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The finite section method for infinite Vandermonde matrices

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Dedicated to the memory of Israel Gohberg

Abstract

The finite section method for infinite Vandermonde matrices is the focus of this paper. In particular, it is shown that for a large class of infinite Vandermonde matrices the finite section method converges in l_1 sense if the right hand side of the equation is in a suitably weighted $l_1(\alpha)$ space. Some explicit results are obtained for a wide class of examples.

Keywords: Finite section method; Infinite Vandermonde matrix; Infinite systems of equations

1. Introduction

Already in the nineteenth century are there cases in the mathematical literature where an infinite system of linear equations in an infinite number of unknowns needs to be solved. The situation gave rise to the 1913 book of Riesz [5] and in later years greatly influenced the development of functional analysis and operator theory.

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A natural approach to finding a solution of a system containing countably many equations and unknowns is the following. Take the first n equations and n unknowns, neglect the rest; then we have a finite system, which we solve. As n grows larger, we expect the solutions of the finite systems to approximate a solution of the infinite system.

This method, which is called the *finite section method*, appears already in the work of Fourier (cited in [5]). Fourier looks for a solution of the Laplace equation

$$v_{xx} + v_{yy} = 0$$

satisfying certain boundary conditions and in the course of his calculations he is led to the infinite system

$$\begin{pmatrix} 1 & 1 & 1 & \cdots \\ 1^2 & 3^2 & 5^2 & \cdots \\ 1^4 & 3^4 & 5^4 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \tag{1}$$

The numbers x_k he finds by applying the finite section method are appropriate for the solution of his original problem; however, in a strict sense, they do not solve the infinite system above.

Therefore, the question arises: Under what conditions is it possible to apply the finite section method to obtain a solution of such an infinite system? The particular problem above admits the natural generalization

$$\begin{pmatrix} 1 & 1 & 1 & \cdots \\ a_0 & a_1 & a_2 & \cdots \\ a_0^2 & a_1^2 & a_2^2 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \end{pmatrix}, \tag{2}$$

i.e., it is a system described by an *infinite Vandermonde matrix*, where we take $a_k = (2k+1)^2$ in (1). We shall examine in this paper how the finite section method works for this class.

We have to define what we mean precisely when we say that the finite section method works, as this differs from the interpretation in e.g. [1,3] where the operator is a bounded invertible operator.

We start by introducing some concepts and notations for sequence spaces; see e.g. [6]. Let ω be the vector space of all complex valued sequences, let X be a linear subspace of ω and let τ be a vector space topology on X. We assume that the set Φ of finitely supported complex sequences is contained in X ($\Phi = \{x \in \omega | \exists n_0(x) \in \mathbb{N}\ \forall n > n_0(x) : x_n = 0\}$).

We denote by $\pi_n : \omega \mapsto \mathbb{C}$ the projection onto the n'th coordinate, that is $\pi_n(x) = \pi_n(x_0, x_1, \ldots) = x_n$ and by $P_n : \omega \mapsto \omega$ the projection $P_n(x_0, x_1, x_2, \ldots, x_n, x_{n+1}, \ldots) = (x_0, x_1, \ldots, x_n, 0, 0, \ldots)$. Whenever convenient, we shall view P_n as a map from ω to $\mathbb{C}^n$.

Further, let $A(X \mapsto \omega)$ be a matrix mapping. That is,

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots \\ a_{10} & a_{11} & a_{12} & \cdots \\ a_{20} & a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix},$$

with $a_{ij} \in \mathbb{C}$. The notation $A(X \mapsto \omega)$ used here indicates that we do not assume that A is defined on the whole of X; we use the notation $A : X \mapsto Y$ to indicate that A is defined on the whole of X. We denote by

$$D_{\max}(A) = \left\{ x \in \omega \mid \sum_j a_{ij} x_j \text{ converges } \forall i \right\}$$

the maximal domain of A. The following subset of $D_{\max}(A)$ will also be used:

$$D_{\text{abs}}(A) = \left\{ x \in \omega \mid \sum_j |a_{ij} x_j| < \infty \ \forall i \right\}.$$

Definition 1.1. Let y be a fixed vector in ω. We say that the finite section method is applicable to the equation $Ax = y$ with right-hand side $y = (y_0, y_1, \ldots)$ in ω in the sense of (X, τ), if for any $n \in \mathbb{N}$ there is a unique solution $x^{(n)} = (x_0^{(n)}, x_1^{(n)}, \ldots, x_n^{(n)})$ in $\mathbb{C}^{n+1}$ to the truncated system $A_n x^{(n)} = y^{(n)}$, where

$$A_n = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nn} \end{pmatrix}, \qquad y^{(n)} = P_n y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix};$$

moreover, $x^{(n)} \to x$ in the topology τ, with $x \in X$, and $Ax = y$.

Note that the definition depends on the topology τ. Note also that in the definition above it is not necessary that A is defined on the whole of X, that is X need not be a subset of $D_{\max}(A)$.

