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# Comparable Means and Generalized Convexity

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## 1. INTRODUCTION

Let $I(a, b)$ denote the set of all continuous, strictly increasing functions on the closed interval $[a, b]$. For every $\phi \in I(a, b)$, every positive integer $n$, every sequence $q_1, q_2, ..., q_n$ of positive numbers satisfying $q_1 + q_2 + \cdots + q_n = 1$, and every sequence $x_1, x_2, ..., x_n$ of elements of $[a, b]$, we consider the weighted mean

$$\mathfrak{M}_\phi(x_1, x_2, ..., x_n; q_1, q_2, ..., q_n) = \phi^{-1}\left\{\sum_{k=1}^{n} q_k\phi(x_k)\right\}, \tag{1}$$

where $\phi^{-1}$ denotes the inverse function of $\phi$. For a suitable choice of $[a, b]$, $\mathfrak{M}_\phi$ reduces to the arithmetic mean, the geometric mean, and the harmonic mean when $\phi(x)$ is equal to $x$, $\log x$, and $-1/x$, respectively, and $q_k = 1/n$ $(k = 1, 2, ..., n)$.

If $\psi(x) = \log x$ and $\chi(x) = x$, then the familiar inequality between the weighted geometric mean and the weighted arithmetic mean asserts that

$$\mathfrak{M}_\psi(x_1, ..., x_n; q_1, ..., q_n) \leqslant \mathfrak{M}_\chi(x_1, ..., x_n; q_1, ..., q_n) \tag{2}$$

holds whenever $n$, $q_1, ..., q_n$, and $x_1, ..., x_n$ satisfy the foregoing conditions. More generally, if $\psi$ and $\chi$ are any two elements of $I(a, b)$ such that (2) always holds, then we write

$$\mathfrak{M}_\psi \leqslant \mathfrak{M}_\chi, \tag{3}$$

and we say that the means are *comparable* if either (3) or the reverse inequality is true. (One can also consider (1) when $\phi$ is continuous and strictly decreasing; but since $\mathfrak{M}_{-\phi} = \mathfrak{M}_\phi$, we restrict our attention to elements of $I(a, b)$. For a complete discussion of this and related matters, consult Chapter III of [1].)

A well-known necessary and sufficient condition for (3) to hold is that the

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composite function $\chi \circ \psi^{-1}$ be convex on the interval $[\psi(a), \psi(b)]$, and, when this is true, one says that $\chi$ *is convex with respect to* $\psi$ (see [1], p. 75). The convexity of $\chi$ with respect to $\psi$ is also necessary and sufficient for the validity of (3) when the finite means are replaced by the corresponding integral means (see [1], p. 169). Moreover, the question of the convexity of one function with respect to another need not arise from the consideration of comparable means—a classical example in this connection is Hadamard's three-circles theorem (see [2], pp. 154–156).

Several natural questions suggest themselves. For example, if $\psi$ and $\chi$ are elements of $I(a, b)$, can one give simple (necessary and) sufficient conditions for $\chi$ to be convex with respect to $\psi$? Of course, if $\phi = \chi \circ \psi^{-1}$ possesses a second derivative, then convexity obtains if and only if $\phi''(x) \geqslant 0$ on the open interval $(\psi(a), \psi(b))$; but this derivative is usually rather complicated. Elsewhere, O. Shisha and the author [3] have proved that, if $\chi'/\psi'$ exists and is nondecreasing on $(a, b)$, then $\chi$ is convex with respect to $\psi$. In Section 2 of this paper, we give a simple derivation of this criterion and show that it has a natural geometric interpretation, and we also establish some other (necessary and) sufficient conditions.

A second question that we consider is the following: If $\psi$ is an arbitrary function in $I(a, b)$, do functions which are convex with respect to $\psi$ always exist? Although Hardy, Littlewood, and Pólya do not mention this question in their book, it is inconceivable that they were not aware of the answer. In Section 3 of this paper, we characterize in several ways the class of all functions which are convex with respect to a given element of $I(a, b)$.

## 2. Parametric Equations

For completeness, we first prove a result which is stated in [1, p. 75].

Lemma 1. *If $\psi, \chi \in I(a, b)$, then $\chi$ is convex with respect to $\psi$ if and only if the function whose graph is described by the parametric equations $x = \psi(t)$, $y = \chi(t)$ $(a \leqslant t \leqslant b)$ is convex.*

Proof. If $\phi = \chi \circ \psi^{-1}$, $\psi(a) \leqslant x \leqslant \psi(b)$, and $t = \psi^{-1}(x)$, then $(x, \phi(x)) = (\psi(t), \chi \circ \psi^{-1}(x)) = (\psi(t), \chi(t))$, and conversely. In other words, $x = \psi(t)$, $y = \chi(t)$ $(a \leqslant t \leqslant b)$ is a parametric representation of the graph of the function $\phi$, and this establishes the lemma.

