Minimum Permanents on Certain Doubly Stochastic Matrices*

Seok-Zun Song

Department of Mathematics
Cheju National University
Cheju City 690-756, Republic of Korea

Submitted by Richard A. Brualdi

ABSTRACT

We determine the minimum permanents and minimizing matrices on the faces of
\( \Omega_{n+2} \) for the fully indecomposable \((0,1)\) matrices of order \( n+2 \), which includes an
identity submatrix of order \( n \).

Let \( D = [d_{ij}] \) be an \( n \)-square \((0,1)\) matrix, and let
\[ a(D) = \{ X = [x_{ij}] \in \mathbb{R}^{n \times n} : x_{ij} = 0 \text{ whenever } d_{ij} = 0 \} . \]

Then \( \Omega(D) \) is a face of \( \Omega_n \), the polytope of \( n \)-square doubly stochastic
matrices, and \( \Omega(D) \) contains a minimizing matrix \( A \) such that \( \text{per} \ A \leq \text{per} \ X \)
for all \( X \in \Omega(D) \).

Recall that an \( n \)-square nonnegative matrix is said to be \textit{fully indecomposable} if it contains no \( k \times (n-k) \) zero submatrix for \( k = 1, \ldots, n-1 \).

Brualdi [1] defined an \( n \)-square \((0,1)\)-matrix \( D \) to be \textit{cohesive} if there is
a matrix \( Z \) in the interior of \( \Omega(D) \) for which
\[ \text{per} \ Z = \min \{ \text{per} \ X \mid X \in \Omega(D) \} . \]

*This research was supported by Korea Science & Engineering Foundation
#893-0101-009-1.

LINEAR ALGEBRA AND ITS APPLICATIONS 143:49–56 (1991) 49

655 Avenue of the Americas, New York, NY 10010 0024-3795/91/$3.50
And he defined an $n$-square $(0,1)$ matrix $D$ to be barycentric if

$$\text{per } b(D) = \min \{ \text{per } X | X \in \Omega(D) \},$$

where the barycenter $b(D)$ of $\Omega(D)$ is given by

$$b(D) = \frac{1}{\text{per} \ D} \sum_{P \leq D} P,$$

where the summation extends over the set of all permutation matrices $P$ with $P \leq D$ and per $D$ is their number.

Let $K_2, K_{2,n},$ and $K_{n,2}$ be the $2 \times 2, 2 \times n,$ and $n \times 2$ matrices with all entries equal to 1, respectively. For $n \geq 3$, let

$$U_{2,n} = \begin{bmatrix} 0_2 & K_{2,n} \\ K_{n,2} & I_n \end{bmatrix}, \quad V_{2,n} = \begin{bmatrix} K_2 & K_{2,n} \\ K_{n,2} & I_n \end{bmatrix}$$

be $(n+2)$-square $(0,1)$ matrices that contain the identity matrix $I_n$ of order $n$ as a submatrix.

In the paper [1], Brualdi obtained the minimum permanent on the face $\Omega(V_{1,n-1})$, where $V_{1,n-1}$ is the $n$-square $(0,1)$ matrix which contains the identity matrix $I_{n-1}$ of order $n-1$ as a submatrix. Minc [3] obtained the minimum permanent on the face $\Omega(V_{2,2})$. Song [4] obtained the minimum permanent on the face $\Omega(V_{2,3})$ and the local minimum permanents on the face $\Omega(U_{2,n})$ at its barycenter.

The purpose of this paper is to determine the minimum permanents and minimizing matrices on the faces $\Omega(U_{2,n})$ and $\Omega(V_{2,n})$, respectively. This enhances the theorems of Brualdi [1, Theorem 5] and Song [4, Theorem 2.5].

**Lemma 1 [3].** If $A = (a_{ij})$ is a minimizing matrix on $\Omega(U_{2,n})$ (or $\Omega(V_{2,n})$), then $\text{per } A(p \mid q) = \text{per } A$ for $a_{pq} > 0$, and $A$ is fully indecomposable. Here $A(p \mid q)$ denotes the submatrix obtained from $A$ by deleting row $p$ and column $q$. 
LEMMA 2 [4]. For \( n \geq 3 \), we have that a minimizing matrix on the face \( \Omega(U_{2,n}) \) is the following form:

\[
A = \begin{bmatrix}
0 & b_1 & b_2 & \cdots & b_n \\
b_1 & b_1 & x_1 & 0 \\
b_2 & b_2 & x_2 & & \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
b_n & b_n & 0 & \cdots & x_n \\
\end{bmatrix},
\]

(2)

where \( x_i \) and \( b_i \) are positive for \( i = 1, \ldots, n \). That is, \( U_{2,n} \) is a cohesive matrix.

LEMMA 3. If \( A \) in (2) of Lemma 2 is a minimizing matrix on the face \( \Omega(U_{2,n}) \), then \( x_i > (n-3)b_i \) (i.e. \( b_i < 1/(n-1) \)) for all \( i = 1, \ldots, n \).

