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Oscillation Properties for Systems of Hyperbolic Differential Equations of Neutral Type[1]

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Sufficient conditions are established for the oscillations of systems of hyperbolic differential equations of the form

$$\frac{\partial^2}{\partial t^2}\left(p(t)u_i(x,t) + \sum_{r=1}^{d} \lambda_r(t)u_i(x,t-\tau_r)\right)$$
$$= a_i(t)\Delta u_i(x,t) + \sum_{j=1}^{m}\sum_{k=1}^{s} a_{ijk}(t)\Delta u_j(x,\rho_k(t))$$
$$- q_i(x,t)u_i(x,t) - \sum_{j=1}^{m}\sum_{h=1}^{l} q_{ijh}(x,t)u_j(x,\sigma_h(t)),$$
$$(x,t) \in \Omega \times [0,\infty) \equiv G, \qquad i = 1,2,\ldots,m,$$

where Ω is a bounded domain in R^n with a piecewise smooth boundary $\partial\Omega$, and Δ is the Laplacian in Euclidean n-space R^n. © 2000 Academic Press

1. INTRODUCTION

Oscillation theory of partial functional differential equations has been studied extensively for the past few years. For example, see [1–6] and the references therein. However, only [7–9] have been published on the oscillation theory of systems of partial functional differential equations.

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In this paper, we study the oscillation of systems of hyperbolic differential equations of neutral type of the form

$$\frac{\partial^2}{\partial t^2}\left(p(t)u_i(x,t) + \sum_{r=1}^{d} \lambda_r(t)u_i(x,t-\tau_r)\right)$$
$$= a_i(t)\Delta u_i(x,t) + \sum_{j=1}^{m}\sum_{k=1}^{s} a_{ijk}(t)\Delta u_j(x,\rho_k(t))$$
$$- q_i(x,t)u_i(x,t) - \sum_{j=1}^{m}\sum_{h=1}^{l} q_{ijh}(x,t)u_j(x,\sigma_h(t)),$$
$$(x,t) \in \Omega \times [0,\infty) \equiv G, \qquad i = 1,2,\dots,m, \quad (1)$$

where Ω is a bounded domain in R^n with a piecewise smooth boundary $\partial\Omega$, and $\Delta u_i(x,t) = \sum_{r=1}^{n} \partial^2 u_i(x,t)/\partial x_r^2$, $i = 1,2,\dots,m$.

Suppose that the following conditions hold:

(H1) $p \in C^2([0,\infty);[0,\infty))$, $\lambda_r \in C^2([0,\infty);[0,\infty))$, τ_r are positive constants, $r \in I_d = \{1,2,\dots,d\}$;

(H2) $q_i \in C(\overline{G};[0,\infty))$, $q_i(t) = \min_{x\in\overline{\Omega}} q_i(x,t)$, $q(t) = \min_{1\le i\le m} q_i(t)$, $i \in I_m = \{1,2,\dots,m\}$;

(H3) $q_{ijh} \in C(\overline{G};R)$, $q_{iih}(x,t) > 0$, $q_{iih}(t) = \min_{x\in\overline{\Omega}} q_{iih}(x,t)$, and

$$\bar{q}_{ijh}(t) = \max_{x\in\overline{\Omega}} |q_{ijh}(x,t)|,$$
$$Q_h(t) = \min_{1\le i\le m}\left\{q_{iih}(t) - \sum_{j=1,\,j\ne i}^{m} \bar{q}_{jih}(t)\right\} \ge 0,$$
$$i,j \in I_m, \qquad h \in I_1 = \{1,2,\dots,l\};$$

(H4) $a_i \in C([0,\infty);[0,\infty))$, $a_{ijk} \in C([0,\infty);R)$, $a_{iik}(t) > 0$, and

$$A_k(t) = \min_{1\le i\le m}\left\{a_{iik}(t) - \sum_{j=1,\,j\ne i}^{m} |a_{jik}(t)|\right\} \ge 0,$$
$$i,j \in I_m, \qquad k \in I_s = \{1,2,\dots,s\};$$

(H5) σ_j, $\rho_k \in C([0,\infty);R)$, $\sigma_j(t) \le t$, $\rho_k(t) \le t$, σ_j, ρ_k are nondecreasing functions and $\lim_{t\to\infty} \sigma_j(t) = \lim_{t\to\infty} \rho_k(t) = \infty$, $j \in I_m$, $k \in I_s$.

We consider two kinds of boundary conditions,

$$\frac{\partial u_i(x,t)}{\partial N} + g_i(x,t)u_i(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,\infty), \qquad i \in I_m, \tag{2}$$

where N is the unit exterior normal vector to $\partial\Omega$, $g_i(x,t)$ is a nonnegative continuous function on $\partial\Omega \times [0,\infty)$, $i \in I_m$, and

$$u_i(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,\infty), \qquad i \in I_m. \tag{3}$$

DEFINITION 1.1. The vector function $u(x, t) = \{u_1(x, t), u_2(x, t), \dots, u_m(x,t)\}^T$ is said to be a solution of the problem (1), (2) (or (1), (3)) if it satisfies (1) in $G = \Omega \times [0,\infty)$ and boundary condition (2) (or (3)).

