Nonlinear stability and D-convergence of Runge–Kutta methods for delay differential equations

Zhang Chengjian*, Zhou Shuzi
Department of Mathematics, Hunan University, Changsha 410082, People's Republic of China

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Abstract

This paper deals with the stability and convergence of Runge–Kutta methods with the Lagrangian interpolation (RKLMs) for nonlinear delay differential equations (DDEs). Some new concepts, such as strong algebraic stability, GDN-stability and D-convergence, are introduced. We show that strong algebraic stability of a RKM for ODEs implies GDN-stability of the corresponding RKLM for DDEs, and that a strongly algebraically stable and diagonally stable RKM with order \( p \), together with a Lagrangian interpolation of order \( q \), leads a D-convergent RKLM of order \( \min\{p, q + 1\} \).

Keywords: Strong algebraic stability; GDN-stability; D-convergence; DDE; Runge–Kutta method

1. Introduction

Consider the following nonlinear DDEs:

\[
\begin{align*}
\dot{y}(t) &= f(t, y(t), y(t - \tau)), & t \in [t_0, T], \\
y(t) &= \varphi(t), & t \in [t_0 - \tau, t_0]
\end{align*}
\]

and

\[
\begin{align*}
\dot{z}(t) &= f(t, z(t), z(t - \tau)), & t \in [t_0, T], \\
z(t) &= \psi(t), & t \in [t_0 - \tau, t_0],
\end{align*}
\]

where \( f : [t_0, T] \times C^N \times C^N \to C^N \) and \( \varphi, \psi : [t_0 - \tau, t_0] \to C^N \) are continuous functions such that (1.1) and (1.2) has a unique solution, respectively. Moreover, we assume that there exist some inner

* Corresponding author.
product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$ such that

$$\text{Re}\{f(t, y_1, u) - f(t, y_2, u), y_1 - y_2\| \leq \sigma\|y_1 - y_2\|^2,$$

$$\|f(t, y, u_1) - f(t, y, u_2)\| \leq \gamma\|u_1 - u_2\|,$$

$$\forall t \in [t_0, T], \quad \forall y, y_1, y_2, u, u_1, u_2 \in C^N,$$

where $\sigma, \gamma$ are constants with

$$0 \leq \gamma \leq -\sigma.$$

Torelli [13] pointed out that the analytic solutions of (1.1) and (1.2), under the conditions (1.3)-(1.5), satisfy

$$\|y(t) - z(t)\| \leq \max_{t_0 \leq t \leq t_0 + h} \|f(x) - y(x)\|, \quad \forall t \in [t_0, T].$$

In recent years, many authors, such as in’t Hout, Spijker and Jackiewicz (cf. [9-14, 2]), have adapted RKMs $(A, b, c)$ (for ODEs):

$$c \begin{bmatrix} A \\ b^T \end{bmatrix},$$

where $A = (a_{ij}) \in R^{s \times s}$, $b = (b_1, b_2, \ldots, b_s)^T$ and $c = (c_1, c_2, \ldots, c_s)^T \in R^s$, to DDEs (1.1). Especially, in’t Hout [9, 10] presented RKLMs for DDEs (1.1), which defined by

$$y_{n+1} = y_n + h \sum_{j=1}^{s} b_j f(t_n + c_j h, y_j^{(n)}, \tilde{y}_j^{(n)}),$$

$$y_i^{(n)} = y_n + h \sum_{j=1}^{s} a_{ij} f(t_n + c_j h, y_j^{(n)}, \tilde{y}_j^{(n)}), \quad i = 1, 2, \ldots, s,$$

where coefficients $a_{ij}, b_j$ and $c_j$ satisfy

$$\sum_{j=1}^{s} b_j = 1, \quad \sum_{j=1}^{s} a_{ij} = c_i, \quad 0 \leq c_i \leq 1, \quad i = 1, 2, \ldots, s;$$

$t_n = t_0 + nh \in [t_0, T]$; $y_n, y_j^{(n)}, \tilde{y}_j^{(n)}$ are approximations to the analytic solution $y(t_n), y(t_n + c_j h), y(t_n + c_j h - \tau)$ of (1.1), respectively, and the argument $\tilde{y}_j^{(n)}$ is determined by

$$\tilde{y}_j^{(n)} = \begin{cases} \varphi(t_n + c_j h - \tau), & t_n + c_j h - \tau \leq t_0, \\ \sum_{i=-r}^{r} L_i(\delta) y_j^{(n-m+i)}, & t_n + c_j h - \tau > t_0, \end{cases}$$

with $\tau = (m - \delta) h$, $\delta \in [0, 1)$, integer $m \geq v + 1, r, v \geq 0$ and

$$L_i(\delta) = \prod_{q=-r}^{v} \left( \frac{\delta - q}{l - q} \right), \quad l = -r, -r + 1, \ldots, v.$$
What we assume \( m \geq v + 1 \) is to guarantee that no (unknown) values \( y^{(i)}_j \) with \( i \geq n \) are used in the interpolation procedure. In addition, we always put \( y^{(n)}_j = \varphi(t_n + c_i h) \) whenever \( n < 0 \), and \( y_n = \varphi(t_n) \) whenever \( n \leq 0 \). In't Hout [9, 10] gave some nice results on asymptotic stability of methods (1.7) for linear DDEs.

