Subgraph induced by the set of degree 5 vertices in a contraction critically 5-connected graph

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An edge of a 5-connected graph is said to be contractible if the contraction of the edge results in a 5-connected graph. A 5-connected graph with no contractible edge is said to be contraction critically 5-connected. Let $G$ be a contraction critically 5-connected graph and let $H$ be a component of the subgraph induced by the set of degree 5 vertices of $G$. Then it is known that $|V(H)| \geq 4$. We prove that if $|V(H)| = 4$, then $H \cong K_4^*$, where $K_4^*$ stands for the graph obtained from $K_4$ by deleting one edge. Moreover, we show that either $|N_G(V(H))| = 5$ or $|N_G(V(H))| = 6$ and around $H$ there is one of two specified structures called a $K_4^*$-configuration and a split $K_4^*$-configuration.

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1. Introduction

We deal with finite undirected graphs with neither loops nor multiple edges. For a graph $G$, let $V(G)$ and $E(G)$ denote the set of vertices of $G$ and the set of edges of $G$, respectively. For an edge $e$ of $G$, we denote the set of end vertices of $e$ by $V(e)$. Let $V_k(G)$ be the set of vertices of degree $k$. Let $V_{\geq k}(G)$ be the set of vertices of degree greater than or equal to $k$. If there is no ambiguity we write $V_k$ and $V_{\geq k}$ for $V_k(G)$ and $V_{\geq k}(G)$, respectively. We denote the degree of $x \in V(G)$ by $\deg_G(x)$. We denote the minimum degree of $G$ by $\delta(G)$. Let $G[S]$ denote the subgraph induced by $S \subseteq V(G)$. For two graphs $G$ and $H$, we denote the join of $G$ and $H$ by $G + H$. Let $K^*_x$ denote the graph obtained from $K_x$ by deleting one edge. Hence $K^*_4 \cong 2K_1 + K_2$. Let $G$ be a connected graph. A subset $S \subseteq V(G)$ is said to be a cutset of $G$, if $G - S$ is not connected. A cutset $S$ is said to be a $k$-cutset if $|S| = k$.

Let $k$ be an integer such that $k \geq 2$ and let $G$ be a $k$-connected graph. An edge $e$ of $G$ is said to be $k$-contractible if the contraction of the edge results in a $k$-connected graph. An edge which is not $k$-contractible is called a non-$k$-contractible edge. If the contraction of $e \in E(G)$ results in a graph with minimum degree $k - 1$, then $e$ is said to be trivially non-contractible. In other words, $e$ is trivially non-contractible if and only if the end vertices of $e$ have a common neighbor of degree $k$. A $k$-connected graph with no $k$-contractible edge is said to be contraction critically $k$-connected.

It is known that every 3-connected graph of order 5 or more contains a 3-contractible edge (Tutte [9]). The classification of contraction critically 4-connected graphs was obtained by Fontet and, independently, by Martinov.

Theorem A (Fontet [4], Martinov [6]). If $G$ is a 4-connected graph with no 4-contractible edge, then $G$ is either the square of a cycle or the line graph of a cyclically 4-connected 3-regular graph.

Egawa proved the following minimum degree condition for a $k$-connected graph to have a $k$-contractible edge.

Theorem B (Egawa [3]). Let $k \geq 2$ be an integer, and let $G$ be a $k$-connected graph with $\delta(G) \geq \lceil \frac{4k}{3} \rceil$. Then $G$ has a $k$-contractible edge, unless $2 \leq k \leq 3$ and $G$ is isomorphic to $K_{k+1}$.
Kriesell extended Egawa’s Theorem and proved the following degree sum condition for a \( k \)-connected graph to have a \( k \)-contractible edge.

**Theorem C** (Kriesell [5]). Let \( k \geq 2 \) be an integer, and let \( G \) be a non-complete \( k \)-connected graph. If \( \deg_{G}(x) + \deg_{G}(y) \geq \left\lceil \frac{k+1}{2} \right\rceil \) for any pair of distinct vertices \( x, y \) of \( G \), then \( G \) has a \( k \)-contractible edge.

From **Theorem A**, we know that each contraction critically 4-connected graph is 4-regular. When \( k \) is greater than 4, there is a contraction critically \( k \)-connected graph which is not \( k \)-regular. However, from **Theorem B**, we see that the minimum degree of a contraction critically 5-connected graph is 5.

The following theorem says that each contraction critically 5-connected graph has many vertices of degree 5.

**Theorem D** (Su [8]). Let \( G \) be a 5-connected graph which does not have a 5-contractible edge. Then each vertex of \( G \) has at least two neighbors of degree 5 and thus \( G \) has at least \( \frac{5}{2} |V(G)| \) vertices of degree 5.

Recently, we got a local structure theorem of 5-connected graphs. Before we state the theorem, we need to introduce some specified configurations.

Let \( x \) be a vertex of a 5-connected graph. A configuration which consists of two triangles with nothing in common but \( x \) is called an \( x \)-bowtie. Hence, an \( x \)-bowtie is isomorphic to \( 2K_{2} + K_{1} \) whose vertex of degree 4 is \( x \). A \( K_{4}^{−} \) is called a reduced \( x \)-bowtie if one of the vertices of degree 3 is \( x \). If, in each triangle of an \( x \)-bowtie, there is a vertex of degree 5 other than \( x \), then the \( x \)-bowtie is said to be an \( x^{+} \)-bowtie. If a reduced \( x \)-bowtie has at least two vertices of degree 5 other than \( x \), then it is called a reduced \( x^{+} \)-bowtie. Hence, in **Fig. 1**, (1) is an \( x^{+} \)-bowtie if neither \( \{y_{1}, y_{2}\} \cap V_{5}(G) \) nor \( \{z_{1}, z_{2}\} \cap V_{5}(G) \) is empty, and (2) is a reduced \( x^{+} \)-bowtie if \( \{|u_{1}, u_{2}, u_{3}\} \cap V_{5}(G)| \geq 2 \).

Let \( S = \{a_{1}, a_{2}, x, b_{1}, b_{2}\} \) be a 5-cutset of a 5-connected graph \( G \) and let \( A \) be a component of \( G - S \) such that \( V(A) \subset V_{5}(G) \), \( |V(A)| = 4 \) and \( G[V(A)] \cong K_{4}^{−} \), say \( V(A) = \{u_{1}, u_{2}, v_{1}, v_{2}\} \), with edges within \( A \) and between \( V(A) \) and \( S \) exactly as in **Fig. 2**; there may be edges between vertices of \( S \). We call this configuration, \( G[V(A) \cup S], a K_{4}^{−} \)-configuration with center \( x \). Note that \( \{u_{1}, u_{2}, v_{1}, v_{2}\} \subset V_{5}(G) \) and edges in **Fig. 2** other than \( xu_{1} \) and \( xv_{1} \) are all trivially non-contractible. Moreover, we can find two non-trivial 5-cutsets, \( \{u_{1}, u_{2}, x, b_{1}, b_{2}\} \) and \( \{v_{1}, v_{2}, x, a_{1}, a_{2}\} \) which contain \( V(xu_{1}) \) and \( V(xv_{1}) \), respectively. Hence all edges in **Fig. 2** are non-contractible. Finally, we observe that if there is an edge between vertices of \( S \), then it clearly is non-contractible since \( S \) is a 5-cutset of \( G \).