Proof. Suppose that \( n > 3 \), since there is nothing to prove for \( n = 3 \). Assume that \( (n-3)b_i \geq x_i \) for all \( i = 1, \ldots, n \). Then \( b_i \geq 1/(n-1) \) for all \( i = 1, \ldots, n \), and hence we have a contradiction as follows:

\[
1 = \sum_{i=1}^{n} b_i \leq \frac{n}{n-1} > 1.
\]

Therefore, we have some \( b_i \) such that \( x_i > (n-3)b_i \) for at least one \( i \). Say, let \( x_n > (n-3)b_n \). If \( x_i \leq (n-3)b_i \) for at least two \( i \), say \( i = 1, 2 \), then \( b_1 \geq 1/(n-1) \) and \( 0 < b_n < 1/(n-1) \). Assume \( b_2 = \max\{b_1, b_2\} \) without loss of generality. Then we have a contradiction as follows:

\[
0 = \per A(1|3) - \per A(1|n+2)
\]

\[
= 2(b_1 - b_n) \left\{ \sum_{i=2}^{n-1} \frac{b_i^2}{x_i} x_2 x_3 \cdots x_{n-1} - b_1 b_n x_2 \cdots x_{n-1} \right\}
\]

\[
> 0,
\]

(3)
since $b_1 - b_n > 0$ and the quantity in the braces of (3) is greater than

$$\left( b_2^2 - b_1 b_n x_2 \right) x_3 \cdots x_{n-1}$$

$$> b_2^2 \left( b_2 - b_1 \frac{1}{n-1} (n-3) \right) x_3 \cdots x_{n-1}$$

$$> 0.$$  \hfill (4)

Now, assume that for some one $i$, say $i = 2$, $x_2 < (n-3)b_2$ and $x_j > (n-3)b_j$ for $j = 1, 3, 4, \ldots, n$ without loss of generality. Then the quantity in braces in (3) is positive by (4). So we obtain $b_1 = b_n$ from (3). Similarly, we obtain $b_1 = b_i$ from

$$0 = \text{per} A(1|3) - \text{per} A(1|i+2)$$

for $i = 3, 4, \ldots, n-1$. Then we have a contradiction as follows:

$$0 = \text{per} A(4|4) - \text{per} A(3|3)$$

$$= 4 (b_1 - b_2) \left[ (n-2) b_1^2 \left( (b_1 + b_2) x_1^{n-3} - (n-3) b_1^2 x_1^{n-4} \right) \right]$$

$$< 0,$$

since $b_1 - b_2 < 0$ and $(b_1 + b_2) x_1^{n-3} - (n-3) b_1^2 x_1^{n-4} > (n-3) b_1 x_1^{n-4} (b_1 + b_2 - b_1) > 0$. Hence $x_i > (n-3)b_i$ for all $i = 1, \ldots, n$.

**Lemma 4.** If $A$ in (2) of Lemma 2 is a minimizing matrix on the face $\Omega(U_{2,n})$, then $b_i + b_j > b_k$ for $i, j, k = 1, \ldots, n$.

**Proof.** We prove that $b_1 + b_2 > b_k$ for $k = 1, \ldots, n$ without loss of generality. Assume that

$$\max\{b_{m+1}, \ldots, b_n\} < b_1 + b_2 \leq \min\{b_3, \ldots, b_m\}$$
without loss of generality for \( m = 3, \ldots, n - 1 \). Then

\[
1 = \sum_{i=1}^{n} b_i = \sum_{i=1}^{2} b_i + \sum_{i=3}^{m} b_i + \sum_{i=m+1}^{n} b_i
\]

\[
< \frac{1}{m-2} \sum_{i=3}^{m} b_i + \sum_{i=3}^{m} b_i + \frac{n-m}{m-2} \sum_{i=3}^{m} b_i
\]

\[
= \left( \frac{1}{m-2} + 1 + \frac{n-m}{m-2} \right) \sum_{i=3}^{m} b_i
\]

\[
= \frac{n-1}{m-2} \sum_{i=3}^{m} b_i.
\]

Since all \( b_i \) are positive by Lemma 2, it follows that there exists at least one value of \( i \) such that \( b_i > 1/(n-1) \). This contradicts the result of Lemma 3. Hence \( b_1 + b_2 > b_k \) for \( k = 1, \ldots, n \). \( \blacksquare \)

**Theorem 5.** For \( n \geq 3 \), \( U_{2,n} \) is barycentric and the minimum permanent on the face \( \Omega(U_{2,n}) \) is

\[
\frac{2(n-1)(n-2)^{n-2}}{n^{n+1}}.
\]