For discussing nonlinear stability of numerical methods, Torelli [13] presented the concept of GRN-stability. However, it is difficult to verify that a method for DDEs is GRN-stable, and only the Euler method has been proved to be GRN-stable so far (cf. [13]). In view of this, we gave a new stability concept, i.e. GDN-stability, which is different from GRN-stability by \( \sigma, \gamma \) in (1.3)-(1.5) are constants independent of \( t \) and positive constant \( C \) in (2.1) is not necessarily equal to one. A convenient criterion for GDN-stability is presented, and the convergence behaviour of methods (1.7) is revealed by introducing the concept of D-convergence in this paper.

2. GDN-stability

Some new stability concepts are introduced as follows.

**Definition 2.1.** A numerical method (1.7) for DDEs is called GDN-stable if, under the conditions (1.3)-(1.5), numerical approximations \( y_n \) and \( z_n \) to the solution of (1.1) and (1.2), respectively, satisfy

\[
\| y_n - z_n \| \leq C \max_{t_0 - \tau \leq t \leq t_0} \| \varphi(t) - \psi(t) \|, \quad n \geq 0,
\]

where constant \( C > 0 \) depends only on the method, the parameter \( \sigma \) and the interval length \( T - t_0 \).

**Definition 2.2.** A RKM \((A, b, c)\) for ODEs is called strongly algebraically stable if matrices

\[
M_i = (\text{diag} A_i)A + A^T(\text{diag} A_i) - A_i A_i^T, \quad i = 0, 1, 2, \ldots, s
\]

are nonnegative definite, where

\[
A_0 = b, \quad A_i = (a_{i1}, a_{i2}, \ldots, a_{is})^T, \quad i = 1, 2, \ldots, s
\]

with \( a_{ij} \geq 0 \) and \( b_i \geq 0 \) (\( i, j = 1, 2, \ldots, s \)).

In particular, algebraic stability can be defined by that matrix \( M_0 \) which is nonnegative definite together with \( b_i \geq 0 \) (\( i = 1, 2, \ldots, s \)).

Let \( \{y_n, y_j^{(n)}\}_{j=1}^{\infty} \) and \( \{z_n, z_j^{(n)}\}_{j=1}^{\infty} \) be two sequences of approximations to problems (1.1) and (1.2), respectively, by method (1.7) with the same stepsize \( h \), and write

\[
T_i^{(n)} = t_n + c_i h, \quad U_i^{(n)} = y_i^{(n)} - z_i^{(n)}, \quad \tilde{U}_i^{(n)} = \tilde{y}_i^{(n)} - \tilde{z}_i^{(n)}, \quad U_0^{(n)} = y_n - z_n,
\]

\[
Q_i^{(n)} = h[f(T_i^{(n)}, y_i^{(n)}, \tilde{y}_i^{(n)}) - f(T_i^{(n)}, z_i^{(n)}, \tilde{z}_i^{(n)})], \quad i = 1, 2, \ldots, s.
\]
Then, (1.7a) and (1.7b) read

\[ U_i^{(n)} = U_0^{(n)} + \sum_{j=1}^{s} a_{ij} Q_j^{(n)}, \quad i = 1, 2, \ldots, s, \]  
\[ U_0^{(n+1)} = U_0^{(n)} + \sum_{j=1}^{s} b_j Q_j^{(n)}, \]  

(2.2a)  
(2.2b)

**Theorem 2.1.** Assume RKM (1.6) is strongly algebraically stable. Then the corresponding RKLM (1.7) is GDN-stable, and satisfies

\[ \| y_n - z_n \| \leq \exp \left[ -\frac{1}{2} (T - t_0) \sigma L_0 \right] \max_{t_0 - t \leq t_0} \| \varphi(t) - \psi(t) \|, \quad n \geq 0, \]  

(2.3)

where \( L_0 = \sup_{\delta \in [0, 1)} (\sum_{i=-r}^{n} |L_i(\delta)|)^2 \).

**Proof.** From (2.2) we get

\[ \| U_0^{(n+1)} \|^2 = \left( U_0^{(n)} + \sum_{i=1}^{s} b_i Q_i^{(n)}, U_0^{(n)} + \sum_{j=1}^{s} b_j Q_j^{(n)} \right) \]

\[ = \| U_0^{(n)} \|^2 + 2 \sum_{i=1}^{s} b_i \text{Re} \langle Q_i^{(n)}, U_0^{(n)} \rangle + \sum_{i,j=1}^{s} b_i b_j \langle Q_i^{(n)}, Q_j^{(n)} \rangle \]

\[ = \| U_0^{(n)} \|^2 + 2 \sum_{i=1}^{s} b_i \text{Re} \left( Q_i^{(n)}, U_i^{(n)} - \sum_{j=1}^{s} a_{ij} Q_j^{(n)} \right) + \sum_{i,j=1}^{s} b_i b_j \langle Q_i^{(n)}, Q_j^{(n)} \rangle \]

\[ = \| U_0^{(n)} \|^2 + \sum_{i=1}^{s} b_i \text{Re} \langle Q_i^{(n)}, U_i^{(n)} \rangle - \sum_{i,j=1}^{s} (b_i a_{ij} + b_j a_{ji} - b_i b_j) \langle Q_i^{(n)}, Q_j^{(n)} \rangle. \]

It follows from the above equality, algebraic stability of method (1.6) and Lemma 3.4 in [5] that

\[ \| U_0^{(n+1)} \|^2 \leq \| U_0^{(n)} \|^2 + 2 \sum_{i=1}^{s} b_i \text{Re} \langle Q_i^{(n)}, U_i^{(n)} \rangle. \]  