Our local structure theorem of 5-connected graphs is the following.

**Theorem E** (Ando [1]). Let \( x \) be a vertex of a 5-connected graph \( G \) such that \( x \) is incident with no 5-contractible edge and each neighbor of \( x \) is incident with no 5-contractible edge. If \( G \) has neither an \( x^{+} \)-bowtie nor a reduced \( x^{+} \)-bowtie, then \( G \) has a \( K_{4}^{−} \)-configuration with center \( x \).

Since in each configuration of **Theorem E**, \( x \) has at least two neighbors of degree 5, **Theorem D** is an immediate corollary of **Theorem E**.
Recently the lower bound of the number of degree 5 vertices in a contraction critically 5-connected graph has been improved as follows.

**Theorem F** (Qin, Yuan and Su [7]). Every 5-connected graph $G$ with no contractible edge has at least $\frac{4}{9}|V(G)|$ vertices of degree 5.

Concerning the lower bound of the number of degree 5 vertices in a contraction critically 5-connected graph, we pose the following problem.

**Problem.** Determine the smallest constant $c$ so that every 5-connected graph $G$ with no contractible edge has at least $c|V(G)|$ vertices of degree 5.

From **Theorem F** we know that $c \geq \frac{4}{9}$. In Fig. 2, we call $S = \{a_1, a_2, x, b_1, b_2\}$ “the 5-cutset part” of a $K_4^−$-configuration. Take two $K_4^−$-configurations and identify their 5-cutset parts. Join all three vertices $a_1, a_2, x$ and join all three vertices $b_1, b_2, x$. So the 5-cutset part is the $x$-bowtie with the two triangles $a_2a_1x$ and $b_1b_2x$. Then the resulting graph $G$ is contraction critically 5-connected. Since $|V(G)| = 13$ and $|V_5(G)| = 8$, this graph shows that $\frac{8}{13} \geq c$.

To solve the problem, we need to investigate more detailed structure of contraction critically 5-connected graphs. In this paper we prove some result concerning the structure of the subgraph induced by the set of degree 5 vertices of a contraction critically 5-connected graph.

Let $G$ be a contraction critically 5-connected graph. It was shown that for any given graph, there is a contraction critically 5-connected graph $G'$ such that $G'[V_{=5}]$ is isomorphic to that given graph ([2]). Hence, in this sense, there is no restriction on the subgraph induced by the set of more than 5 degree vertices, $G[V_{=5}] = G - V_5$.

From **Theorem F**, we know that a contraction critically 5-connected graph $G$ has many vertices of degree 5. Let $G_S$ be the subgraph of $G$ induced by the set of degree 5 vertices, that is, $G_S = G[V_5]$. Let $H$ be a component of $G_S$. Then it was shown that $|V(H)| \geq 4$ [7]. We prove that if $|V(H)| = 4$, then $H \cong K_4^−$ and there is one of two specified configurations around $H$ in $G$. Before we state the result, we need to introduce one more specified configuration in 5-connected graphs. Let $S = \{a_1, a_2, x_1, x_2, b_1, b_2\}$ be a 6-cutset of a 5-connected graph $G$ and let $A$ be a component of $G - S$ such that $V(A) \subset V_5(G)$, $|V(A)| = 4$ and $G[V(A)] \cong K_4^−$, say $V(A) = \{u_1, u_2, v_1, v_2\}$, with edges within $A$ and between $V(A)$ and $S$ exactly as in Fig. 3; there may be edges between vertices of $S$. We call this configuration, $G[V(A) \cup S]$, a split $K_4^−$-configuration.

Now we can state our result.

**Theorem 1.** Let $G$ be a contraction critically 5-connected graph. Let $H$ be a component of the subgraph $G[V_5]$. If $|V(H)| = 4$, then $H \cong K_4^−$ and there is either a $K_4^−$-configuration or a split $K_4^−$-configuration which has $H$ as its $K_4^−$-part.

The organization of the paper is as follows. Section 2 contains preliminary results. In Section 3, we give a proof of **Theorem 1**.

2. Preliminaries

In this section we give some more definitions and prove preliminary results.

For a graph $G$, we write $|G|$ for $|V(G)|$. For a subset $S \subset V(G)$, let $N_G(S) = \cup_{x \in S} N_G(x) - S$. For subgraphs $A$ and $B$ of a graph $G$, when there is no ambiguity, we write simply $A$ for $V(A)$ and $B$ for $V(B)$. So $N_G(A)$ and $A \cap B$ mean $N_G(V(A))$ and $V(A) \cap V(B)$, respectively. Also for a subgraph $A$ of $G$ and a subset $S$ of $V(G)$, we write $A \cap S$ and $A \cup S$ for $V(A) \cap S$ and $V(A) \cup S$, respectively. For $S \subset V(G)$, let $G - S$ denote the graph obtained from $G$ by deleting the vertices in $S$ together with the edges incident with them; thus $G - S = G[V(G) - S]$. When there is no ambiguity, we write $E(S)$ for $E(G[S])$. For subsets $S$ and $T$
of \( V(G) \), we denote by \( E_C(S, T) \) the set of edges between \( S \) and \( T \). Namely, \( E_C(S, T) = \{xy \mid x \in S, y \in T\} \). If \( S = \{x\} \), then we simply write \( E_C(x, T) \) instead of \( E_C(\{x\}, T) \).

From now on through this paper, we concern only 5-connected graphs. A subgraph \( A \) of a 5-connected graph \( G \) is called a fragment if \( |N_C(A)| = 5 \) and \( V(G) - (A \cup N_C(A)) \neq \emptyset \). In other words, a fragment \( A \) is a non-empty union of components of \( G - S \) where \( S \) is a 5-cutset of \( G \) such that \( V(G) - (A \cup S) \neq \emptyset \). For a fragment \( A \) of \( G \), we let \( \bar{A} = G - N_C(A) - A \). Then we observe that if \( \bar{A} \) is a fragment of \( G \), then \( \bar{A} \) is also a fragment of \( G \).

For an edge \( e \) of \( G \), a fragment \( A \) is said to be a fragment with respect to \( e \) if \( V(e) \subset N_C(A) \). For \( F \subset E(G) \), \( A \) is said to be a fragment with respect to \( F \) if \( A \) is a fragment with respect to some \( e \in F \). A fragment \( A \) with respect to \( F \) is said to be minimum if there is no fragment \( B \) with respect to \( F \) such that \( |B| < |A| \). A fragment \( A \) with respect to \( F \) is said to be minimal if there is no fragment \( B \) other than \( A \) with respect to \( F \) such that \( B \subset A \).

The following is a simple observation.