DEFINITION 1.2. A nontrivial component $u_i(x,t)$ of the vector function $u(x,t) = \{u_1(x,t), u_2(x,t), \dots, u_m(x,t)\}^T$ is said to oscillate in $\Omega \times [\mu_0, \infty)$ if for each $\mu > \mu_0$ there is a point $(x_0, t_0) \in \Omega \times [\mu, \infty)$ such that $u_i(x_0, t_0) = 0$.

DEFINITION 1.3. The vector solution $u(x, t) = \{u_1(x, t), u_2(x, t), \dots, u_m(x,t)\}^T$ of the problem (1), (2) (or (1), (3)) is said to be oscillatory in the domain $G = \Omega \times [0,\infty)$ if at least one of its nontrivial components is oscillatory in G. Otherwise, the vector solution $u(x,t)$ is said to be nonoscillatory.

We note that conditions for the oscillation of system (1) for $p(t) = 1$, $\lambda_r(t) = 0$, $s = 1$, $l = 1$ have been obtained in [7].

2. OSCILLATION OF THE PROBLEM (1), (2)

THEOREM 2.1. *Suppose that $p(t) \geq 1$ is a monotone decreasing function, and*

$$\sum_{r=1}^{d} \lambda_r(t) < 1. \tag{4}$$

If there exists some $h_0 \in I_t$ such that $\sigma'_{h_0}(t) \geq 0$, and

$$\int_{t_0}^{\infty} Q_{h_0}(t)\left[1 - \sum_{r=1}^{d} \lambda_r\big(\sigma_{h_0}(t)\big)\right] dt = \infty, \qquad t_0 > 0, \tag{5}$$

then every solution $u(x,t)$ of the problem (1), (2) *is oscillatory in G.*

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t) = \{u_1(x,t), u_2(x,t), \dots, u_m(x,t)\}^T$ of the problem (1), (2). We assume that $|u_i(x,t)| > 0$ for $t \geq t_0 \geq 0$, $i \in I_m$. Let $\delta_i = \operatorname{sgn} u_i(x,t)$,

$Z_i(x,t) = \delta_i u_i(x,t)$; then $Z_i(x,t) > 0$, $(x,t) \in \Omega \times [t_0, \infty)$, $i \in I_m$. From (H1), (H5) there exists a number $t_1 \geq t_0$ such that $Z_i(x,t) > 0$, $Z_i(x, t - \tau_r) > 0$, $Z_i(x, \rho_k(t)) > 0$ and $Z_i(x, \sigma_h(t)) > 0$ in $\Omega \times [t_1, \infty)$, $i \in I_m$, $r \in I_d$, $k \in I_s$, $h \in I_l$.

Integrating (1) with respect to x over the domain Ω, we have

$$\begin{aligned}\frac{d^2}{dt^2}&\left(p(t)\int_\Omega u_i(x,t)\,dx + \sum_{r=1}^{d}\lambda_r(t)\int_\Omega u_i(x,t-\tau_r)\,dx\right)\\ &= a_i(t)\int_\Omega \Delta u_i(x,t)\,dx + \sum_{j=1}^{m}\sum_{k=1}^{s} a_{ijk}(t)\int_\Omega \Delta u_j(x,\rho_k(t))\,dx\\ &\quad -\int_\Omega q_i(x,t)u_i(x,t)\,dx - \sum_{j=1}^{m}\sum_{h=1}^{l}\int_\Omega q_{ijh}(x,t)u_j(x,\sigma_h(t))\,dx,\\ &\qquad t \geq t_1, \quad i \in I_m. \qquad (6)\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d^2}{dt^2}&\left(p(t)\int_\Omega Z_i(x,t)\,dx + \sum_{r=1}^{d}\lambda_r(t)\int_\Omega Z_i(x,t-\tau_r)\,dx\right)\\ &= a_i(t)\int_\Omega \Delta Z_i(x,t)\,dx + \sum_{j=1}^{m}\sum_{k=1}^{s} a_{ijk}(t)\frac{\delta_j}{\delta_i}\int_\Omega \Delta Z_j(x,\rho_k(t))\,dx\\ &\quad -\int_\Omega q_i(x,t)Z_i(x,t)\,dx - \sum_{j=1}^{m}\sum_{h=1}^{l}\frac{\delta_j}{\delta_i}\int_\Omega q_{ijh}(x,t)Z_j(x,\sigma_h(t))\,dx,\\ &\qquad t \geq t_1, \quad i \in I_m. \qquad (7)\end{aligned}$$

From Green's formula and boundary condition (2), it follows that

$$\int_\Omega \Delta Z_i(x,t)\,dx = \int_{\partial\Omega}\frac{\partial Z_i(x,t)}{\partial N}\,dS = -\int_{\partial\Omega} g_i(x,t)Z_i(x,t)\,dS \leq 0 \qquad (8)$$

and

$$\begin{aligned}\int_\Omega \Delta Z_j(x,\rho_k(t))\,dx &= \int_{\partial\Omega}\frac{\partial Z_j(x,\rho_k(t))}{\partial N}\,dS\\ &= -\int_{\partial\Omega} g_j(x,\rho_k(t))Z_j(x,\rho_k(t))\,dS,\\ &\qquad t \geq t_1, \quad j \in I_m, \quad k \in I_s, \qquad (9)\end{aligned}$$

where dS is the surface element on $\partial\Omega$.