(2.4)

Furthermore, by conditions (1.3)-(1.5) and Schwartz inequality we have

\[ \text{Re} \langle Q_i^{(n)}, U_i^{(n)} \rangle = h [\text{Re} \langle f(T_i^{(n)}, y_i^{(n)}, \bar{y}_i^{(n)}) - f(T_i^{(n)}, z_i^{(n)}, \bar{z}_i^{(n)}), U_i^{(n)} \rangle \]

\[ + \text{Re} \langle f(T_i^{(n)}, z_i^{(n)}, \bar{y}_i^{(n)}) - f(T_i^{(n)}, z_i^{(n)}, \bar{z}_i^{(n)}), U_i^{(n)} \rangle] \]

\[ \leq h \sigma \| U_i^{(n)} \|^2 + h \| f(T_i^{(n)}, z_i^{(n)}, \bar{y}_i^{(n)}) - f(T_i^{(n)}, z_i^{(n)}, \bar{z}_i^{(n)}) \| \cdot \| U_i^{(n)} \| \]

\[ \leq h \sigma \| U_i^{(n)} \|^2 + h \gamma \| \bar{U}_i^{(n)} \| \| U_i^{(n)} \| \]

\[ \leq h \sigma \| U_i^{(n)} \|^2 + \frac{1}{2} h \gamma [\| \bar{U}_i^{(n)} \|^2 + \| U_i^{(n)} \|^2] \]
Substituting (2.5) in (2.4) yields
\[
\|U_0^{(n+1)}\|^2 \leq \|U_0^{(n)}\|^2 - h\sigma \sum_{i=1}^{s} b_i \|\tilde{U}_i^{(n)}\|^2. \tag{2.6}
\]

In addition, with (1.7d) we have
\[
\|\tilde{U}_i^{(n)}\|^2 \leq \left[ \sum_{l=-r}^{v} |L_l(\delta)| \|U_i^{(n-m+l)}\| \right]^2 \leq L_0 \max_{-r \leq l \leq v} \|U_i^{(n-m+l)}\|^2. \tag{2.7}
\]

Combining (2.6) with (2.7) and using (1.7c) we arrive at
\[
\|U_0^{(n+1)}\|^2 \leq (1 - h\sigma L_0) \max\{\|U_0^{(n)}\|^2, \max_{(i,l) \in E} \|U_i^{(n-m+l)}\|^2\}, \tag{2.8}
\]
where \( E = \{(i,l) \mid 1 \leq i \leq s, -r \leq l \leq v\} \). Similar to (2.8), the inequalities
\[
\|U_i^{(n)}\|^2 \leq (1 - h\sigma L_0) \max\{\|U_0^{(n)}\|^2, \max_{(i,l) \in E} \|U_i^{(n-m+l)}\|^2\}, \quad i = 1, 2, \ldots, s, \tag{2.9}
\]
can be obtained by considering the RKLM corresponding to the schemes \((A, A^T, c) (i = 1, 2, \ldots, s)\) and the strong algebraic stability of the method.

In the following, with the help of inequalities (2.8), (2.9) and an induction we shall prove the inequalities
\[
\|U_i^{(n)}\|^2 \leq (1 - h\sigma L_0)^{n+1} \max_{t_0 - r \leq t \leq t_0} \|\varphi(t) - \psi(t)\|^2, \quad n \geq 0, \quad i = 1, 2, \ldots, s. \tag{2.10}
\]
In fact, it is clear from (2.8), (2.9) and \( m \geq v + 1 \) that
\[
\|U_i^{(0)}\|^2 \leq (1 - h\sigma L_0) \max_{t_0 - r \leq t \leq t_0} \|\varphi(t) - \psi(t)\|^2, \quad i = 0, 1, 2, \ldots, s.
\]
Suppose for \( n \leq k (k \geq 0) \) that
\[
\|U_i^{(n)}\|^2 \leq (1 - h\sigma L_0)^{n+1} \max_{t_0 - r \leq t \leq t_0} \|\varphi(t) - \psi(t)\|^2, \quad i = 0, 1, 2, \ldots, s.
\]
Then, from (2.8), (2.9), \( m \geq v + 1 \) and \( 1 - h\sigma L_0 > 1 \), we conclude that
\[
\|U_i^{(n+1)}\|^2 \leq (1 - h\sigma L_0)^{k+2} \max_{t_0 - r \leq t \leq t_0} \|\varphi(t) - \psi(t)\|^2, \quad i = 0, 1, 2, \ldots, s.
\]
This completes the proof of inequalities (2.10). In view of (2.10), we get for \( n \geq 0 \) that
\[
\|U_0^{(n)}\|^2 \leq (1 - h\sigma L_0)^{n+1} \max_{b_0 - \tau \leq t \leq b_0} \|\varphi(t) - \psi(t)\|^2 \\
\leq \exp[-(n + 1)h\sigma L_0] \max_{b_0 - \tau \leq t \leq b_0} \|\varphi(t) - \psi(t)\|^2 \\
\leq \exp[ (T - t_0)\sigma L_0] \max_{t_0 - \tau \leq t \leq t_0} \|\varphi(t) - \psi(t)\|^2 .
\]
As a result, we know that method (1.7) is GDN-stable.

An analogous result for ODES, based on classical Lipschitz condition, can be seen from Gear [7].