**Lemma 2.1.** Let \( A \) be a fragment of a 5-connected graph \( G \). If there is \( S \subset N_C(A) \) such that \( |A \cap N_C(S)| < |S| \), then \( A = A \cap N_C(S) \). In particular, if there are two vertices \( x, y \in N_C(A) \) such that \( N_C(x, y) \cap A = \{z\} \), then \( A = \{z\} \).

**Proof.** Assume that \( A \neq A \cap N_C(S) \), which means \( A - A \cap N_C(S) \neq \emptyset \). Then, we observe that \((N_C(A) - S) \cup (A \cap N_C(S))\) separates \( A - A \cap N_C(S) \) from \( A \cup S \). Since \( |A \cap N_C(S)| < |S| \), we also observe that \( |(N_C(A) - S) \cup (A \cap N_C(S))| = |N_C(A)| - |S| + |A \cap N_C(S)| < 5 \). This implies that \( G \) is not 5-connected, which contradicts the assumption that \( G \) is 5-connected, and **Lemma 2.1** is proved. ■

The following lemma states some elementary properties of fragments of a 5-connected graph.

**Lemma 2.2.** Let \( G \) be a 5-connected graph. Let \( A \) and \( B \) be fragments of \( G \) and let \( S = N_C(A) \) and \( T = N_C(B) \).

| \( B \) | \( \bar{A} \cap B \) | \( S \cap B \) | \( A \cap B \) |
| \hline
| \( T \) | \( \bar{A} \cap T \) | \( S \cap T \) | \( A \cap T \) |
| \hline
| \( \bar{A} \) | \( \bar{A} \cap \bar{B} \) | \( S \cap \bar{B} \) | \( A \cap \bar{B} \) |

Then the following hold.

1. If \( |A \cap T| > |S \cap B| \), then \( \bar{A} \cap B = \emptyset \).
2. If \( |(A \cap T) \cup (S \cap T) \cup (S \cap \bar{B})| \geq 6 \), then \( \bar{A} \cap B = \emptyset \).
3. If \( |(A \cap T) \cup (S \cap T) \cup (S \cap B)| \geq 6 \) and \( |(A \cap T) \cup (S \cap T) \cup (S \cap B)| \geq 6 \), then \( |\bar{A}| = 1 \).

**Proof.** (1) If \( |A \cap T| > |S \cap B| \), then \( |\bar{A} \cap T| \cup (S \cap T) \cup (S \cap B)| = |T| - |A \cap T| + |S \cap B| < |T| = 5 \). This implies \( \bar{A} \cap B = \emptyset \) since \( G \) is 5-connected.

(2) Assume \( |(A \cap T) \cup (S \cap T) \cup (S \cap \bar{B})| \geq 6 \). Then \( |(A \cap T) \cup (S \cap T) \cup (S \cap B)| = |S| + |T| - |(A \cap T) \cup (S \cap T) \cup (S \cap \bar{B})| \leq 5 + 5 - 6 = 4 \), which implies \( \bar{A} \cap B = \emptyset \) since \( G \) is 5-connected.
Lemma 2.3 [Yuan [10]]. Let $x$ be a vertex of a contraction critically 5-connected graph $G$. Let $A$ be a fragment with respect to $E(x)$ such that $|A| \geq 2$. Then $N_C(x) \cap (N_C(A) \cup A) \cap V_5 \neq \emptyset$.

Proof. Assume $|A| \leq 1$. We show that $S \cap B \neq \emptyset$. Assume $S \cap B = \emptyset$. Then $|S \cap B| = |S \cap B| + |A \cap T| = |S \cap T| + |A \cap T| \leq |T| = 5$. Hence, if $A \cap T \neq \emptyset$, then $A \cap T$ is a fragment with respect to $E(x)$ since $x, y \in (S \cap T) \cup (A \cap T)$. Since $y \in A \cap T$, we observe that $A \cap B = \emptyset$, which contradicts the minimality of $A$. This contradiction shows $A \cap T = \emptyset$. On the other hand, since $|A \cap T| \geq 1$, we know that $S \cap B \neq A \cap T$. Hence, Lemma 2.2 assures us that $A \cap T = \emptyset$. Now we know that $B = (A \cap T) \cup (S \cap B) \cup (A \cap B) = \emptyset$, which contradicts the choice of $B$. This contradiction proves $S \cap B \neq \emptyset$. Since $|B| \leq 1$, we observe that $|B| = |S \cap B| = 1$, say $B = S \cap B = \{z\}$. Then $z \in N_C(x) \cap S \cap V_5$, which contradicts the assumption. This contradiction proves Claim 2.3.1.

If both $|S \cap B| \leq 6$ and $|S \cap \bar{B}| \cap (A \cap T) \geq 6$, then Lemma 2.2 assures us that $|A| \leq 1$, which contradicts the assumption. Hence, without loss of generality we may suppose that $|S \cap B| \leq 5$, which implies $A \cap B = \emptyset$ since $A$ is minimal.

Claim 2.3.2. $A \cap \bar{B} = \emptyset$.

Proof. Assume $A \cap \bar{B} \neq \emptyset$. Then, since $A$ is minimal, we know that $|S \cap \bar{B}| \cup (S \cap T) \cup (A \cap T)| \geq 6$. Thus Lemma 2.2 assures us that $A \cap \bar{B} = \emptyset$. Now we know $A \cap B = \emptyset$ and $A \cap B = \emptyset$, which means $B \cap S = \emptyset$. By Claim 2.3.1, we see that $|B| = |S \cap B| \geq 2$. We show that $A \cap T \leq 1$. Then since $|S \cap B| \geq 2$, we have $A \cap T \leq |S \cap B|$, which implies $A \cap B = \emptyset$. Thus $A = A \cap T$ and $|A| = |A \cap T| \leq 1$, which contradicts the assumption. This is shown that $|A \cap T| \geq 2$. Then, we observe that $|S \cap B| \leq 6$, and $|S \cap \bar{B}| \cap (A \cap T) \geq 6$, then Lemma 2.2 assures us that $|A| \leq 1$, which contradicts the assumption. This contradiction proves Claim 2.3.2.

Claim 2.3.3. $(1) \ |S \cap B| = |S \cap \bar{B}| = 2$, and $S \cap T = \{x\}$, (2) $|A| = 2$.

Proof. (1) Assume $|S \cap B| \leq 1$. By Claim 2.3.2, we know that $A = A \cap T$. Hence $|A \cap T| = |A| \geq 2$. Then, since $|S \cap B| < |A \cap T|$, we have $A \cap B = \emptyset$, which implies $B = S \cap \bar{B}$ and $|B| = |S \cap B| = 1$. This contradicts Claim 2.3.1 and it is shown that $|S \cap B| \geq 2$.

By symmetry, we have $|S \cap B| \geq 2$. Since |S| = 5 and $S \cap T \neq \emptyset$, we can conclude that $|S \cap B| = |S \cap B| = 2$, and $S \cap T = \{x\}$.

(2) Assume $A \cap T = |A| \geq 3$. Then, we observe that both $|S \cap B| \cup (S \cap T) \cup (A \cap T) \geq 6$ and $(S \cap \bar{B}) \cup (S \cap T) \cup (A \cap T) \geq 6$, and hence Lemma 2.2 assures us that $|A| \leq 1$, which contradicts the assumption. This contradiction proves Claim 2.3.3.