Noting conditions (H2) and (H3) and combining (7)–(9), we get

$$\begin{aligned}
&\frac{d^2}{dt^2}\left(p(t)\int_\Omega Z_i(x,t)\,dx + \sum_{r=1}^{d}\lambda_r(t)\int_\Omega Z_i(x,t-\tau_r)\,dx\right)\\
&\quad + \sum_{j=1}^{m}\sum_{k=1}^{s} a_{ijk}(t)\frac{\delta_j}{\delta_i}\int_{\partial\Omega} g_j(x,\rho_k(t))Z_j(x,\rho_k(t))\,dS\\
&\quad + q_i(t)\int_\Omega Z_i(x,t)\,dx + \sum_{h=1}^{l} q_{iih}(t)\int_\Omega Z_i(x,\sigma_h(t))\,dx\\
&\quad - \sum_{h=1}^{l}\sum_{j=1, j\neq i}^{m} \bar{q}_{ijh}(t)\int_\Omega Z_j(x,\sigma_h(t))\,dx \le 0, \qquad t \ge t_1, \qquad i \in I_m.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{d^2}{dt^2}\left(p(t)\int_\Omega Z_i(x,t)\,dx + \sum_{r=1}^{d}\lambda_r(t)\int_\Omega Z_i(x,t-\tau_r)\,dx\right)\\
&\quad + \sum_{k=1}^{s} a_{iik}(t)\int_{\partial\Omega} g_i(x,\rho_k(t))Z_i(x,\rho_k(t))\,dS\\
&\quad - \sum_{j=1, j\neq i}^{m}\sum_{k=1}^{s} |a_{ijk}(t)|\int_{\partial\Omega} g_j(x,\rho_k(t))Z_j(x,\rho_k(t))\,dS\\
&\quad + q_i(t)\int_\Omega Z_i(x,t)\,dx + \sum_{h=1}^{l} q_{iih}(t)\int_\Omega Z_i(x,\sigma_h(t))\,dx\\
&\quad - \sum_{h=1}^{l}\sum_{j=1, j\neq i}^{m} \bar{q}_{ijh}(t)\int_\Omega Z_j(x,\sigma_h(t))\,dx \le 0, \qquad t \ge t_1, \qquad i \in I_m.
\end{aligned} \tag{10}$$

Setting

$$V_i(t) = \int_\Omega Z_i(x,t)\,dx, \qquad Y_i(t) = \int_{\partial\Omega} g_i(x,t)Z_i(x,t)\,dS, \qquad t \ge t_1, \qquad i \in I_m,$$

from (10) we have

$$\left[p(t)V_i(t) + \sum_{r=1}^{d} \lambda_r(t)V_i(t-\tau_r)\right]'' + \sum_{k=1}^{s}\left[a_{iik}(t)Y_i(\rho_k(t)) - \sum_{j=1,\, j\neq i}^{m} |a_{ijk}(t)|Y_j(\rho_k(t))\right] + q_i(t)V_i(t) + \sum_{h=1}^{l}\left[q_{iih}(t)V_i(\sigma_h(t)) - \sum_{j=1,\, j\neq i}^{m} \bar{q}_{ijh}(t)V_j(\sigma_h(t))\right] \le 0, \quad t \ge t_1, \qquad i \in I_m. \tag{11}$$

Letting

$$V(t) = \sum_{i=1}^{m} V_i(t), \qquad Y(t) = \sum_{i=1}^{m} Y_i(t), \qquad t \ge t_1,$$

from (11) we have

$$\left[p(t)V(t) + \sum_{r=1}^{d} \lambda_r(t)V(t-\tau_r)\right]'' + \sum_{k=1}^{s}\left\{\sum_{i=1}^{m}\left[a_{iik}(t)Y_i(\rho_k(t)) - \sum_{j=1,\, j\neq i}^{m} |a_{ijk}(t)|Y_j(\rho_k(t))\right]\right\} + q(t)V(t) + \sum_{h=1}^{l}\left\{\sum_{i=1}^{m}\left[q_{iih}(t)V_i(\sigma_h(t)) - \sum_{j=1,\, j\neq i}^{m} \bar{q}_{ijh}(t)V_j(\sigma_h(t))\right]\right\} \le 0, \quad t \ge t_1. \tag{12}$$

