\) are even. Fig. 3 gives the analogous tiles for \(|V(G)|\) even and \(n\) odd. Fig. 4 gives the tiles for \(|V(G)|\) odd and \(n\) odd. Unfortunately the case \(|V(G)|\) odd and \(n\) even is a little trickier. Fig. 5 gives the tiles for \(|V(G)|\) odd and \(n\) even. The first tile is only \(3 \times 4\), which means we could weaken the notion of even data slightly in this parity case (the gap between the last and first directed paths need only be at least 3 vertices). However the third tile is more complicated than previously. In this case the third tile is \(4 \times 6\) (rather than \(4 \times 4\) as one might have expected). Rows 2 and 3 (counting down from the top) can be repeated any number of times (including zero times) to extend this tile. Rows 4 and 5 are more complicated and must be included exactly once. Still these figures complete the proof of Theorem 3.1.

4. Existence of even data

The previous section shows that even data is a convenient technical condition to impose on the graph \(G\) since it limits the number of types of tiles we must construct. The existence of even data forces the graph \(G\) to be not too dense, but other than this, even data does not seem to be a very restrictive condition. This is evidenced by the following two theorems.

**Theorem 4.1.** If \(m, n \geq 3\), then \(C_m \times C_n\) has a Hamilton decomposition with even data.

**Proof.** The Hamilton decompositions of \(C_m \times C_n\) in [3,4] are built from decompositions of \(C_3 \times C_3\), \(C_3 \times C_4\) or \(C_4 \times C_4\) by showing how to add two rows or two columns. In this case any decomposition can be extended. Thus, it suffices to find decomposition of the three examples above which contain a square with three edges in one Hamilton cycle and one in the other. The cycle with three edges can be taken as the base cycle and the fourth edge as the even data for the other cycle. Such decompositions are easy to build and are left as an exercise to the reader. (The examples in [4, Figs. 2 and 7] have this property.) \(\Box\)

**Theorem 4.2.** If \(G\) has a Hamilton decomposition \((H_1, \ldots, H_m)\) with even data and \(|V(G)| \geq 9m - 6\), then \(G \times C_n\) \((n \geq 3)\) has a Hamilton decomposition with even data.

**Proof.** A simplified version of the construction in [4] for Hamilton decompositions of the product \(G \times C_n\) proceeds as follows. One chooses vertex-disjoint 2-paths \(A_i \in H_i\). (This is a weakening of the results of [4]. For example, for \(n\) even, one only needs to choose edges in \(H_i\) and they need to be completely vertex-disjoint.) For \(H_i\), \(1 \leq i \leq m - 1\), we choose simple cycles \(H_i\) in \(H_i \times C_n\) which use every copy of every edge of \(H_i - A_i\) as shown in Fig. 6. The remaining edges give a graph \(\Gamma\) which can be thought of as a modified version of \(H_m \times C_n\). The graph \(\Gamma\) is shown to decompose into two Hamilton cycles \(\tilde{H}_m\) and \(\tilde{H}_{m+1}\) by starting with a Hamilton decomposition of \(H_m \times C_n\) which is repetitive outside \(A_m \times C_n\) and showing how to modify it to match the modifications of \(\Gamma\). The resulting cycles \(\tilde{H}_m\) and \(\tilde{H}_{m+1}\) have a technical property we will use below. If \(e\) is any edge of \(H_m - A_m\), then both \(\tilde{H}_m\) and \(\tilde{H}_{m+1}\) contain edges of
the form $e \times \{k\}$ for some $k$. This property follows immediately from the Hamilton decomposition of $H_m \times C_n$ chosen in Figs. 2 and 7 to start the construction in [4].

For $H_1$, choose the 2-path $A_1$ in the stretch of vertices of length at least four between the last directed path and the first directed path of the even data. Note that the resulting cycle $\tilde{H}_1$ has $n$ copies of every even directed path from $H_1$. Further all but one of the separations between these directed paths consist of an even number of vertices of $\tilde{H}_1$. The one exception occurs only if $|V(G)|$ and $n$ are odd. In this case, it is between the last directed path in $H_1 \times \{n\}$ and the first directed path in $H_1 \times \{1\}$. Thus any subset of these directed paths will have at most one odd separation between them.