The definition given above differs from the interpretation in e.g., [1,3]. There the operator $A : X \to Y$ is assumed to be bounded and boundedly invertible, and then the finite section method is said to converge when $A_n^{-1} P_n y$ converges to $A^{-1} y$.

For $\alpha > 0$, we define $l_1(\alpha) = \{x \in \omega \mid \sum_r |x_r| \alpha^r < \infty\}$, with the norm given by $\|x\|_{1,\alpha} = \sum_r |x_r| \alpha^r$.

The following theorem is one of our main results.

Theorem 1.2. *Let A be an infinite Vandermonde matrix determined by the sequence of complex numbers $a_0, a_1, \ldots$. Assume that*

$$0 < |a_0| < |a_1| < \cdots, \tag{3}$$

$$\alpha = \sum_k \frac{1}{|a_k|} < \infty. \tag{4}$$

Denote

$$b_k = \left| \prod_{\substack{i=0 \\ i \neq k}}^{\infty} \frac{1}{1 - \left| \frac{a_k}{a_i} \right|} \right|,$$

and suppose further that for any non-negative integer j

$$\sum_k |a_k^j| b_k < \infty. \tag{5}$$

Then the finite section method is applicable to the equation $Ax = d$ as in (2) in the sense of l_1 convergence for any $d \in l_1(\alpha)$.

If a_k is positive for all k, and we take for the right hand side $d = (\delta_{j1})_{j=1}^{\infty}$ the infinite vector with a one on the first position and zeros elsewhere, then for the solution $x = (x_k)_{k=1}^{\infty}$ we have $|x_k| = b_k$ (see the paragraph just before the proof of Theorem 1.2 below). Thus the condition $\sum |a_k^j| b_k < \infty$ is not very restrictive, as in the case where $a_k > 0$ for all k it is an obvious necessary condition.

As a corollary to this theorem we shall obtain that the finite section method gives an actual solution of (2) for the following special case.

Theorem 1.3. *Assume that $a_k \sim k^p$ for some $p > 2$, and $0 < |a_0| < |a_1| < \cdots$. Let $\alpha = \sum_k \frac{1}{|a_k|}$. Then the finite section method is applicable to the equation $Ax = d$ as in (2) in the sense of l_1 convergence for any $d \in l_1(\alpha)$.*

This is in contrast with the case (1), where the finite section method does not give a solution. (Compare however Remark 3.5.)

In the final section of the paper we shall show that for another particular case the finite section method is applicable to an even wider class of right hand sides.

Theorem 1.4. *Suppose that for some complex number a with $|a| > 1$ we have $a_k = a^k$. Then the finite section method is applicable to $Ax = d$ as in (2) in the sense of l_1 convergence for any $d \in l_\infty$.*

The proof of Theorem 1.2 is given in Section 2. The proof of the result on the special case presented in Theorem 1.3 is given in Section 3, while the proof of Theorem 1.4 is given in Section 4.

We finish this introduction by considering the surjectivity and injectivity of the Vandermonde matrix A given in (2) when viewed as a linear map from ω to itself. That is, we consider the situation $A(\omega \mapsto \omega)$. The map A from its domain $D_{\max}(A)$ to ω is surjective, but A is not injective. Indeed, it follows from Polya's theorem (see e.g., [2, Theorem 5.3.1]) that any infinite Vandermonde matrix A is surjective, and moreover, for any $d \in \omega$ there is even an $x \in D_{\text{abs}}(A)$ such that $Ax = d$. Now let C be the Vandermonde matrix formed by taking the second, third, etc. columns of A, so $A = \begin{pmatrix} B & C \end{pmatrix}$, where B is the first column of A. Then C is also a Vandermonde matrix and so is surjective. It is now easy to see that A is not injective: take any non-zero number y. Then solve $Cz = -By$, and put $x = \binom{y}{z}$. Then $Ax = 0$.

2. Proof of the main result

In this section, we consider the finite section method for the Vandermonde matrix A, and we shall prove Theorem 1.2.

To simplify matters, we first take a special right-hand side, namely, we examine the system

$$\begin{pmatrix} 1 & 1 & 1 & \cdots \\ a_0 & a_1 & a_2 & \cdots \\ a_0^2 & a_1^2 & a_2^2 & \cdots \\ \vdots & \vdots & \vdots & \\ a_0^r & a_1^r & a_2^r & \cdots \\ a_0^{r+1} & a_1^{r+1} & a_2^{r+1} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_r \\ x_{r+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix},$$

that is, the right-hand vector has its r-th coordinate 1 and it has all other coordinates 0. The truncated system is of the form

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_0^r & a_1^r & \cdots & a_n^r \\ \vdots & \vdots & & \vdots \\ a_0^n & a_1^n & \cdots & a_n^n \end{pmatrix} \begin{pmatrix} x_0^{(n)} \\ x_1^{(n)} \\ \vdots \\ x_r^{(n)} \\ \vdots \\ x_n^{(n)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix},$$

for $n \geq r$ (which we shall assume henceforth). To simplify the notation we put

$$D = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_0^n & a_1^n & \cdots & a_n^n \end{vmatrix} \quad \text{and} \quad D_r = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{r-1} & a_2^{r-1} & \cdots & a_n^{r-1} \\ a_1^{r+1} & a_2^{r+1} & \cdots & a_n^{r+1} \\ \vdots & \vdots & & \vdots \\ a_1^n & a_2^n & \cdots & a_n^n \end{vmatrix}.$$