Noting that

$$\sum_{i=1}^{m}\left[q_{iih}(t)V_i(\sigma_h(t)) - \sum_{j=1,\, j\neq i}^{m} \bar{q}_{ijh}(t)V_j(\sigma_h(t))\right] = \left[q_{11h}(t)V_1(\sigma_h(t)) - \sum_{j=1,\, j\neq 1}^{m} \bar{q}_{1jh}(t)V_j(\sigma_h(t))\right] + \left[q_{22h}(t)V_2(\sigma_h(t)) - \sum_{j=1,\, j\neq 2}^{m} \bar{q}_{2jh}(t)V_j(\sigma_h(t))\right] + \cdots + \left[q_{mmh}(t)V_m(\sigma_h(t)) - \sum_{j=1,\, j\neq m}^{m} \bar{q}_{mjh}(t)V_j(\sigma_h(t))\right]$$

$$= \left[q_{11h}(t) - \sum_{j=1, j\neq 1}^{m} \bar{q}_{j1h}(t) \right] V_1(\sigma_h(t))$$
$$+ \left[q_{22h}(t) - \sum_{j=1, j\neq 2}^{m} \bar{q}_{j2h}(t) \right] V_2(\sigma_h(t))$$
$$+ \cdots$$
$$+ \left[q_{mmh}(t) - \sum_{j=1, j\neq m}^{m} \bar{q}_{jmh}(t) \right] V_m(\sigma_h(t))$$
$$\geq \min_{1\leq i\leq m} \left\{ q_{iih}(t) - \sum_{j=1, j\neq i}^{m} \bar{q}_{jih}(t) \right\} \sum_{i=1}^{m} V_i(\sigma_h(t))$$
$$= Q_h(t) V(\sigma_h(t)), \qquad t \geq t_1, \qquad h \in I_l,$$

and similarly that

$$\sum_{i=1}^{m} \left[a_{iik}(t) Y_i(\rho_k(t)) - \sum_{j=1, j\neq i}^{m} |a_{ijk}(t)| Y_j(\rho_k(t)) \right]$$
$$\geq \min_{1\leq i\leq m} \left\{ a_{iik}(t) - \sum_{j=1, j\neq i}^{m} |a_{jik}(t)| \right\} \sum_{i=1}^{m} Y_i(\rho_k(t))$$
$$= A_k(t) Y(\rho_k(t)), \qquad t \geq t_1, \qquad k \in I_s.$$

Then from (12), we get

$$\left[p(t)V(t) + \sum_{r=1}^{d} \lambda_r(t) V(t - \tau_r) \right]'' + \sum_{k=1}^{m} A_k(t) Y(\rho_k(t))$$
$$+ q(t)V(t) + \sum_{h=1}^{l} Q_h(t) V(\sigma_h(t)) \leq 0, \qquad t \geq t_1.$$

It is easy to see that

$$Y(\rho_k(t)) = \sum_{i=1}^{m} Y_i(\rho_k(t)) \geq 0, \qquad t \geq t_1, \qquad k \in I_s.$$

Therefore,

$$\left[p(t)V(t) + \sum_{r=1}^{d} \lambda_r(t) V(t - \tau_r) \right]'' + q(t)V(t)$$
$$+ \sum_{h=1}^{l} Q_h(t) V(\sigma_h(t)) \leq 0, \qquad t \geq t_1. \tag{13}$$

Now let $W(t) = p(t)V(t) + \sum_{r=1}^{d} \lambda_r(t)V(t - \tau_r)$, $t \ge t_1$; then $W(t) > 0$, $t \ge t_1$, and the inequality (13) shows that $W''(t) \le 0$ for $t \ge t_1$. Hence $W'(t)$ is monotone decreasing in the interval $[t_1, \infty)$. We can claim that $W'(t) \ge 0$ for $t \ge t_1$. In fact, if $W'(t) < 0$ for $t \ge t_1$, then there exists a $T > t_1$ such that $W'(T) = -L < 0$. This implies that

$$W'(t) \le W'(T) = -L \qquad \text{for } t \ge T.$$

Hence

$$W(t) \le W(T) - L(t - T), \qquad t \ge T.$$

Therefore,

$$\lim_{t\to\infty} W(t) = -\infty,$$

which contradicts the fact that $W(t) = p(t)V(t) + \sum_{r=1}^{d} \lambda_r(t)V(t - \tau_r) > 0$.

From (13) we obtain that there exists some $h_0 \in I_l$ such that

$$W''(t) + Q_{h_0}(t)V\big(\sigma_{h_0}(t)\big) \le 0, \qquad t \ge t_1. \tag{14}$$

Thus we obtain

$$W''(t) + \frac{Q_{h_0}(t)}{p\big(\sigma_{h_0}(t)\big)}\left[W\big(\sigma_{h_0}(t)\big) - \sum_{r=1}^{d} \lambda_r\big(\sigma_{h_0}(t)\big)V\big(\sigma_{h_0}(t) - \tau_r\big)\right] \le 0, \qquad t \ge t_1. \tag{15}$$

Since $W(t) \ge p(t)V(t) \ge V(t)$, $W'(t) \ge 0$, from (15) we have

$$W''(t) + Q_{h_0}(t)\left[1 - \sum_{r=1}^{d} \lambda_r\big(\sigma_{h_0}(t)\big)\right]\frac{W\big(\sigma_{h_0}(t)\big)}{p\big(\sigma_{h_0}(t)\big)} \le 0, \qquad t \ge t_1. \tag{16}$$

Integrating the inequality (16), we have

$$W'(t) - W'(t_1) + \int_{t_1}^{t} Q_{h_0}(s)\left[1 - \sum_{r=1}^{d} \lambda_r\big(\sigma_{h_0}(s)\big)\right]\frac{W\big(\sigma_{h_0}(s)\big)}{p\big(\sigma_{h_0}(s)\big)}\, ds \le 0, \qquad t \ge t_1. \tag{17}$$