Recalling the rule from calculus for computing the derivative of a function in terms of a parametric description, and making use of the fact (see [4], p. 205) that a differentiable function is convex on an open interval if and only if its derivative is nondecreasing (also, see [1], p. 76, footnote a), we obtain the following theorems.

THEOREM 1. *Suppose that* $\psi, \chi \in I(a, b)$ *and that* $\chi'/\psi'$ *is defined throughout the interval* $(a, b)$. *Then* $\chi$ *is convex with respect to* $\psi$ *if and only if* $\chi'/\psi'$ *is non-decreasing on* $(a, b)$.

THEOREM 2. *Suppose that* $\psi, \chi \in I(a, b)$ *and that* $\psi''/\psi'$ *and* $\chi''/\chi'$ *are defined throughout* $(a, b)$. *Then* $\chi$ *is convex with respect to* $\psi$ *if and only if*

$$\frac{\psi''(t)}{\psi'(t)} \leqslant \frac{\chi''(t)}{\chi'(t)}$$

*whenever* $a < t < b$.

Instead of enunciating theorems of a more sophisticated nature (cf. [5], p. 23), let us prove one particularly simple result.

THEOREM 3. *If* $\psi, \chi \in I(a, b)$, *then* $\chi$ *is convex with respect to* $\psi$ *if and only if*

$$\begin{vmatrix} 1 & \psi(t_1) & \chi(t_1) \\ 1 & \psi(t_2) & \chi(t_2) \\ 1 & \psi(t_3) & \chi(t_3) \end{vmatrix} \geqslant 0 \tag{4}$$

*whenever* $a \leqslant t_1 < t_2 < t_3 \leqslant b$.

PROOF. Theorem 3 follows at once from Lemma 1, or, alternatively, from a similar condition for convex functions (see [1], Theorem 122). Inequality (4), of course, can be interpreted in terms of the orientation (or signed area) of certain inscribed triangles.

## 3. EXISTENCE THEOREMS

We now turn to the second question mentioned in the introduction.

LEMMA 2. *Let* $\psi \in I(a, b)$. *Then a function* $\chi$ *on* $[a, b]$ *is convex with respect to* $\psi$ *if and only if* $\chi = \mathscr{K} \circ \psi$ *where* $\mathscr{K}$ *is a convex function in* $I(\psi(a), \psi(b))$.

PROOF. Suppose that $\chi$ is convex with respect to $\psi$, and denote the function $\chi \circ \psi^{-1}$ by $\mathscr{K}$. By the definition of generalized convexity, $\mathscr{K}$ is convex on $[\psi(a), \psi(b)]$. If $x \in [a, b]$, let $y = \psi(x)$. Then $\chi \circ \psi^{-1}(y) = \mathscr{K}(y)$ and $\chi(x) = \mathscr{K}\{\psi(x)\}$, as desired.

Conversely, if $\chi = \mathscr{K} \circ \psi$ where $\mathscr{K}$ is a convex function in $I(\psi(a), \psi(b))$, then $\chi \in I(a, b)$. Corresponding to each $y \in [\psi(a), \psi(b)]$, there exists a unique $x \in [a, b]$ such that $\psi(x) = y$. Consequently, $\chi(x) = \mathscr{K} \circ \psi(x)$ reduces to $\chi\{\psi^{-1}(y)\} = \mathscr{K}(y)$, and $\chi \circ \psi^{-1}$ is convex.

THEOREM 4. *Let $\psi \in I(a, b)$. Then a function $\chi$ on $[a, b]$ is convex with respect to $\psi$ if and only if*

$$\chi(x) - \chi(a) = \int_{\psi(a)}^{\psi(x)} \mu(t)\, dt \qquad (a \leqslant x \leqslant b) \tag{5}$$

*where $\mu$ is a nondecreasing, positive, summable function on $(\psi(a), \psi(b))$.*

PROOF. Let $A = \psi(a)$ and $B = \psi(b)$. In view of Lemma 2, it will suffice to prove that $\mathscr{K}$ is a convex function in $I(A, B)$ if and only if

$$\mathscr{K}(y) - \mathscr{K}(A) = \int_A^y \mu(t)\, dt \qquad (A \leqslant y \leqslant B) \tag{6}$$

where $\mu$ is a nondecreasing, positive, summable function on $(A, B)$.