**Proof.** By Lemma 2, we have that a minimizing matrix on the face \( \Omega(U_{2,n}) \) is of the form \( A \) in (2). Define

\[
\Phi = x_1 x_2 \cdots x_{n-2}, \quad \Phi_i = \frac{\Phi}{x_i}, \quad \Phi_{ij} = \frac{\Phi}{x_i x_j}
\]

for \( i, j = 1, \ldots, n-2, \; i \neq j, \; n > 3 \). If \( n = 3 \), use \( \Phi = x_1, \; \Phi_1 = 1, \; \Phi_{ij} = 0 \). Since
\( x_{n-1}, x_n \) are positive, we have

\[
0 = \text{per} A(n + 2|n + 2) - \text{per} A(n + 1|n + 1) \\
= 4(b_{n-1} - b_n) \\
\times \left[ b_1^2 \left( (b_{n-1} + b_n) \Phi_1 - \sum_{i=2}^{n-2} b_i^2 \Phi_{1i} \right) \\
+ b_2^2 \left( (b_{n-1} + b_n) \Phi_2 - \sum_{i=1 \atop i \neq 2}^{n-2} b_i^2 \Phi_{2i} \right) + \cdots \\
+ b_{n-2}^2 \left( (b_{n-1} + b_n) \Phi_{n-2} - \sum_{i=1}^{n-3} b_i^2 \Phi_{(n-2)i} \right) \right] \\
= 4(b_{n-1} - b_n) \\
\times \left[ b_1^2 \left( \sum_{i=2}^{n-2} \left( \frac{1}{n-3} (b_{n-1} + b_n) x_i - b_i^2 \right) \right) \Phi_{1i} \\
+ b_2^2 \left( \sum_{i=1 \atop i \neq 2}^{n-2} \left( \frac{1}{n-3} (b_{n-1} + b_n) x_i - b_i^2 \right) \right) \Phi_{2i} + \cdots \\
+ b_{n-2}^2 \left( \sum_{i=1}^{n-3} \left( \frac{1}{n-3} (b_{n-1} + b_n) x_i - b_i^2 \right) \right) \Phi_{(n-2)i} \right]. \tag{5}
\]

Since \( x_i > (n-3)b_i \) and \( b_i + b_j > b_k \) for \( i, j, k = 1, \ldots, n \) by Lemmas 3 and 4, we have that

\[
\frac{1}{n-3} (b_{n-1} + b_n) x_i - b_i^2 > b_i (b_{n-1} + b_n - b_i) > 0.
\]
Therefore the quantity in the large bracket of (5) is positive, and hence \( b_{n-1} - b_n \) must be zero in (5). Similarly, we have \( b_i = b_n = 1/n \) from the equality of \( \text{per} \ A(n+2\,|\,n+2) \) and \( \text{per} \ A(i+2\,|\,i+2) \) for \( i = 1, \ldots, n-1 \). In this case, the minimizing matrix \( A \) in (2) is the barycenter of the face \( \Omega(U_{2,n}) \), and the minimum permanent on \( \Omega(U_{2,n}) \) is

\[
\frac{2(n-1)(n-2)^{n-2}}{n^{n+1}}.
\]

**Theorem 6.** For \( n \geq 4 \), the minimum permanent on the face \( \Omega(V_{2,n}) \) is

\[
\frac{2(n-1)(n-2)^{n-2}}{n^{n+1}},
\]

which occurs uniquely at the barycenter \( b(U_{2,n}) \) of the face \( \Omega(U_{2,n}) \).

**Proof.** Choose \( A \) so that it has the minimum permanent on the face \( \Omega(V_{2,n}) \). Then \( A \) is fully indecomposable by Lemma 1. Since the first two columns and rows of \( V_{2,n} \) are the same respectively, we can replace each of the first two columns and rows by their average (by Theorem 1 in Minc [3]). Then the resulting matrix \( B \) has the same permanent as \( A \) and has the following form:

\[
B = \begin{bmatrix}
  a & a & b_1 & b_2 & \cdots & b_n \\
  a & a & b_1 & b_2 & \cdots & b_n \\
  b_1 & b_1 & x_1 & x_2 & \cdots & 0 \\
  b_2 & b_2 & x_2 & x_3 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  b_n & b_n & 0 & 0 & \cdots & x_n \\
\end{bmatrix}.
\]

Then all \( b_i \) and \( x_i \) are positive for \( i = 1, \ldots, n \) by the same method as the proof of Lemma 2 (see Theorem 2.7 in [4]). If all \( x_i \) are less than \( 2b_i \) for \( i = 1, \ldots, n \), then all \( b_i \) are greater than \( \frac{1}{4} \) from

\[
1 = x_i + 2b_i < 4b_i.
\]
Then we have a contradiction as follows:

$$1 < \frac{n}{4} < \sum_{i=1}^{n} b_i \leq 1$$

for $n \geq 4$. Hence there exists some $i$ such that $x_i \geq 2b_i$. We may assume that $x_n \geq 2b_n$ without loss of generality. If $a$ is positive, then we have

$$0 = \text{per } B(1|1) - \text{per } B(1|n+2)$$

$$= (x_n - 2b_n)(ax_2 \cdots x_{n-1} + b_1^2x_2x_3 \cdots x_{n-1} + \cdots + b_{n-1}^2x_1x_2 \cdots x_{n-2})$$

$$+ b_n^2x_1x_2 \cdots x_{n-1}$$

$$> 0.$$  

This is a contradiction. So $a$ must be zero. Therefore, the minimizing matrix with $a = 0$ in (6) is the barycenter of the face $\Omega(U_{2,n})$ from Theorem 5. Hence we have the required minimum permanent by Theorem 5.

REFERENCES


Received 1 May 1989; final manuscript accepted 21 December 1989