Then we obtain

$$\int_{t_1}^{t} Q_{h_0}(s)\left[1 - \sum_{r=1}^{d} \lambda_r\big(\sigma_{h_0}(s)\big)\right] ds \le \frac{p\big(\sigma_{h_0}(t_1)\big)}{W\big(\sigma_{h_0}(t_1)\big)}[-W'(t) + W'(t_1)] \le \frac{p\big(\sigma_{h_0}(t_1)\big)W'(t_1)}{W\big(\sigma_{h_0}(t_1)\big)}, \qquad t \ge t_1, \tag{18}$$

which contradicts the condition (5). This completes the proof of Theorem 2.1.

THEOREM 2.2. *Let $p(t) \geq 1$ be a monotone decreasing function, and let the condition* (4) *hold. If*

$$\int^{\infty} q(t)\left[1 - \sum_{r=1}^{d} \lambda_r(t)\right] dt = \infty, \tag{19}$$

then every solution $u(x,t)$ of the problem (1), (2) *oscillates in G.*

Proof. As in the proof of Theorem 2.1, we obtain (13). Therefore,

$$W''(t) + q(t)V(t) \leq 0, \qquad t \geq t_1. \tag{20}$$

The remainder of the proof is similar to that of Theorem 2.1 and we omit it.

COROLLARY 2.1. *If the inequality* (13) *has no eventually positive solution, then every solution $u(x,t)$ of the problem* (1), (2) *is oscillatory in G.*

THEOREM 2.3. *Suppose that there exists a positive constant M such that $0 < M \leq p(t) < 1$, and*

$$\sum_{r=1}^{d} \lambda_r(t) < M. \tag{21}$$

If there exists some $h_0 \in I_l$ such that $\sigma'_{h_0}(t) \geq 0$ and

$$\int_{t_0}^{\infty} Q_{h_0}(t)\left[1 - M^{-1} \sum_{r=1}^{d} \lambda_r\big(\sigma_{h_0}(t)\big)\right] dt = \infty, \qquad t_0 > 0, \tag{22}$$

then every solution $u(x,t)$ of the problem (1), (2) *is oscillatory in G.*

Proof. As in the proof of Theorem 2.1, we obtain (15). Using that $p(t) < 1$, from (15) we have

$$W''(t) + Q_{h_0}(t)\left[W\big(\sigma_{h_0}(t)\big) - \sum_{r=1}^{d} \lambda_r\big(\sigma_{h_0}(t)\big)V\big(\sigma_{h_0}(t) - \tau_r\big)\right] \leq 0, \qquad t \geq t_1. \tag{23}$$

Since $W(t) \geq p(t)V(t) \geq MV(t)$, $W'(t) \geq 0$, from (23) we obtain

$$W''(t) + Q_{h_0}(t)\left[1 - M^{-1} \sum_{r=1}^{d} \lambda_r\big(\sigma_{h_0}(t)\big)\right] W\big(\sigma_{h_0}(t)\big) \leq 0, \qquad t \geq t_1. \tag{24}$$

Integrating inequality (24), we get

$$W'(t) - W'(t_1) + \int_{t_1}^{t} Q_{h_0}(s)\left[1 - M^{-1}\sum_{r=1}^{d}\lambda_r\big(\sigma_{h_0}(s)\big)\right]W\big(\sigma_{h_0}(s)\big)\,ds \le 0, \qquad t \ge t_1. \quad (25)$$

Therefore,

$$\int_{t_1}^{t} Q_{h_0}(s)\left[1 - M^{-1}\sum_{r=1}^{d}\lambda_r\big(\sigma_{h_0}(s)\big)\right]ds \le \frac{1}{W\big(\sigma_{h_0}(t_1)\big)}\left[-W'(t) + W'(t_1)\right] \le \frac{W'(t_1)}{W\big(\sigma_{h_0}(t_1)\big)}, \qquad t \ge t_1,$$

which contradicts the condition (22). This completes the proof.

THEOREM 2.4. *Suppose that there exists a positive constant M such that $0 < M \le p(t) < 1$, and the condition* (21) *holds. If*

$$\int_{t_0}^{\infty} q(t)\left[1 - M^{-1}\sum_{r=1}^{d}\lambda_r(t)\right]dt = \infty, \qquad t_0 > 0, \qquad (26)$$

then every solution $u(x,t)$ of the problem (1), (2) *is oscillatory in G.*

Proof. As in the proof of Theorem 2.2, we obtain (20). The remainder of the proof is similar to that of Theorem 2.3 and we omit it.

3. OSCILLATION OF THE PROBLEM (1), (3)

The following fact will be used.

The smallest eigenvalue α_0 of the Dirichlet problem

$$\Delta\omega(x) + \alpha\omega(x) = 0 \qquad \text{in } \Omega,$$
$$\omega(x) = 0 \qquad \text{on } \partial\Omega,$$

where α is a constant, is positive and the corresponding eigenfunction $\varphi(x)$ is positive in Ω.