For $H_k$, $2 \leq k \leq m$, we must choose a 2-path $A_k$ not containing the edge $e_k$ and vertex-disjoint from the previously selected 2-paths. The $(k-1)$ vertices of $A_1, \ldots, A_{k-1}$ block at most $9(k-1)$ 2-paths and $e_k$ blocks exactly two 2-paths. Since $|V(G)| \geq 9m - 6 > 9(k-1) + 2$ it must be possible to choose an $A_k$ meeting these criteria. For $2 \leq k \leq m-1$ we get cycles $\tilde{H}_k$ and the edges $e_k \times (1)$ provide even data for these cycles. For $H_m$ we get two Hamilton cycles, $\tilde{H}_m$ and $\tilde{H}_{m+1}$, both of which contain edges of the form $e_m \times \{k\}$. Choosing $e_m \times \{1\}$ for one of these Hamilton cycles and $e_m \times \{k_0\}$ for the other gives us a collection of even directed paths with (all but at most one) even separation. The only condition that remains to be checked is that we need a sufficiently large separation between the first and last directed paths. If $|V(G \times C_n)|$ is even, then all separations are even and we need only show that at least one of them contains four vertices of $\tilde{H}_1$. But this is clear since the directed paths occur in only two horizontal levels $H_1 \times \{1\}$ and $H_1 \times \{k_0\}$. If $|V(G \times C_n)|$ is odd, then we must show that the one odd separation contains at least 5 vertices of $\tilde{H}_1$. But this odd separation is between the last directed path in $H_1 \times \{n\}$ and the first directed path in $H_1 \times \{1\}$. This separation is longer than the separation between the last and first directed paths in $H_1$, hence has length at least 5. \qed

Corollary 4.2.1. If $m \geq 2$ and $n_1, \ldots, n_m \geq 3$, then $C_{n_1} \times \cdots \times C_{n_m}$ has a Hamilton decomposition with even data.

Proof. Without loss we may suppose $n_1 \geq n_2 \geq \cdots \geq n_m$. The corollary follows immediately from Theorems 4.1 and 4.2 and induction, provided only that the condition $|V(G)| \geq 9m - 6$ is always satisfied. The only case where this condition fails is if $G = C_3 \times C_3$. However, for this example it is easy to find a Hamilton decomposition, even data and 2-paths $A_1$ and $A_2$ as required by the proof of Theorem 4.2. (Take $H_1$ to be $\{11, 12, 22, 21, 31, 32, 33, 23, 13, 11\}$ and $H_2$ to be $\{11, 21, 23, 22, 32, 12, 13, 33, 31, 11\}$). Then $e_1 = \{11, 21\}$, $A_1 = \{33, 23, 13\}$, and $A_2 = \{22, 32, 12\}$ works.) \qed

In fact with a little more care, one can show that $C_{n_1} \times \cdots \times C_{n_m}$ has a Hamilton decomposition with even data in which all of the directed paths have length exactly 4. This allows one to omit the discussion on lengthening tile 3 of Fig. 5 above.

Corollary 4.2.2. (a) If $m \geq 2$ and $n_1, \ldots, n_m \geq 3$, then $\overrightarrow{C_{n_1}} \times \cdots \times \overrightarrow{C_{n_m}} \times \overrightarrow{Q_1}$ has a decomposition into $2m + 1$ directed Hamilton cycles.

(b) If $m \geq 2$, then $\overrightarrow{Q_{2m+1}}$ has a decomposition into $2m + 1$ directed Hamilton cycles.

Proof. For $m \geq 3$, this follows immediately from Corollary 4.2.1 and Theorem 3.1. The only case that requires a little extra thought is $m = 2$, i.e., when $G = C_{n_1}$ is a single cycle. In this case, a Hamilton decomposition of $G$ consists of
Fig. 7. (a) The first cycle, (b) the second cycle, (c) the third cycle, (d) the fourth cycle and (e) the fifth cycle for $C_3 \times C_3 \times Q_1$. 
just $G$ itself. Provided $n_1 \geq 4$ we may regard the empty set as even data and apply the construction above. The only case which is not covered is $n_1 = 3$. For this exceptional case $\overline{C}_3 \times \overline{C}_3 \times \overline{Q}_1$ a Hamilton decomposition is given in Fig. 7. □

References