By Claims 2.3.2 and 2.3.3, we know that $A = A \cap T$ and $|A| = 2$, say $A = A \cap T = \{y, z\}$. Moreover, by Claim 2.3.3 we know $|S \cap B| = |S \cap B| = 2$, say $S \cap B = \{w, v\}$ and $S \cap B = \{w', v'\}$. Recall $xw \in E(G)$. We show $vz \in E(G)$. Since $A = A \cap T = \{y, z\}$, then $x \notin N_C(z)$, and $x \notin E(G)$. This implies $S \cap B = \emptyset$. Hence, $S \cap B = \emptyset$. Now we observe that $E(G)$ is a fragment with respect to $vz$ and let $R = N_C(C)$.

Claim 2.3.4. (1) $y \in A \cap R$, (2) $|S \cap C| = |S \cap \bar{C}| \leq 2$ and $S \cap R = \{v\}$.

Proof. (1) Assume $y \notin A \cap R$, then without loss of generality we may suppose $y \in A \cap C$. Then $A \cap C = \emptyset$, since $A = \{y, z\}$. Since $y \in V_5 \cap 6$ and $|A| = 2$, we see that $N_C(y) = S \cup \{x\}$, which implies that $S \cap C = \emptyset$. Then, since $z \in A \cap R$, we know that $|S \cap C| < |A \cap R|$, which implies $A \cap C = \emptyset$. Now we have $C = \emptyset$, which contradicts the choice of $C$. This contradiction proves $y \in A \cap R$.

(2) Assume $|S \cap C| \leq 1$. Then since $|S \cap C| < |A \cap R| = 2$, Lemma 2.2(1) assures us $A \cap C = \emptyset$, which implies that $|C| = |S \cap C| = 1$. Since $S \cap C \subseteq S \cap N_C(x)$, either $S \cap C = \{w\}$ or $S \subseteq \{x\}$. If $S \subseteq \{w\}$, then $w \in N_C(x) \cap S \cap V_5$, which contradicts the assumption. Hence $S \cap C = \{x\}$, which implies $x \in V_5$ and $[y, z, v, w, u] \subset N_C(x)$. Let $N_C(x) = \{y, z, v, w, u\}$. Then, since neither $N_C(x) \cap A$ nor $N_C(x) \cap B$ is empty, $u \in A \cap B$. Hence $N_C(x) \cap (A \cap B) = \emptyset$, which implies $A \cap B = \emptyset$ since $|S \cap B| \cup (S \cap T) \cup (A \cap T) = \{x, v, w\} \cup [x ; y, z] = 5$. Hence $B = S \cap \bar{B} = \{w, v\}$. Since $[y, z] \subset N_C(x)$ and $N_C(x) \cap \cap V_5 = \emptyset$, we know that $y, z \in V_6$, $yz \in E(G)$ and $N_C(y)$. $N_C(z) \supset S$. Furthermore, since $[w, v] \subset N_C(x)$ and $N_C(x) \cap S \cap V_5 = \emptyset$, we know
Lemma 2.3. By is proved. Claim 2.4.3 assures us that \( S \cap C \geq 2 \). Hence by symmetric, we observe that \( |S \cap C| = |S| = 2 \) and \( S \cap T = \{ u \} \).

Now we are in a position to complete the proof of Lemma 2.3. Claim 2.3.4 assures us that \( S \cap R = \{ v \} \), which means that \( x \notin S \cap R \). Hence, without loss of generality, we may suppose that \( x \in S \cap R \). Then since \( xw \in E(G) \) and \( |S \cap C| = |S| = 2 \), we have \( S \cap C = \{ x, w \} \) and \( S \cap R = \{ v \} \). We observe that \( v \in R \cap C \). Since \( B \cap C \neq \emptyset \), \( (T \cap C) \cup (T \cap R) \cup (B \cap R) \geq 5 \), which implies \( (T \cap C) \cup (T \cap R) \cup (B \cap R) \leq 5 \). Since \( B \cap C \neq \emptyset \), we see that \( (T \cap C) \cup (T \cap R) \cup (B \cap R) = 5 \) and hence \( B \cap C \) is a fragment of \( G \). Now we observe that \( N_C((y, z)) = S = \{ x, w, v, v', v'' \} \), since \( w, v \in B \cap C \) and \( x, v \in T \cap R \). Since \( N_C((y, z)) \cap (B \cap C) = \{ w \} \), Lemma 2.1 assures us that \( B \cap C = \{ w \} \), which means \( w \in V_5 \). Hence \( w \in N_C(x) \cap S \cap V_5 \), which contradicts the assumption. This is the final contraction and the proof of Lemma 2.3 is completed.

In the situation of Lemma 2.3, if \( x \) has no neighbors of degree 5 in the fragment \( A \), then we have a stronger conclusion as follows.

Lemma 2.4. Let \( x \) be a vertex of a contraction critically 5-connected graph \( G \). Let \( A \) be a fragment with respect to \( E(x) \) such that \( |A|, |\tilde{A}| \geq 2 \) and \( N_C(x) \cap A \cap V_5 = \emptyset \). Then there is a vertex \( z \) such that (1) \( z \in N_C(x) \cap N_C(A) \cap V_5 \), (2) \( N_C(x) \cap N_C(z) \cap A = \emptyset \) and (3) \( |N_C(z) \cap A| \geq 2 \).

Proof. Let \( A \) be a fragment with respect to \( E(x) \) such that \( |A|, |\tilde{A}| \geq 2 \) and \( N_C(x) \cap A \cap V_5 = \emptyset \). If \( A' \subset A \), then also \( |\tilde{A}'| \geq 2 \) and \( N_C(x) \cap A' \cap V_5 = \emptyset \). Hence we may assume that \( A \) is a minimal fragment with respect to \( E(x) \) such that \( |A| \geq 2 \). Let \( S = N_C(A) \). We call a vertex \( z \in N_C(x) \cap S \) desirable if \( z \in V_5 \), \( N_C(x) \cap N_C(z) \cap A = \emptyset \) and \( |N_C(z) \cap A| \geq 2 \). By way of contradiction, assume that there is no desirable vertex.

Claim 2.4.1. \( |A| \geq 3 \).

Proof. Let \( A \) be a fragment with respect to \( E(x) \) such that \( |A| = 2 \), say \( A = \{ y, y' \} \). Since \( N_C(x) \cap A \cap V_5 = \emptyset \), Lemma 2.3 assures us that \( N_C(x) \cap \tilde{A} \cap V_5 = \emptyset \) say \( z \in N_C(x) \cap \tilde{A} \cap V_5 \). If \( y \notin N_C(z) \), then \( y \in N_C(x) \cap V_5 \), which contradicts the assumption that \( N_C(x) \cap \tilde{A} \cap V_5 = \emptyset \). Hence \( y \in N_C(z) \). Similarly we have \( y' \in N_C(z) \). Hence \( y, y' \subset N_C(z) \), which means \( z \) is a desirable vertex, a contradiction. 