THEOREM 3.1. *If all conditions of Theorem* 2.1 *hold, then every solution of the problem* (1), (3) *is oscillatory in G.*

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t) = \{u_1(x,t), u_2(x,t), \dots, u_m(x,t)\}^T$ of the problem (1), (3). We assume that $|u_i(x,t)| > 0$ for $t \ge t_0 \ge 0$, $i \in I_m$. Let $\delta_i = \operatorname{sgn} u_i(x,t)$, $Z_i(x,t) = \delta_i u_i(x,t)$; then $Z_i(x,t) > 0$, $(x,t) \in \Omega \times [t_0,\infty)$, $i \in I_m$. From (H1), (H5) there exists a number $t_1 \ge t_0$ such that $Z_i(x,t) > 0$, $Z_i(x,t-\tau_r) > 0$, $Z_i(x,\rho_k(t)) > 0$, and $Z_i(x,\sigma_h(t)) > 0$ in $\Omega \times [t_1,\infty)$, $i \in I_m$, $r \in I_d$, $k \in I_s$, $h \in I_l$.

Multiplying both sides of (1) by $\varphi(x) > 0$ and integrating with respect to x over the domain Ω, we have

$$\begin{aligned}
&\frac{d^2}{dt^2}\left(p(t)\int_\Omega u_i(x,t)\,\varphi(x)\,dx + \sum_{r=1}^{d}\lambda_r(t)\int_\Omega u_i(x,t-\tau_r)\,\varphi(x)\,dx\right)\\
&\quad = a_i(t)\int_\Omega \Delta u_i(x,t)\,\varphi(x)\,dx\\
&\qquad + \sum_{j=1}^{m}\sum_{k=1}^{s} a_{ijk}(t)\int_\Omega \Delta u_j(x,\rho_k(t))\,\varphi(x)\,dx\\
&\qquad - \int_\Omega q_i(x,t)\,u_i(x,t)\,\varphi(x)\,dx\\
&\qquad - \sum_{j=1}^{m}\sum_{h=1}^{l}\int_\Omega q_{ijh}(x,t)\,u_j(x,\sigma_h(t))\,\varphi(x)\,dx,\\
&\qquad\qquad t \ge t_1, \qquad i \in I_m. \qquad (27)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{d^2}{dt^2}\left(p(t)\int_\Omega Z_i(x,t)\,\varphi(x)\,dx + \sum_{r=1}^{d}\lambda_r(t)\int_\Omega Z_i(x,t-\tau_r)\,\varphi(x)\,dx\right)\\
&\quad = a_i(t)\int_\Omega \Delta Z_i(x,t)\,\varphi(x)\,dx + \sum_{k=1}^{s} a_{iik}(t)\int_\Omega \Delta Z_i(x,\rho_k(t))\,\varphi(x)\,dx\\
&\qquad + \sum_{j=1,\,j\neq i}^{m}\sum_{k=1}^{s} a_{ijk}(t)\frac{\delta_j}{\delta_i}\int_\Omega \Delta Z_j(x,\rho_k(t))\,\varphi(x)\,dx\\
&\qquad - \int_\Omega q_i(x,t)\,Z_i(x,t)\,\varphi(x)\,dx\\
&\qquad - \sum_{j=1}^{m}\sum_{h=1}^{l}\frac{\delta_j}{\delta_i}\int_\Omega q_{ijh}(x,t)\,Z_j(x,\sigma_h(t))\,\varphi(x)\,dx,\\
&\qquad\qquad t \ge t_1, \qquad i \in I_m. \qquad (28)
\end{aligned}$$

Green's formula and boundary (3) yield

$$\begin{aligned}
\int_\Omega \Delta Z_i(x,t)\,\varphi(x)\,dx &= \int_\Omega Z_i(x,t)\,\Delta\varphi(x)\,dx\\
&= -\alpha_0\int_\Omega Z_i(x,t)\,\varphi(x)\,dx \le 0 \qquad (29)
\end{aligned}$$

and

$$\int_\Omega \Delta Z_j(x, \rho_k(t))\varphi(x)\,dx$$
$$= \int_\Omega Z_j(x, \rho_k(t))\Delta\varphi(x)\,dx$$
$$= -\alpha_0 \int_\Omega Z_j(x, \rho_k(t))\varphi(x)\,dx, \qquad t \ge t_1, \qquad j \in I_m, \qquad k \in I_s. \tag{30}$$

Combining (28)–(30), we have

$$\frac{d^2}{dt^2}\left(p(t)\int_\Omega Z_i(x,t)\varphi(x)\,dx + \sum_{r=1}^{d} \lambda_r(t)\int_\Omega Z_i(x, t-\tau_r)\varphi(x)\,dx\right)$$
$$\le -\alpha_0 \sum_{k=1}^{s} a_{iik}(t)\int_\Omega \Delta Z_i(x, \rho_k(t))\varphi(x)\,dx$$
$$+ \alpha_0 \sum_{j=1, j\ne i}^{m} \sum_{k=1}^{s} |a_{ijk}(t)| \int_\Omega \Delta Z_j(x, \rho_k(t))\varphi(x)\,dx$$
$$- q_i(t)\int_\Omega Z_i(x,t)\varphi(x)\,dx - \sum_{h=1}^{l} q_{iih}(t)\int_\Omega Z_i(x, \sigma_h(t))\varphi(x)\,dx$$
$$+ \sum_{j=1, j\ne i}^{m} \sum_{h=1}^{l} \bar{q}_{ijh}(t)\int_\Omega Z_j(x, \sigma_h(t))\varphi(x)\,dx,$$
$$t \ge t_1, \qquad i \in I_m. \tag{31}$$