Suppose that $\mathscr{K}$ is defined by (6). Then $\mathscr{K}$ is convex (cf. [1], p. 130) since, if $A \leqslant y_1 < y_2 \leqslant B$, then

$$\mathscr{K}\left(\frac{y_1 + y_2}{2}\right) - \mathscr{K}(y_1) \leqslant \mathscr{K}(y_2) - \mathscr{K}\left(\frac{y_1 + y_2}{2}\right),$$

and

$$\mathscr{K}\left(\frac{y_1 + y_2}{2}\right) \leqslant \frac{\mathscr{K}(y_1) + \mathscr{K}(y_2)}{2}.$$

The continuity and monotonicity of $\mathscr{K}$ are direct consequences of elementary properties of the Lebesgue integral.

Conversely, suppose that $\mathscr{K}$ is a convex function in $I(A, B)$. Then there exists a nondecreasing, summable function $\mu$ on $(A, B)$ such that (6) holds (see [5], p. 24, and [1], p. 130). Moreover, if $\mu(t_0) \leqslant 0$ for some $t_0 \in (A, B)$, then

$$\mathscr{K}(t_0) - \mathscr{K}(A) = \int_A^{t_0} \mu(t)\, dt \leqslant (t_0 - A)\mu(t_0) \leqslant 0,$$

which contradicts the assumption that $\mathscr{K}$ is strictly increasing. Thus $\mu(t) > 0$ if $A < t < B$.

If either $\mu$ or $\psi$ is sufficiently smooth (e.g., absolutely continuous), then the integral in (5) can be written in a simplified form (cf. [6], Theorems 321, 277, and 322.1).

## 4. SOME APPLICATIONS

It is easy to formulate additional results in connection with generalized convexity. We conclude this paper by proving a few representative theorems.

First, we observe that if $\psi \in I(a, b)$, then the set $C_\psi$ of all functions in $I(a, b)$ which are convex with respect to $\psi$ is a subcone of $I(a, b)$, that is, if $\chi_1 \in C_\psi$, $\chi_2 \in C_\psi$, $\alpha_1 > 0$, and $\alpha_2 > 0$, then $\alpha_1\chi_1 + \alpha_2\chi_2 \in C_\psi$. This follows directly from Theorem 4 or Theorem 3, or from Lemma 2 and the fact that the sum of two convex functions is again convex.

The following result is somewhat deeper.

THEOREM 5. *Suppose that $\psi, \chi \in I(a, b)$ and that $\chi$ is convex with respect to $\psi$. Then $\chi$ is absolutely continuous on $[a, b]$ if and only if $\psi$ is absolutely continuous there.*

PROOF. By Lemma 2, $\chi = \mathscr{K} \circ \psi$ where $\mathscr{K}$ is a convex function in $I(\psi(a), \psi(b))$. Since $\mathscr{K}$ is continuous on $[\psi(a), \psi(b)]$, it is absolutely continuous there (see [6], p. 189, and [1], p. 130), and, consequently, the absolute continuity of $\psi$ on $[a, b]$ implies that of $\chi$ (see [7], p. 103). The same conclusion follows from Theorem 4 and [6, Theorems 272 and 275].

In order to prove that $\psi$ is absolutely continuous on $[a, b]$ if $\chi$ is absolutely continuous there, we first use Theorem 3, together with an analogous result, to infer that $\psi$ is concave with respect to $\chi$ (obvious definition) by merely interchanging the last two columns in the determinant appearing in (4). Then we modify the foregoing argument in order to complete the proof of the theorem.

If $\phi$ is an unbounded, strictly increasing, positive, continuous function on $(0, \infty)$ and if $x_1 > 0, x_2 > 0, \ldots, x_n > 0$, then we consider the sum

$$\mathfrak{S}_\phi(x_1, x_2, \ldots, x_n) = \phi^{-1}\left\{\sum_{k=1}^{n} \phi(x_k)\right\}$$

(see [1], p. 84). Omitting some obvious definitions, we have the following theorem.

THEOREM 6. *Suppose that $\psi$ and $\chi$ are unbounded, strictly increasing, continuous functions on $[0, \infty)$ and that $\psi(0) = \chi(0) = 0$. Then $\mathfrak{S}_\psi \geqslant \mathfrak{S}_\chi$ if $\mathfrak{M}_\psi \leqslant \mathfrak{M}_\chi$.*

PROOF. Setting $t_1 = 0$ in (4), we conclude from Theorem 3 that $\chi/\psi$ is is nondecreasing on $(0. \infty)$. The desired conclusion then follows from [1, Theorem 105].

In this connection, we note that Theorems 1 and 3 of this paper furnish a simple proof, as well as a geometric interpretation, of Theorem 148 of [1] when $f(0) = g(0) = 0$.

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