Let \( y \in N_C(x) \cap A \). Let \( B \) be a fragment with respect to \( xy \) and let \( T = N_C(B) \).

Claim 2.4.2. Either \( A \cap B = \emptyset \) or \( A \cap \tilde{B} = \emptyset \).

Proof. Suppose neither \( A \cap B = \emptyset \) nor \( A \cap \tilde{B} = \emptyset \). Since \( \{ x, y \} \in (S \cap T) \cup (A \cap T) \) and \( A \) is minimal, either \( |S \cap B| \cup (S \cap T) \cup (A \cap T) | 

\geq 6 \) or \( |A \cap B| \leq 1 \). If \( |S \cap B| \cup (S \cap T) \cup (A \cap T) \leq 5 \), then \( |A \cap B| \leq 1 \), say \( A \cap B = \{ y' \} \), then \( y' \in N_C(x) \cap V_5 \), which contradicts the assumption that \( N_C(x) \cap A \cap V_5 = \emptyset \). Hence \( y' \in N_C(z) \). Similarly we have \( y' \in N_C(z) \). Hence \( y, y' \subset N_C(z) \), which means \( z \) is a desirable vertex, a contradiction. 

Claim 2.4.3. \( A \cap \tilde{B} \neq \emptyset \).

Proof. Suppose \( A \cap \tilde{B} = \emptyset \), then \( A = A \cap T \). From Claim 2.4.1, we know that \( |A \cap T| = |A| \geq 3 \). By Lemma 2.2(1), neither \( S \cap B \) nor \( S \cap \tilde{B} \) can be empty, we show that either \( |S \cap B| = 1 \) or \( |S \cap \tilde{B}| = 1 \). Suppose neither \( |S \cap B| = 1 \) nor \( |S \cap \tilde{B}| = 1 \). Then, since \( |S| = 5 \) and \( S \cap T \neq \emptyset \), \( |S \cap B| = |S \cap \tilde{B}| = 2 \). Now we observe both \( |S \cap B| = 1 \) or \( |S \cap \tilde{B}| = 1 \). We may assume that \( |S \cap B| = 1 \), say \( S \cap B = \{ z \} \). Then, since \( |S \cap B| < |A \cap T| \), Lemma 2.2 assures us that \( A \cap B = \emptyset \), which means \( B = S \cap B = \{ z \} \). Since \( y \in N_C(x) \cap N_C(z) \cap A \cap T \subset N_C(z) \) and \( |A \cap T| \geq 3 \), \( z \) is a desirable vertex, which contradicts the assumption. This contradiction proves Claim 2.4.3.

Claim 2.4.4. (1) \( |S \cap B| \cup (S \cap T) \cup (A \cap T) | \geq 6 \), (2) \( |S \cap B| = 1 \).

Proof. (1) By Claim 2.4.3, we have \( A \cap \tilde{B} \neq \emptyset \), which implies \( |S \cap B| \cup (S \cap T) \cup (A \cap T) | \geq 5 \). Assume \( |S \cap B| \cup (S \cap T) \cup (A \cap T) | = 5 \). Then the minimality of \( A \) assures us that \( |A \cap B| = 1 \), say \( A \cap B = \{ y' \} \). Then \( y' \in N_C(x) \cap V_5 \), which contradicts the assumption that \( N_C(x) \cap A \cap V_5 = \emptyset \).

(2) Suppose \( |S \cap B| \geq 2 \). Then \( |S \cap B| \cup (S \cap T) \cup (A \cap T) | \geq 6 \). By this together with (1), Lemma 2.2 assures us \( |A| = |A \cap T| \leq 1 \), which contradicts the assumption \( |A| \geq 2 \) and Claim 2.4.4 is proved.

Now it is shown \( |S \cap B| = 1 \), say \( S \cap B = \{ z \} \). Since \( |S \cap B| \cup (S \cap T) \cup (A \cap T) | \geq 6 \), Lemma 2.2 assures us that \( B = S \cap B \) and \( |A \cap T| \geq |S \cap B| + 1 = 2 \). Since \( y \in N_C(x) \cap N_C(z) \cap A \), \( N_C(z) \cap A = |A \cap T| \geq 2 \), we know that \( z \) is a desirable vertex, which contradicts the assumption. This is the final contradiction and the proof of Lemma 2.4 is completed.
3. A proof of Theorem 1

In this section we give a proof of Theorem 1. Let $G$ be a contraction critically 5-connected graph and let $H$ be a component of $G[V_S]$. Then it is known that the minimum degree of $H$ is at least 2 [8,1]. Moreover, it is shown that the maximum degree of $H$ is at least 3 [7]. Hence $|V(H)|$ is at least 4 and if $|V(H)| = 4$, then either $H \cong K_4$ or $H \cong K_5$. Let $H$ be a component of $G[V_S]$ such that $|V(H)| = 4$. A vertex $x \in V(H)$ is called a proper vertex if $H - x \cong K_3$. Thus if $H \cong K_4$, then all four vertices are proper, and if $H \cong K_5$, then two vertices are proper and the other two are not proper. For a proper vertex of $x \in V(H)$, we observe that if $H \cong K_4$, then $|N_G(x) \setminus V(H)| = |E_G(x, V(G) - V(H))| = 2$ and if $H \cong K_5$, then $|N_G(x) \setminus V(H)| = |E_G(x, V(G) - V(H))| = 3$. Let $A$ be a fragment with respect to $E(x)$. Then, if $x$ is proper then $H - x \cong K_3$, which implies that either $A \cap V(H) = \emptyset$ or $A \cap V(H) = \emptyset$. For a proper vertex of $x \in V(H)$, a fragment $A$ is said to be a proper fragment with respect to $x$ if $(1)$ $A$ is a fragment with respect to $E_G(x, V(G) - V(H))$ and $(2)$ $A \cap V(H) = \emptyset$. A fragment $A$ of $G$ is said to be a proper fragment with respect to $H$ if $A$ is a proper fragment with respect to some proper vertex of $H$.

Let $x$ be a proper vertex of $H$. Let $A$ be a proper fragment with respect to $x$. Choose $A$ so that $|A|$ to be as small as possible. Let $S = N_C(A)$.

Claim 3.1. $N_C(x) \cap S \cap V(H) \neq \emptyset$.

Proof. If $|A| = 1$, then since $A \cap V(H) = \emptyset$, we have $|S \cap V(H)| \geq 3$, which implies $N_C(x) \cap S \cap V(H) \neq \emptyset$.

Hence suppose $|A| \geq 2$. Then, Lemma 2.2 assures us that there exists a vertex $x' \in N_C(x) \cap A \cap V_S$. Then $x' \in V_S$ and $xx' \in E(G)$, which means that $x' \in V(H)$. Since $A$ is a proper fragment, we know that $A \cap V(H) = \emptyset$, which implies that $x' \in S$. It is shown that $x' \in N_C(x) \cap S \cap V(H)$. Now Claim 3.1 is proved.