Setting $V_i(t) = \int_\Omega Z_i(x,t)\varphi(x)\,dx$, $t \ge t_1$, $i \in I_m$, from (31) we have

$$\left[p(t)V_i(t) + \sum_{r=1}^{d} \lambda_r(t)V_i(t-\tau_r)\right]''$$
$$+ \alpha_0 \sum_{k=1}^{s} a_{iik}(t)V_i(\rho_k(t)) - \alpha_0 \sum_{j=1, j\ne i}^{m} \sum_{k=1}^{s} |a_{ijk}(t)| V_j(\rho_k(t))$$
$$+ q_i(t)V_i(t)$$
$$+ \sum_{h=1}^{l}\left[q_{iih}(t)V_i(\sigma_h(t)) - \sum_{j=1, j\ne i}^{m} \bar{q}_{ijh}(t)V_j(\sigma_h(t))\right] \le 0,$$
$$t \ge t_1, \qquad i \in I_m. \tag{32}$$

Letting $V(t) = \sum_{i=1}^{m} V_i(t)$, $t \geq t_1$, from (32) we have

$$\left[p(t)V(t) + \sum_{r=1}^{d} \lambda_r(t)V(t-\tau_r) \right]'' + \alpha_0 \sum_{k=1}^{s} \left\{ \sum_{i=1}^{m} \left[a_{iik}(t)V_i(\rho_k(t)) - \sum_{j=1, j\neq i}^{m} |a_{ijk}(t)| V_j(\rho_k(t)) \right] \right\} + q(t)V(t) + \sum_{h=1}^{l} \left\{ \sum_{i=1}^{m} \left[q_{iih}(t)V_i(\sigma_h(t)) - \sum_{j=1, j\neq i}^{m} \bar{q}_{ijh}(t)V_j(\sigma_h(t)) \right] \right\} \leq 0, \quad t \geq t_1. \quad (33)$$

As in the proof of Theorem 2.1, from (33) we obtain

$$W''(t) + \alpha_0 \sum_{k=1}^{s} A_k(t)V(\rho_k(t)) + q(t)V(t) + \sum_{h=1}^{l} Q_h(t)V(\sigma_h(t)) \leq 0, \qquad t \geq t_1. \quad (34)$$

The remainder of the proof is similar to that of Theorem 2.1 and we omit it.

COROLLARY 3.1. *If the differential inequality* (34) *has no eventually positive solution, then every solution* $u(x,t)$ *of the problem* (1), (3) *oscillates in* G.

It is not difficult to see that the following theorems are true.

THEOREM 3.2. *Suppose that* $p(t) \geq 1$ *is a monotone decreasing function, and the condition* (4) *holds.*

If there exists some $k_0 \in I_s$ *such that* $\rho'_{k_0}(t) \geq 0$ *and*

$$\int_{t_0}^{\infty} \alpha_0 A_{k_0}(t) \left[1 - \sum_{r=1}^{d} \lambda_r(\rho_{k_0}(t)) \right] dt = \infty, \qquad t_0 > 0, \quad (35)$$

then every solution $u(x,t)$ *of the problem* (1), (3) *is oscillatory in* G.

THEOREM 3.3. *Suppose that there exists a positive constant* M *such that* $0 < M \leq p(t) < 1$, *and the condition* (21) *holds.*

If there exists some $k_0 \in I_s$ *such that* $\rho'_{k_0}(t) \geq 0$ *and*

$$\int_{t_0}^{\infty} \alpha_0 A_{k_0}(t) \left[1 - M^{-1} \sum_{r=1}^{d} \lambda_r\left(\rho_{k_0}(t) \right) \right] dt = \infty, \qquad t_0 > 0, \tag{36}$$

then every solution $u(x,t)$ *of the problem* (1), (3) *is oscillatory in* G.

THEOREM 3.4. *If the conditions of Theorem* 2.2 *hold, then every solution* $u(x,t)$ *of the problem* (1), (3) *is oscillatory in* G.

THEOREM 3.5. *If the conditions of Theorem* 2.3 *hold, then every solution* $u(x,t)$ *of the problem* (1), (3) *is oscillatory in* G.

THEOREM 3.6. *If the conditions of Theorem* 2.4 *hold, then every solution* $u(x,t)$ *of the problem* (1), (3) *is oscillatory in* G.