3. D-convergence

In this section, we start discussing the convergence of RKLMs (1.7) for DDEs (1.1) with conditions (1.3)–(1.5). It is always assumed that the analytic solution \( y(t) \) of (1.1) is smooth enough and its derivatives used later are bounded by
\[
\left\| \frac{d^i y(t)}{dt^i} \right\| \leq \tilde{M}_i, \quad t \in [t_0 - \tau, T].
\]
For convenience, we review some concepts on RKMs \((A,b,c)\) (cf. [6]). A given RKM \((A,b,c)\) is called diagonally stable if there exists a positive diagonal matrix \( D \) such that \( DA + A^TD \) is nonnegative definite. Moreover, positive integer \( \max\{q | B(q), C(q), \text{hold} \} \) is called stage order of the method \((A,b,c)\), where
\[
B(q): b^Tc^{j-1} = \frac{1}{j}, \quad j = 1, 2, \ldots, q, \\
C(q): Ac^{j-1} = \frac{1}{j}c^j, \quad j = 1, 2, \ldots, q.
\]
are two simplifying order conditions of the method. The following, notations are adopted:
\[
y^{(n)} = \begin{pmatrix}
\hat{y}_1^{(n)} \\
\hat{y}_2^{(n)} \\
\vdots \\
\hat{y}_s^{(n)}
\end{pmatrix}, \quad \tilde{y}^{(n)} = \begin{pmatrix}
\tilde{y}_1^{(n)} \\
\tilde{y}_2^{(n)} \\
\vdots \\
\tilde{y}_s^{(n)}
\end{pmatrix}, \quad Y^{(n)} = \begin{pmatrix}
y(t_n + c_1 h) \\
y(t_n + c_2 h) \\
\vdots \\
y(t_n + c_s h)
\end{pmatrix}, \\
\hat{y}^{(n)} = \begin{pmatrix}
y(t_n + c_1 h - \tau) \\
y(t_n + c_2 h - \tau) \\
\vdots \\
y(t_n + c_s h - \tau)
\end{pmatrix}, \quad F(t, y, \tilde{y}) = \begin{pmatrix}
f(t_n + c_1 h, y_1, \tilde{y}_1) \\
f(t_n + c_2 h, y_2, \tilde{y}_2) \\
\vdots \\
f(t_n + c_s h, y_s, \tilde{y}_s)
\end{pmatrix},
\]

\[
\frac{ dy^{(n)} }{ dt^i } = \begin{pmatrix}
\frac{ d\hat{y}_1^{(n)} }{ dt^i } \\
\frac{ d\hat{y}_2^{(n)} }{ dt^i } \\
\vdots \\
\frac{ d\hat{y}_s^{(n)} }{ dt^i }
\end{pmatrix}, \quad \frac{ d\tilde{y}^{(n)} }{ dt^i } = \begin{pmatrix}
\frac{ d\tilde{y}_1^{(n)} }{ dt^i } \\
\frac{ d\tilde{y}_2^{(n)} }{ dt^i } \\
\vdots \\
\frac{ d\tilde{y}_s^{(n)} }{ dt^i }
\end{pmatrix}.
\]
\( e = (1, 1, \ldots, 1)^T \in \mathbb{R}^s \), \( I_N \) denotes \( N \times N \) identity matrix, \( \tilde{G} = G \otimes I_N \) is the Kronecker product of matrix \( G \) and \( I_N \). Moreover, we also introduce the set
\[
D_\sigma = \{ z \in \mathbb{C}^{NS \times NS} \mid z = \text{blockdiag}(z_i) \text{ with } \mu(z_i) \leq \sigma, z_i \in \mathbb{C}^N, \; i = 1, 2, \ldots, s \},
\]
where \( \mu(\cdot) \) denotes the logarithmic norm [6] defined by
\[
\mu(G) = \max_{x \neq 0} \frac{\text{Re} \langle Gx, x \rangle}{\|x\|}, \quad \forall G \in \mathbb{C}^{N \times N}.
\]