Let $y \in N_C(x) \cap A$. Let $B$ be a fragment with respect to $xy$ and let $T = N_C(B)$.

Claim 3.2. If $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| \leq 5$, then $A \cap B = \emptyset$.

Proof. Assume that $A \cap B \neq \emptyset$. Then $(S \cap B) \cup (S \cap T) \cup (A \cap T)$ is a cutset of $G$. Since $G$ is 5-connected, we see that $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| = 5$. Since $A$ is proper, we know that $xy \in E_G(x, V(G) - V(H))$. Then, since $x \in S \subseteq B \cup (S \cap T) \cup (A \cap T)$ and $A \cap B \subseteq A$, we see that $A \cap B$ is a proper fragment with respect to $x$. Furthermore since $y \in A \cap T$, we observe that $|A \cap B| \leq |A|$, which contradicts the minimality of $A$. Now Claim 3.2 is proved.

Claim 3.3. (1) $S \cap B \neq \emptyset$ and $S \cap \overline{B} \neq \emptyset$, (2) $\overline{A} \cap T \neq \emptyset$, (3) $|A \cap T| \geq 2$.

Proof. (1) Assume that $S \cap B = \emptyset$. Then, since $|S \cap B| < |A \cap T|$, Lemma 2.2 assures us that $\overline{A} \cap B = \emptyset$. Since $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| = |(S \cap B) \cup (S \cap T) \cup (A \cap T)| \leq |T| = 5$, Claim 3.2 assures us that $A \cap B = \emptyset$. Now we have $B = \emptyset$, which contradicts the choice of $B$. This contradiction shows that $S \cap B \neq \emptyset$. By symmetry, we have $S \cap \overline{B} \neq \emptyset$.

(2) Assume that $\overline{A} \cap T = \emptyset$. Then, since $|A \cap T| < |S \cap B|$, we know that $|A \cap T| < |S \cap B|$. Hence Lemma 2.2 assures us that $\overline{A} \cap B = \emptyset$. Similarily, since $|A \cap T| < |S \cap B|$, we have $A \cap B = \emptyset$, which implies that $\overline{A} = \emptyset$ contradicting the choice of $A$. This contradiction shows that $\overline{A} \cap T \neq \emptyset$.

(3) Since $y \in A \cap T$, $|A \cap T| \geq 1$. Assume that $|A \cap T| = 1$. Then, since $S \cap B \neq \emptyset$, we observe that $|A \cap T| \leq |S \cap B|$, which implies that $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| \leq 5$. Hence, Claim 3.2 assures us that $A \cap B = \emptyset$. By similar argument, we have $A \cap B = \emptyset$. Hence we have $A = A \cap T$ and $|A| = |A \cap T| = 1$, which means that $A = \{y\}$ and $y \in V_S$. Then, since $y \in V_S$ and $xy \in E(G)$, we have $y \in V(H)$, which contradicts the choice of $A$ to be a proper fragment. This contradiction shows that $|A \cap T| \geq 2$ and now Claim 3.3 is proved.

Claim 3.4. (1) If $|S \cap B| = 1$, then $B = S \cap B$ and $N_C(x) \cap B \cap V(H) \neq \emptyset$.

(2) If neither $|S \cap B| = 1$ nor $|S \cap B| = 2$ and $S \cap T = \{x\}$.

Proof. (1) Suppose $|S \cap B| = 1$, say $S \cap B = \{x'\}$. By Claim 3.3(3), we know that $|A \cap T| \geq 2$. Then, since $|S \cap B| < |A \cap T|$, Lemma 2.2 assures us that $A \cap B = \emptyset$. By Claim 3.2(2), we know that $|A \cap T| \geq 1$. Then, since $|S \cap B| < |A \cap T|$, we have $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| \leq 5$. Hence, by Claim 3.2, we have $A \cap B = \emptyset$. Now we have $A \cap B = A \cap B = \emptyset$, which means $B = S \cap B = \{x'\}$. Then, since $x' \in V_S$ and $xx' \in E(G)$, we see that $x' \in V(H)$. Now (1) is proved.

(2) Assume that neither $|S \cap B| = 1$ nor $|S \cap B| = 1$. From Claim 3.3(1), we also have $|S \cap B| \geq 1$ and $|S \cap \overline{B}| \geq 1$, which implies that $|S \cap B| \geq 2$ and $|S \cap \overline{B}| \geq 2$. Since $|S| = 5$ and $S \cap T \neq \emptyset$, we have $|S \cap B| = |S \cap \overline{B}| = 2$ and $S \cap T = \{x\}$ and Claim 3.4 is proved.

Claim 3.5. $|A \cap T| = 2$.

Proof. Assume that $|A \cap T| \neq 2$. By Claim 3.3(2) and (3), we have $\overline{A} \cap T \neq \emptyset$ and $|A \cap T| \geq 2$, which implies $|\overline{A} \cap T| = |A \cap T| = 1$ and $|A \cap T| = 3$.

We show $|S \cap B| \neq 1$. Assume $|S \cap B| = 1$, say $S \cap B = \{x'\}$. Then Claim 3.4 assures us that $B = S \cap B = \{x'\}$ and $x' \in V(H)$. Since $x$ is a proper vertex, we see that $\overline{B} \cap V(H) = \emptyset$. Hence we have $V(H) \subseteq V(G) - (A \cup \overline{B})$. Since $A \cap B = \emptyset$, $|V(G) - (A \cup \overline{B})| = |\overline{A} \cap T| + |A \cap T| + |S \cap B| = 1 + 1 + 1 = 3$, which contradicts the fact that $|V(H)| = 4$. This contradiction shows that $|S \cap B| \neq 1$. By symmetry, we have $|S \cap B| \neq 1$. 

Now we know that neither $|S \cap B| = 1$ nor $|\bar{S} \cap \bar{B}| = 1$. Then from Claim 3.3(2), we know that $|S \cap B| = |\bar{S} \cap \bar{B}| = 2$ and $S \cap T = \{x\}$. Since $x$ is a proper vertex of $H$ and $B$ is a fragment with respect to $xy$, we observe that either $B$ or $\bar{B}$ is a proper fragment with respect to $x$. Without loss of generality, we may suppose that $B$ is a proper fragment with respect to $x$. Since $|S \cap B| = 2$ and $|A \cap T| \geq 3$, we know that $|S \cap B| < |A \cap T|$, which implies that $A \cap \bar{B} = \emptyset$. Thus we have $|B| = |S \cap B| + |A \cap \bar{B}| = |A| - |A \cap T| + |A \cap \bar{B}| = |A|$, which contradicts the minimality of $|A|$. This contradiction proves Claim 3.5.

Claim 3.6. $|N_C(x) \cap A| \neq 1$.

Proof. Assume that $N_C(x) \cap A = \{y\}$. By Claim 3.5, we know $|A \cap T| = 2$, say $A \cap T = \{y, z\}$. Since $N_C(x) \cap A = \{y\}$, note that $xz \notin E(G)$.