4. EXAMPLES

EXAMPLE 4.1. Consider the system of hyperbolic differential equations

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}\left((1+e^{-t})u_1(x,t) + \frac{1}{2}u_1(x,t-\pi)\right) \\
&\quad = \Delta u_1(x,t) + (3e^{-t}+2)\Delta u_1\left(x, t-\frac{3\pi}{2}\right) \\
&\quad\quad - \frac{3}{2}u_1(x,t) - 3u_1(x,t-\pi) - e^{-t}u_2(x,t-\pi) \\
&\quad\quad - 2u_1\left(x, t-\frac{\pi}{2}\right) - u_2\left(x, t-\frac{\pi}{2}\right), \\
&\frac{\partial^2}{\partial t^2}\left((1+e^{-t})u_2(x,t) + \frac{1}{2}u_2(x,t-\pi)\right) \\
&\quad = e^{-t}\Delta u_2(x,t) + 2(e^{-t}+1)\Delta u_2\left(x, t-\frac{3\pi}{2}\right) \\
&\quad\quad - \frac{5}{2}u_2(x,t) - u_1(x,t-\pi) - (e^{-t}+1)u_2(x,t-\pi) \\
&\quad\quad - u_1\left(x, t-\frac{\pi}{2}\right) - 3u_2\left(x, t-\frac{\pi}{2}\right), \\
&\qquad\qquad (x,t) \in (0,\pi) \times [0,\infty)
\end{aligned} \tag{37}$$

with boundary condition

$$\frac{\partial}{\partial x}u_i(0,t) = \frac{\partial}{\partial x}u_i(\pi,t) = 0, \qquad t \geq 0, \qquad i = 1,2. \tag{38}$$

Here $n = 1$, $m = 2$, $d = 1$, $s = 1$, $l = 2$, $p(t) = 1 + e^{-t}$, $\lambda_1(t) = \frac{1}{2}$, $\tau_1 = \pi$, $a_1(t) = 1$, $a_{111}(t) = 3e^{-t} + 2$, $a_{121}(t) = 0$, $\rho_1(t) = t - \frac{3\pi}{2}$, $q_1(x,t) = \frac{3}{2}$, $q_{111}(x,t) = 3$, $q_{121}(t) = e^{-t}$, $\sigma_1(t) = t - \pi$, $q_{112}(x,t) = 2$, $q_{122}(x,t) = 1$, $\sigma_2(t) = t - \frac{\pi}{2}$, $a_2(t) = e^{-t}$, $a_{211}(t) = 0$, $a_{221}(t) = 2(1 + e^{-t})$, $q_2(x,t) = \frac{5}{2}$, $q_{211}(x,t) = 1$, $q_{221}(x,t) = e^{-t} + 1$, $q_{212}(x,t) = 1$, and $q_{222}(x,t) = 3$. It is easy to see that $\lambda_1(t) = \frac{1}{2} < 1$, $p(t) = 1 + e^{-t} \geq 1$, $Q_1(t) = 1$, $Q_2(t) = 1$, and

$$\sigma_1'(t) = (t - \pi)' = 1 \geq 0,$$

$$\int_{t_0}^{\infty} Q_1(t)\big(1 - \lambda_1(\sigma_1(t))\big)\,dt = \int_{t_0}^{\infty} \frac{1}{2}\,dt = \infty, \qquad t_0 > 0.$$

Hence all conditions of Theorem 2.1 are fulfilled. Then every solution of the problem (37), (38) oscillates in $(0,\pi) \times [0,\infty)$. In fact, $u_1(x,t) = \cos x \sin t$, $u_2(x,t) = \cos x \cos t$ is such a solution.

EXAMPLE 4.2. Consider the system of hyperbolic differential equations

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}\left((1 + e^{-t})u_1(x,t) + \frac{1}{2}u_1(x,t-\pi)\right) \\
&\quad = 2\Delta u_1(x,t) + 2(e^{-t} + 1)\Delta u_1\left(x, t - \frac{3\pi}{2}\right) \\
&\quad\quad - \frac{3}{2}u_1(x,t) - 2u_1(x,t-\pi) - u_2(x,t-\pi) \\
&\quad\quad - 3u_1\left(x, t - \frac{\pi}{2}\right) - u_2\left(x, t - \frac{\pi}{2}\right), \\
&\frac{\partial^2}{\partial t^2}\left((1 + e^{-t})u_2(x,t) + \frac{1}{2}u_2(x,t-\pi)\right) \\
&\quad = \frac{3}{2}\Delta u_2(x,t) + (2e^{-t} + 3)\Delta u_2\left(x, t - \frac{3\pi}{2}\right) \\
&\quad\quad - u_2(x,t) - u_1(x,t-\pi) - 3u_2(x,t-\pi) \\
&\quad\quad - u_1\left(x, t - \frac{\pi}{2}\right) - 2u_2\left(x, t - \frac{\pi}{2}\right), \\
&\qquad\qquad (x,t) \in (0,\pi) \times [0,\infty),
\end{aligned} \tag{39}$$

with boundary condition

$$u_i(0,t) = u_i(\pi,t) = 0, \qquad t \geq 0, \qquad i = 1,2. \tag{40}$$

It is easy to see that all conditions of Theorem 3.1 are fulfilled. Then every solution of the problem (39), (40) oscillates in $(0,\pi) \times [0,\infty)$. In fact, $u_1(x,t) = \sin x \cos t$, $u_2(x,t) = \sin x \sin t$ is such a solution.

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