With the above notations, methods (1.7) can be written as
\[
\begin{align*}
\dot{y}^{(n+1)} &= y^{(n)} + h \tilde{b}^T F(t_n, y^{(n)}, \tilde{y}^{(n)}), \quad (3.2a) \\
\dot{y}^{(n)} &= \tilde{e} y^{(n)} + h \tilde{A} F(t_n, y^{(n)}, \tilde{y}^{(n)}), \quad (3.2b) \\
\tilde{y}^{(n)} &= \text{determined by (1.7d)}, \quad (3.2c)
\end{align*}
\]
and the local errors in (3.2a)–(3.2c), \( Q_n \in \mathbb{C}^N \), \( r_n = (r_1^{(n)}, r_2^{(n)}, \ldots, r_s^{(n)}) \), \( \rho_n = (\rho_1^{(n)}, \rho_2^{(n)}, \ldots, \rho_s^{(n)}) \in \mathbb{C}^{NS} \) can be defined as
\[
\begin{align*}
\dot{y}(t_{n+1}) &= \dot{y}(t_n) + h \tilde{b} F(t_n, Y^{(n)}, \tilde{y}^{(n)}) + Q_n, \quad (3.3a) \\
\dot{y}^{(n)} &= \tilde{e} y(t_n) + h \tilde{A} F(t_n, Y^{(n)}, \tilde{y}^{(n)}) + r_n, \quad (3.3b) \\
\tilde{y}^{(n)} &= (\tilde{y}_1^{(n)} \dots \tilde{y}_s^{(n)})^T \quad \text{with} \\
\tilde{y}_i^{(n)} &= \left\{ \begin{array}{ll}
\varphi(t_n + c_i h - \tau), & t_n + c_i h - \tau \leq t_0, \\
\sum_{l=-r}^{0} L_i(\delta) Y_i^{(m+1)} + \rho_i^{(n)}, & t_n + c_i h - \tau \geq t_0.
\end{array} \right. \quad (3.3c)
\end{align*}
\]
According to Taylor formula and Lemma 5.1 in [8], \( Q_n \), \( r_n \) and \( \rho_n \) can be determined, respectively, as follows:
\[
\begin{align*}
Q_n &= \sum_{l=1}^{p} \frac{h^l}{(l-1)!} \left( \frac{1}{l} - \sum_{j=1}^{s} b_j c_j^{l-1} \right) y^{(l)}(t_n) + R_0^{(n)}, \quad (3.4a) \\
r_i^{(n)} &= \sum_{l=1}^{p} \frac{h^l}{(l-1)!} \left( \frac{1}{l} c_i^l - \sum_{j=1}^{s} a_{ij} c_j^{l-1} \right) y^{(l)}(t_n) + R_i^{(n)}, \quad (3.4b) \\
\rho_i^{(n)} &= \frac{h^{r+i+1}}{(v + r + 1)!} \left[ \prod_{l=-r}^{v} (\delta - l) \right] y^{(s+r+1)}(\xi_i^{(n)}), \quad \xi_i^{(n)} \in (t_{n-m-r} + c_i h, t_{n-m+v} + c_i h), \quad (3.4c)
\end{align*}
\]
where
\[
\| R_i^{(n)} \| \leq \tilde{M}_i h^{r+1}, \quad i = 0, 1, 2, \ldots, s, \quad h \in (0, h_0],
\]
h_0 depends only on the method, and \( \tilde{M}_i \) (\( i = 0, 1, 2, \ldots, s \)) depend only the method and some \( \tilde{M}_i \) in (3.1).
Definition 3.1. An interpolation scheme is called being of order $q$ if its local error is $O(h^{q+1})$.

In accordance with (3.4c) we know that the interpolation scheme (1.7d) is of order $r + v$.

Definition 3.2. A given RKLM (1.7) with $y_n = y(t_n)$ ($n \leq 0$) and $y^{(n)}_t = y(t_n + c_i h)$ ($n < 0$) is called D-convergent of order $p$ if this method, when applied to any given DDE (1.1) subject to (1.3)–(1.5), produces an approximation sequence $\{y_n\}$, and the global error satisfies a bound of the form

$$\|y(t_n) - y_n\| \leq c(t_n) h^p, \quad 0 < h \leq h_0,$$

where the maximum stepsize $h_0$ depends only on characteristic parameter $\sigma$ and the method; the function $c(t)$ depends only on some $\tilde{M}_i$ in (3.1), delay $\tau$, characteristic parameter $\sigma$ and the method.

It is remarkable that D-convergence for $\tau = 0$ is just B-convergence [6] in ODEs.

To describe the main results, we define the inner product and the induced norm in $(C^N)^s$ as follows:

$$\langle u, v \rangle = \sum_{i=1}^{s} \langle u_i, v_i \rangle, \quad \|u\| = \sqrt{\langle u, u \rangle} = \sqrt{\sum_{i=1}^{s} \|u_i\|^2},$$

where $u = (u_1, u_2, \ldots, u_s), v = (v_1, v_2, \ldots, v_s) \in (C^N)^s, u_i, v_i \in C^N$. In addition, we also present two lemmas, which are obtained by a slight alteration to the proofs of the Lemma 2.1 and 2.2 in [1].

Lemma 3.1. Assume RKM (1.6) is diagonally stable. Then there exist constants $D_1, D_2 > 0$, which depend only on the method, such that for any given $h > 0$ and $v \in D_\sigma$ with $\sigma \leq 0$ we have

1. matrix $\tilde{I}_s - h\tilde{A}$ is nonsingular,
2. $(\tilde{I}_s - h\tilde{A})^{-1} \leq D_1, \|h\tilde{A}_i (\tilde{I}_s - h\tilde{A})^{-1}\| \leq D_2, i = 0, 1, 2, \ldots, s.$

Lemma 3.2. Assume RKM (1.6) is strongly algebraically stable. Then for $\forall h > 0$ and $\forall v \in D\sigma$ with $\sigma \leq 0$ we have

$$\|k_i(z)\| \leq 1, \quad i = 0, 1, 2, \ldots, s.$$

where $k_i(z) = I_N + h\tilde{A}_i^T z (\tilde{I}_s - h\tilde{A})^{-1} \tilde{e}$.

Based on the above lemmas we can derive our main result.

Theorem 3.1. Assume RKM (1.6) with stage order $p$ is strongly algebraically stable and diagonally stable, and the interpolation scheme (1.7d) is of order $q$ (where $q = r + v$). Then the corresponding RKLM (1.7) is D-convergent of order $\min\{p, q + 1\}$.