At first we show $|A| \geq 3$. Assume $|A| = 2$. Then, $A = A \cap T = \{y, z\}$. Since $|A| = 2$ and $xz \notin E(G)$, we know that $N_C(z) = (S - \{x\}) \cup \{y\}$ and hence $z \in V_S$. Claim 3.1 assures us that $S \cap V(H)$ has a vertex $x'$ other than $x$. Thus $x' \in V(H)$, $x'z \in E(G)$ and $z \in V_S$, which implies that $z \in V(H)$. This contradicts the fact that $A$ is a proper fragment and it is shown that $|A| \geq 3$.

Next we show that either $|S \cap B| = 1$ or $|S \cap \bar{B}| = 1$. Assume neither $|S \cap B| = 1$ nor $|S \cap \bar{B}| = 1$. Then, by Claim 3.4, we have $|S \cap B| = |S \cap \bar{B}| = 2$ and $S \cap T = \{x\}$. Therefore $|S \cap B| \cup (S \cap T) \cup (A \cap T) = 2 + 1 + 2 = 5$. Hence, by Claim 3.2, we have $A \cap B = \emptyset$. Similarly, we also have $A \cap \bar{B} = \emptyset$. Hence $|A| = |A \cap T| = 2$, which contradicts the previous assertion that $|A| \geq 3$. Now it is shown either $|S \cap B| = 1$ or $|S \cap \bar{B}| = 1$.

Without loss of generality, we may suppose that $|S \cap B| = 1$, say $S \cap B = \{x'\}$. Then Claim 3.4 assures us that $B = S \cap B = \{x'\}$ and $x' \in V(H)$. Since $x$ is a proper vertex and $|x, y| \subseteq T$, either $B$ or $\bar{B}$ is proper. The fact $B \cap V(H) \neq \emptyset$ forces $B$ is a proper fragment of $H$, which implies $V(H) \subseteq (A \cap T) \cup (S \cap T) \cup (S \cap B)$. Since, then $(|A \cap T| \cup (S \cap T) \cup (S \cap B)) = |T| - |A \cap T| + |S \cap B| = 5 - 2 + 1 + 4 = 4$, we see that $V(H) = (A \cap T) \cup (S \cap T) \cup (S \cap B)$. By Claim 3.2(2), we know that $A \cap T \neq \emptyset$, which implies $|S \cap V(H)| = (|S \cap T| \cup (S \cap B)) = |V(H)| - |A \cap T| \leq 4 - 1 = 3$. Let $A' = A - \{y\}$ and $S' = N_C(A') = (S - \{x\}) \cup \{y\}$. Then, since $e \in V(H)$ and $y \notin V(H)$, $|S'| \cap V(H) = (x', x'' \in V(H))$, which implies $S' \cap V(H) = \{x', x''\}$. Since $x', x'' \in E(G)$, $A'$ is also a fragment with respect to $E'$ such that $A', A' \geq 2$. Moreover, since $A' \cap V(H) = \emptyset$ and $x', x'' \in V(H)$ we have $N_C(x') \cap A' \cap V_S = \emptyset$. Applying Lemma 2.4 with the roles $x$ and $A$ replaced by $x'$ and $A'$, respectively, we see that there is a vertex $w$ such that $w \in N_C(x') \cap S' \cap V_S$ and $|N_C(w) \cap A'| \geq 2$. Since $w \in N_C(x') \cap S' \cap V_S$ and $x' \in V(H)$ we have $w \in S' \cap V(H)$. Then, since $S' \cap V(H) = \{x', x''\}$, the fact $w \in S' \cap V(H)$ implies $w = x'$. However, $|N(x') \cap A'| = |A \cap T - \{y\}| = 2 - 1 = 1$, which means $x' \neq w$. This is a contradiction and the proof of Claim 3.6 is completed.

Claim 3.7. $H \not\cong K_4$.

Proof. Assume $H \cong K_4$. Then, since $|E_C(x, V(G) - V(H))| = 2$, we see that $|N_C(x) \cap S| = |N_C(x) \cap A| = 1$, which contradicts Claim 3.6. This contradiction shows that $H \not\cong K_4$.

By Claim 3.7 it is shown that $H \cong K_4^-$. To complete the proof it remains to show that there is either a $K_4$-configuration or a split $K_4$-configuration which has $H$ as its $K_4$-part. Let $V(H) = \{x_1, x_2, x_3, x_4\}$. Suppose $deg_G(x_1) = deg_G(x_2) = 2$ and $deg_G(x_3) = deg_G(x_4) = 3$. Thus $x_1$ and $x_2$ are proper vertices of $H$. Let $A_1$ be a minimum proper fragment with respect to $x_1$ and let $S_1 = N_C(A_1)$. Let $N_C(x_1) \cap (S \cap N) = \{y, y', w\}$. Suppose $w \in N_C(x_1) \cap S$ and $y \in N_C(x_1) \cap A_1$. Let $B_1$ be a fragment with respect to $x_1$ and let $T_1 = N_C(B_1)$. Then, by Claim 3.5, we know that $|A_1 \cap T_1| = 2$, say $A_1 \cap T_1 = \{y, z\}$. By Claim 3.6, we also know that $|N_C(x_1) \cap A_1| = 2$, which means $N_C(x_1) \cap A_1 = \{y, y'\}$.

Claim 3.8. Either $A_1 \cap T_1 = N_C(x_3) \cap A_1$ or $A_1 \cap T_1 = N_C(x_4) \cap A_1$.

Proof. At first we consider the case that neither $|S_1 \cap B_1| = 1$ nor $|\bar{S}_1 \cap \bar{B}_1| = 1$. Claim 3.1 assures us the existence of a vertex $x'$ in $N_C(x_1) \cap S \cap V(H)$. Note that $x' \notin \{x_1, x_2\}$ since $N_C(x_1) \cap V(H) = \{x_1, x_2\}$. In this case, by Claim 3.4, we have $|S_1 \cap B_1| = |S_1 \cap \bar{B}_1| = 2$ and $S_1 \cap T_1 = \{x\}$. Moreover we know that $|A_1 \cap T_1| = |A_1 \cap T_1| = 2$ because Claim 3.5 assures us that $|A_1 \cap T_1| = 2$. Hence, since $|S_1 \cap B_1| \cup (S_1 \cap T_1) \cup (A_1 \cap T_1) = |S_1 \cap \bar{B}_1| \cup (S_1 \cap T_1) \cup (A_1 \cap T_1)| = 2 + 1 + 2 = 5$, Claim 3.2 assures us that $A_1 \cap \bar{B}_1 = A_1 \cap \bar{B}_1 = \emptyset$. Now we know that $A_1 = A_1 \cap T_1 = \{y, z\}$. Since $\{x', y'\} \subseteq V_S$, we see that $N_C(y) \cap \{z\} = N_C(z) \cap \{y\} = S$, which means $yx', zx' \in E(G)$. Since $x' \in \{x_3, x_4\}$, we have either $\{y, z\} = N_C(x_3) \cap A_1$ or $\{y, z\} = N_C(x_4) \cap A_1$.