Proof. Subtraction of (3.2) from (3.3) yields the following recursion scheme:

$$
\begin{align*}
\tilde{e}^{(n+1)}_0 &= \tilde{e}^{(n)}_0 + h\hat{\tilde{A}}^T [z_n \tilde{e}_n + F(t_n, y^{(n)}, \tilde{y}^{(n)}) - F(t_n, y^{(n)}, \tilde{y}^{(n)})] + Q_n, \quad (3.5a) \\
\tilde{e}_n &= \tilde{e}^{(n)}_0 + h\hat{A} [z_n \tilde{e}_n + F(t_n, y^{(n)}, \tilde{y}^{(n)}) - F(t_n, y^{(n)}, \tilde{y}^{(n)})] + r_n, \quad (3.5b)
\end{align*}
$$
where \( \epsilon_0^{(n)} = y(t_n) - y_n \), \( \epsilon_n = (\epsilon_1^{(n)}, \epsilon_2^{(n)}, \ldots, \epsilon_s^{(n)})^T = Y^{(n)} - y^{(n)} \), \( z_n = \text{blockdiag} (z_i^{(n)}) \) with
\[
\begin{align*}
z_i^{(n)} &= \int_0^1 f_i(t_n + c_i h, y_i(t_n + \theta (Y_i^{(n)} - y_i^{(n)})), \tilde{Y}_i(t_n + \theta)) \, d\theta, \quad i = 1, 2, \ldots, s,
\end{align*}
\]
and \( f_2(t, u, v) \) is the Jacobian matrix \( (\partial f(t, u, v)/\partial u) \) \( (t \in \mathbb{R}, \ u, v \in \mathbb{C}^N) \). From (3.5) we can get
\[
\begin{align*}
\epsilon_0^{(n+1)} &= k_0(z_n) \epsilon_0^{(n)} + h \tilde{T} z_n (I_s - h \tilde{A} z_n)^{-1} r_n + h \tilde{T} [h z_n (I_s - h \tilde{A} z_n)^{-1} \tilde{A} + I_s] \\
&\quad \times [F(t_n, y_i^{(n)}, \tilde{Y}_i^{(n)}) - F(t_n, y_i^{(n)}, \tilde{Y}_i^{(n)})] + \mathcal{Q}_n.
\end{align*}
\]
(3.6)

By conditions (1.3), (3.4), \( B(p) \) and \( C(p) \) we know for \( h \in (0, h_0) \) that
\[
\begin{align*}
z_n &\in \mathcal{D}_n, \quad r_n = (R_1^{(n)}, R_2^{(n)}, \ldots, R_s^{(n)}), \quad \mathcal{Q}_n = R_0^{(n)}.
\end{align*}
\]
(3.7)

A combination of (1.3)-(1.5), (1.7c), (3.4), (3.6), (3.7), as well as Lemmas 3.1 and 3.2, yields for \( h \in (0, h_0) \) that
\[
\begin{align*}
\| \epsilon_0^{(n+1)} \| &\leq \| k_0(z_n) \| \| \epsilon_0^{(n)} \| + \| h \tilde{T} z_n (I_s - h \tilde{A} z_n)^{-1} \| \| r_n \| \\
&\quad + h \{ \| h \tilde{T} z_n (I_s - h \tilde{A} z_n)^{-1} \| \| \tilde{A} [F(t_n, y_i^{(n)}, \tilde{Y}_i^{(n)}) - F(t_n, y_i^{(n)}, \tilde{Y}_i^{(n)})] \| \\
&\quad + \| h \tilde{T} [F(t_n, y_i^{(n)}, \tilde{Y}_i^{(n)}) - F(t_n, y_i^{(n)}, \tilde{Y}_i^{(n)})] \| \} + \| \mathcal{Q}_n \| \\
&\leq \| \epsilon_0^{(n)} \| + D_2 \left( \sum_{i=1}^s \| R_i^{(n)} \| ^2 \right) \\
&\quad + h \left[ D_2 \left( \sum_{i=1}^s \left( \sum_{j=1}^s \| a_{ij} \| \right) \right) \left[ \sum_{i=1}^s \| f(t_n + c_j h, y_j^{(n)}, \tilde{Y}_j^{(n)}) - f(t_n + c_j h, y_j^{(n)}, \tilde{Y}_j^{(n)}) \| ^2 \right] \right] \\
&\quad + \left( \sum_{j=1}^s b_j \| f(t_n + c_j h, y_j^{(n)}, \tilde{Y}_j^{(n)}) - f(t_n + c_j h, y_j^{(n)}, \tilde{Y}_j^{(n)}) \| \right) + \tilde{M}_0 h^{p+1} \\
&\leq \| \epsilon_0^{(n)} \| + \left[ D_2 \left( \sum_{i=1}^s \| M_i^2 + \tilde{M}_0 \| \right) \right] h^{p+1} + h(-\sigma) \left[ D_2 \left( \sum_{i=1}^s \| c_i^2 \| + 1 \right) \right] \\
&\quad \times \max_{1 \leq j \leq s} \| \tilde{Y}_j^{(n)} - \tilde{Y}_j^{(n)} \|.
\end{align*}
\]
(3.8)