Next we consider the case that either $|S \cap B| = 1$ or $|\bar{S} \cap \bar{B}| = 1$. Without loss of generality assume $|S \cap B| = 1$, say $S_1 \cap B_1 = \{x\}$. By Claim 3.4, we see that $B_1 = S_1 \cap B_1 = \{x\}$ and $x' \in V(H) \cap N_C(x_1)$. Then $B_1 = \{x\}$ means $N_C(x') = T_1$, which implies $N_C(x') \cap A_1 = A_1 \cap T_1$. Since $x' \in V(H) \cap N_C(x_1) = \{x_3, x_4\}$, we have either $N_C(x_3) \cap A_1 = A_1 \cap T_1$ or $N_C(x_4) \cap A_1 = A_1 \cap T_1$. Now Claim 3.8 is proved.

Claim 3.9. $y' = z$. 

Proof. Assume \( y' \neq z \). From Claim 3.8, we know that either \( A_1 \cap T_1 = N_C(x_3) \cap A_1 \) or \( A_1 \cap T_1 = N_C(x_4) \cap A_1 \). Without loss of generality, we may suppose that \( A_1 \cap T_1 = N_C(x_3) \cap A_1 \). Then \( x_3 \in S, y \in N_C(x_3) \) and \( y' \neq N_C(x_3) \). Let \( B'_1 \) be a fragment with respect to \( x_1,y' \) and let \( T'_1 = N_C(B'_1) \). Then, applying Claim 3.8 with the role \( T_1 \) replaced by \( T'_1 \), we see that either \( A_1 \cap T'_1 = N_C(x_3) \cap A_1 \) or \( A_1 \cap T'_1 = N_C(x_4) \cap A_1 \). Since \( y' \neq N_C(x_3) \), we know that \( A_1 \cap T'_1 \neq N_C(x_3) \cap A_1 \). Hence we have \( A_1 \cap T'_1 = N_C(x_4) \cap A_1 \), which implies \( x_4 \in S \). Now we know that \( x_3, x_4, w \in S \) and \( y, y' \in A_1 \). Since \( N_C(x_1) = \{x_3, x_4, y, y', w\} \), this implies that \( N_C(x_1) \cap A = \emptyset \), which contradicts the choice of \( A \). This contradiction proves Claim 3.9.

Claim 3.10. \( x_2 \in \tilde{A}_1 \).

Proof. Assume \( x_2 \notin \tilde{A}_1 \). Then, since \( A_1 \) is proper, we have \( x_2 \in S_1 \). By Claim 3.1, we know that \( \{x_3, x_4\} \subseteq S_1 \neq \emptyset \), which implies \( |S \cap V(H)| \geq 3 \) since \( x_1, x_2 \in S_1 \). Since \( \{x_1, y\} \subseteq T_1 \), we observe that either \( B_1 \) or \( B_1' \) is a proper fragment with respect to \( x_1 \). Without loss of generality, we suppose \( B_1 \) is a proper fragment with respect to \( x_1 \). Then, since \( B_1 \cap V(H) = \emptyset \) and \( |S \cap V(H)| \geq 3 \), we observe that \( |S \cap B_1| \leq 2 \). Claim 3.5 assures us \( |A_1 \cap T_1| = 2 \). Hence we see that \( |S \cap B_1| \leq |A_1 \cap T_1| \), which implies \( |A_1 \cap T_1| = 2 \). Since \( \{x_3, x_4\} \subseteq B_1 \), we observe that \( N_C(x_1) \cap (A_1 \cap T_1) = \{x_3, x_4, y, y', w\} \cap (A_1 \cap T_1) = \emptyset \). This together with the fact that \( |(A_1 \cap T_1) \cup (S_1 \cap T_1) \cup (S \cap B_1)| \leq 5 \) implies \( A_1 \cap B_1 = \emptyset \). Then, since \( |S \cap B_1| \leq |A_1 \cap T_1| \), we see that \( |B_1| = |S \cap B_1| + |A_1 \cap B_1| = |A_1 \cap T_1| + |A_1 \cap B_1| | \leq |A_1| \). Since \( B_1 \) is a proper fragment with respect to \( x_1 \), the inequality \( |B_1| \leq |A_1| \) and the minimality of \( A_1 \) assure us that \( B_1 \) is also a minimum proper fragment with respect to \( x_1 \). Then, by Claim 3.6, we see that \( |N_C(x_1) \cap B_1| = 2 \). On the other hand Claim 3.9 assures us that \( |y, y'| \subseteq T_1 \). Hence \( |N_C(x_1) \cap B_1| = |y, y'|, w \subseteq B_1| \leq 1 \), which contradicts the previous assertion. This contradiction proves Claim 3.10.

We proceed with the proof of Theorem 1. From Claims 3.8 and 3.9, we have either \( \{y, y'\} \subset N_C(x_3) \) or \( \{y, y'\} \subset N_C(x_4) \). Without loss of generality we may suppose that \( \{y, y'\} \subset N_C(x_3) \). Then \( N_C(x_3) = \{x_3, x_4, y, y', w\} \). Let \( A_2 \) be a minimum proper fragment with respect to \( x_2 \) and let \( S_2 = N_C(A_2) \). Without loss of generality we may assume that \( w' \notin S_2 \). Applying Claim 3.6 with the roles of \( x_1 \) and \( A_1 \) replaced by \( x_2 \) and \( A_2 \), respectively, we see that \( |N(x_2) \cap A_2| = 2 \). Then, since \( w' \notin S_2 \), we have \( N(x_2) \cap A_2 = \{y, z\} \). Again applying Claims 3.8 and 3.9 with the roles of \( x_1 \) and \( A_1 \) replaced by \( x_2 \) and \( A_2 \), respectively, we have either \( \{z, z'\} \subset N_C(x_3) \) or \( \{z, z'\} \subset N_C(x_4) \). Claim 3.10 assures us that \( x_2 \in \tilde{A}_1 \), which means \( N_C(x_2) \cap \{y, y'\} = \emptyset \). Since \( \{z, z'\} \subset N_C(x_3) \), this implies \( \{y, y'\} \cap \{z, z'\} = \emptyset \). Hence, since \( N_C(x_3) = \{x_1, x_2, x_3, y, y', w\} \) and \( \{y, y'\} \cap \{z, z'\} = \emptyset \), we have \( \{z, z'\} \cap N_C(x_3) = \emptyset \), which forces \( \{z, z'\} \subset N_C(x_2) \). Putting all together we have \( N_C(x_3) - V(H) = \{y, y', w\}, N_C(x_3) - V(H) = \{y, y', w\}, N_C(x_3) - V(H) = \{z, z', w\}, N_C(x_3) - V(H) = \{z, z', w\} \) and \( \{y, y'\} \cap \{z, z'\} = \emptyset \). Hence if \( w = w' \), we have a \( K_4 \)-configuration which has \( H \) as its \( K_4 \)-part, and if \( w \neq w' \) we have a split \( K_4 \)-configuration which has \( H \) as its \( K_4 \)-part. Now the proof of Theorem 1 is completed.

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References

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