It follows from (3.2c), (3.3c) and (3.4c) for \( h \in (0, h_0) \), that
\[
\begin{align*}
\| \tilde{Y}_j^{(n)} - \tilde{Y}_j^{(n)} \| &\leq \sum_{l=-r}^v |L_l(\delta)| \| \epsilon_l^{(n-m+l)} \| + \| \rho_j^{(n)} \| \\
&\leq \sup_{\delta \in (0,1)} \sum_{l=-r}^v |L_l(\delta)| \max_{-r \leq l \leq v} \| \epsilon_l^{(n-m+l)} \| + \frac{h^{p+1}}{(q+1)!} \sup_{\delta \in (0,1)} \left[ \prod_{l=-r}^v |\delta - l| \right] \tilde{M}_{q+1}.
\end{align*}
\]
(3.9)
where \( q = v + r \). Substituting (3.8) in (3.7) we get
\[
\|e_0^{(n+1)}\| \leq \gamma_1 \max_{(i, t) \in E} \|e_0^{(n-m+i)}\| + \gamma_2 h^{p+1} + \gamma_3 h^{q+2}, \quad h \in (0, h_0),
\]
(3.10)

with
\[
\gamma_1 = -\sigma \left( D_2 \sqrt{\sum_{i=1}^s c_i^2 + 1} \right) \sup_{\delta \in [0, 1]} \sum_{l=-r}^v |L(\delta)|,
\]
\[
\gamma_2 = D_2 \sqrt{\sum_{i=1}^s \hat{M}_i^2 + \hat{M}_0},
\]
\[
\gamma_3 = -\sigma \frac{\hat{M}_{q+1}}{(q + 1)!} \left( D_2 \sqrt{\sum_{i=1}^s c_i^2 + 1} \right) \sup_{\delta \in [0, 1]} \prod_{l=-r}^v |\delta - l|.
\]

By (3.10) we have for \( h \in (0, h_0) \) that
\[
\|e_0^{(n+1)}\| \leq (1 + h \gamma_1) \max \{\|e_0^{(n)}\|, \max_{(i, t) \in E} \|e_0^{(n-m+i)}\|\} + \gamma_2 h^{p+1} + \gamma_3 h^{q+2}.
\]
(3.11)

Using Lemma 3.1 and 3.2, similar to (3.11) the inequalities
\[
\|e_i^{(n)}\| \leq (1 + h \gamma_1) \max \{\|e_0^{(n)}\|, \max_{(j, t) \in E} \|e_0^{(n-m+i)}\|\} + \gamma_2 h^{p+1} + \gamma_3 h^{q+2},
\]
\[
i = 1, 2, \ldots, s, \ h \in (0, h_0)
\]
(3.12)
can be obtained by (3.2b), (3.2c), (3.3b) and (3.3c), where
\[
\gamma_2 = D_2 \sqrt{\sum_{i=1}^s \hat{M}_i^2 + \hat{M}_i}, \quad i = 1, 2, \ldots, s.
\]

Putting \( \gamma_2 = \max \{\gamma_2^{(i)} \mid 0 \leq i \leq s\} \) and combining (3.11) with (3.12) we arrive at
\[
\|e_i^{(n)}\| \leq (1 + h \gamma_1) \max \left\{\|e_0^{(n)}\|, \max_{(j, t) \in E} \|e_0^{(n-m+i)}\|\right\} + \gamma_2 h^{p+1} + \gamma_3 h^{q+2},
\]
\[
i = 0, 1, 2, \ldots, s, \ h \in (0, h_0).
\]
(3.13)

Next, with an induction to (3.13) we shall prove inequalities
\[
\|e_i^{(n)}\| \leq \sum_{j=0}^n (1 + h \gamma_1)^j (\gamma_2 h^{p+1} + \gamma_3 h^{q+2}), \quad i = 0, 1, 2, \ldots, s, \ h \in (0, h_0).
\]
(3.14)

In fact, it is apparent from (3.13) and \( m \geq v + 1 \) that
\[
\|e_i^{(0)}\| \leq \gamma_2 h^{p+1} + \gamma_3 h^{q+2}, \quad i = 0, 1, 2, \ldots, s, \ h \in (0, h_0).
\]

Suppose for \( n < k \) \( (k \geq 0) \) that
\[
\|e_i^{(n)}\| \leq \sum_{j=0}^n (1 + h \gamma_1)^j (\gamma_2 h^{p+1} + \gamma_3 h^{q+2}), \quad i = 0, 1, 2, \ldots, s, \ h \in (0, h_0).
\]
Then, from (3.13) and $m \geq v + 1$ we get

$$
\|e^{(k+1)}_i\| \leq \sum_{j=0}^{k+1} (1 + h\gamma_1)^j(\gamma_2 h^{p+1} + \gamma_3 h^{q+2}), \quad i = 0, 1, 2, \ldots, s, \quad h \in (0, h_0].
$$

This completes the proof of inequalities (3.14). By (3.14) we arrive at for $h \in (0, h_0]$ that

$$
\|e^{(n)}_0\| \leq \sum_{j=0}^{n} (1 + h\gamma_1)^j(\gamma_2 h^{p+1} + \gamma_3 h^{q+2})
$$

$$
= \frac{(1 + h\gamma_1)^{n+1} - 1}{h\gamma_1}(\gamma_2 h^{p+1} + \gamma_3 h^{q+2})
$$

$$
\leq \frac{\exp[(n + 1)h\gamma_1] - 1}{\gamma_1}(\gamma_2 h^{p} + \gamma_3 h^{q+1})
$$

$$
\leq c(n,h)h^{\min\{p,q+1\}},
$$

where

$$
c(t) = \begin{cases} 
\frac{\exp[(t-t_0)\gamma_1]\exp(h_0\gamma_1) - 1}{\gamma_1}(\gamma_2 + \gamma_3 h_0^{p+1-p}), & p \leq q, \\
\frac{\exp[(t-t_0)\gamma_1]\exp(h_0\gamma_1) - 1}{\gamma_2}(\gamma_2 h_0^{p-q-1} + \gamma_3), & p > q.
\end{cases}
$$

Hence, method (1.7) is D-convergence of order $\min\{p, q + 1\}$. □

4. Some examples

**Example 4.1.** The two-stage two-order diagonally implicit method (cf. [4])

$$
\begin{array}{ccc}
\lambda & \lambda & 0 \\
1 - \lambda & 1 - 2\lambda & \lambda \\
\frac{1}{2} & \frac{1}{2}
\end{array}
$$

(4.1)

is strongly algebraically stable iff $\frac{1}{4} \leq \lambda \leq \frac{1}{2}$, since

$$
M_0 = \left(\begin{array}{cc}
\frac{1}{4} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{array}\right) \left[\begin{array}{cc}
1 & -1 \\
-1 & 1
\end{array}\right], \quad M_1 = \left[\begin{array}{cc}
\lambda^2 & 0 \\
0 & 0
\end{array}\right], \quad M_2 = \left[\begin{array}{cc}
(1 - 2\lambda)(4\lambda - 1) & 0 \\
0 & \lambda^2
\end{array}\right].
$$

As to the diagonal stability of the method, we can verify it to be true by taking $D = I_2$, such that

$$
DA + A^TD = \left[\begin{array}{cc}
2\lambda & 1 - 2\lambda \\
1 - 2\lambda & 2\lambda
\end{array}\right]
$$

is positive definite for $\lambda > \frac{1}{4}$. Moreover, we find that $B(2), C(1)$ hold. Therefore, it follows from Theorems 2.1 and 3.1 that the RKLM (1.7) corresponding to method (4.1) is GDN-stable and D-convergent of order one if $\frac{1}{4} \leq \lambda \leq \frac{1}{2}$. 
Example 4.2 (cf. Burrage and Butcher [4]). The s-stage method given by

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c}
0 & \lambda & b_1 & \lambda & b_1 & \lambda & \sum_{i=1}^{s-1} b_i + \lambda & b_1 & b_2 & b_3 & \ldots & b_{s-1} & \lambda \\
\sum_{i=1}^{s-1} b_i + \lambda & b_1 & b_2 & \lambda & b_1 & b_2 & b_3 & \ldots & b_{s-1} & \lambda & b_1 & b_2 & b_3 & \ldots & b_{s-1} & b_2
\end{array}
\]

(4.2)

is strongly algebraically stable iff \(0 \leq b_i \leq 2\lambda \ (i = 1, 2, \ldots, s)\), since

\[
M_0 = \text{diag}(b_1(2\lambda - b_1), b_2(2\lambda - b_2), \ldots, b_s(2\lambda - b_s)),
\]

\[
M_i = \text{diag}(b_1(2\lambda - b_1), \ldots, b_{i-1}(2\lambda - b_{i-1}), \lambda^2, 0, \ldots, 0).
\]

Further, if we take \(D = I_s\), then

\[
DA + A^TD =
\begin{bmatrix}
2\lambda & b_1 & b_1 & \ldots & b_1 & b_1 \\
b_1 & 2\lambda & b_2 & \ldots & b_2 & b_2 \\
b_1 & b_2 & 2\lambda & \ldots & b_3 & b_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
b_1 & b_2 & b_3 & \ldots & 2\lambda & b_{s-1} \\
b_1 & b_2 & b_3 & \ldots & b_{s-1} & 2\lambda
\end{bmatrix}
\]

is positive definite whenever

\[
2\lambda > \max \left\{ (s-1)b_1, \max_{2 \leq i \leq s} \left[ \sum_{j=1}^{i-1} b_j + (s-i)b_i \right] \right\} \geq 0.
\]

Consequently, in terms of Theorems 2.1 and 3.1 we know that the RKLM (1.7) corresponding to the method (4.2) is GDN-stable, and D-convergent of order \(\min\{p, q+1\}\) if the interpolation is of order \(q\) and

\[
2\lambda > \max \left\{ (s-1)b_1, \max_{2 \leq i \leq s} \left\{ \left[ \sum_{j=1}^{i-1} b_j + (s-i)b_i \right], b_i \right\} \right\} \geq 0,
\]

\(B(p), C(p)\) hold. In particular, setting \(s = 2, \lambda = b_1 = b_2 = \frac{1}{2}\), the method (4.2) becomes

\[
\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\]

(4.3)

It is easy to testify for (4.3) that \(B(1), C(1)\) hold. Hence, in accordance with above discussion, the method (4.3) together with the linear interpolation scheme produces a GDN-stable and order one
D-convergent RKLM

\[ y_1^{(n)} = y_n + \frac{h}{2} f \left( t_n + \frac{h}{2}, y_1^{(n)}, \tilde{y}_1^{(n)} \right), \]

\[ y_2^{(n)} = y_n + \frac{h}{2} f \left( t_n + \frac{h}{2}, y_1^{(n)}, \tilde{y}_1^{(n)} \right) + \frac{h}{2} f(t_{n+1}, y_2^{(n)}, \tilde{y}_2^{(n)}), \]

\[ \tilde{y}_j^{(n)} = (1 - \delta)y_j^{(n-m)} + \delta y_j^{(n-m+1)}, \quad j = 1, 2 \]

\[ y_{n+1} = y_n + \frac{h}{2} f \left( t_n + \frac{h}{2}, y_1^{(n)}, \tilde{y}_1^{(n)} \right) + \frac{h}{2} f(t_{n+1}, y_2^{(n)}, \tilde{y}_2^{(n)}), \]

where \( n \geq 0, (m - \delta)h = \tau, \) positive integer \( m \geq 2, \delta \in [0, 1). \)

References