Then, by Cramer's rule, $x_0^{(n)} = \frac{(-1)^r D_r}{D}$. Before going on to obtain an expression for D_r, we introduce the following notation. Let C_r^n denote the set of all injective and monotonically increasing mappings from $\{1, 2, \ldots, r\}$ to $\{1, 2, \ldots, n\}$ and let $s = n - r$. Let us consider the determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_1 & a_2 & \cdots & a_n & b \\ \vdots & \vdots & & \vdots & \vdots \\ a_1^r & a_2^r & \cdots & a_n^r & b^r \\ \vdots & \vdots & & \vdots & \vdots \\ a_1^n & a_2^n & \cdots & a_n^n & b^n \end{vmatrix}$$

as a polynomial of b. Then the term b^r has the coefficient $(-1)^{n+r} D_r$. On the other hand, the well-known formula for Vandermonde determinants yields

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_1 & a_2 & \cdots & a_n & b \\ \vdots & \vdots & & \vdots & \vdots \\ a_1^r & a_2^r & \cdots & a_n^r & b^r \\ \vdots & \vdots & & \vdots & \vdots \\ a_1^n & a_2^n & \cdots & a_n^n & b^n \end{vmatrix} = (b - a_1)(b - a_2) \cdots (b - a_n) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix}.$$

By comparing the coefficients of b^r, we obtain

$$D_r = (-1)^{n+r}\left(\sum_{\varphi\in C_s^n}(-1)^s a_{\varphi(1)}a_{\varphi(2)}\cdots a_{\varphi(s)}\right)\begin{vmatrix} 1 & 1 & \cdots & 1\\ a_1 & a_2 & \cdots & a_n\\ \vdots & \vdots & & \vdots\\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1}\end{vmatrix}.$$

Therefore,

$$\begin{aligned} x_0^{(n)} &= (-1)^n\frac{\sum_{\varphi\in C_s^n}(-1)^s a_{\varphi(1)}a_{\varphi(2)}\cdots a_{\varphi(s)}}{\prod_{i=1}^{n}(a_i-a_0)}\\ &= (-1)^{s+n}\sum_{\varphi\in C_r^n}\left(\prod_{i=1}^{n}\frac{1}{1-\frac{a_0}{a_i}}\right)\frac{1}{a_{\varphi(1)}\cdots a_{\varphi(r)}}\\ &= (-1)^r\left(\prod_{i=1}^{n}\frac{1}{1-\frac{a_0}{a_i}}\right)\left(\sum_{\varphi\in C_r^n}\frac{1}{a_{\varphi(1)}\cdots a_{\varphi(r)}}\right). \end{aligned}$$

For the case where $r = 0$ the term $\sum_{\varphi\in C_r^n}\frac{1}{a_{\varphi(1)}\cdots a_{\varphi(r)}}$ has to be replaced by 1.

Calculating $x_0^{(n)}$ instead of any $x_k^{(n)}$ has no restriction to the generality. We denote by $C_{r,k}^n$ the set of all functions $\varphi : \{1, 2, \dots, r\} \mapsto \{0, 1, \dots, k-1, k+1, \dots, n\}$ which are injective and monotonically increasing and we denote by $C_{r,k}$ the set of all functions $\varphi : \{1, 2, \dots, r\} \mapsto \{0, 1, \dots, k-1, k+1, \dots\}$ which are injective and monotonically increasing. Then

$$x_k^{(n)} = (-1)^r\left(\prod_{\substack{i=0\\ i\neq k}}^{n}\frac{1}{1-\frac{a_k}{a_i}}\right)\left(\sum_{\varphi\in C_{r,k}^n}\frac{1}{a_{\varphi(1)}\cdots a_{\varphi(r)}}\right);$$

hence

$$x_k^{(n)} \overset{n\to\infty}{\longrightarrow} (-1)^r\left(\prod_{\substack{i=0\\ i\neq k}}^{\infty}\frac{1}{1-\frac{a_k}{a_i}}\right)\left(\sum_{\varphi\in C_{r,k}}\frac{1}{a_{\varphi(1)}\cdots a_{\varphi(r)}}\right).$$

Note that $\sum_i \frac{1}{|a_i|} < \infty$ implies that the product $\prod_{i=0}^{\infty}(1-\frac{z}{a_i})$ is locally uniformly convergent in $z \in \mathbb{C}$. Furthermore, the inequality

$$\sum_{\varphi\in C_{r,k}}\frac{1}{|a_{\varphi(1)}\cdots a_{\varphi(r)}|} \le \left(\sum_i\frac{1}{|a_i|}\right)\left(\sum_{\varphi\in C_{r-1,k}}\frac{1}{|a_{\varphi(1)}\cdots a_{\varphi(r-1)}|}\right)$$

and induction show that

$$\sum_{\varphi\in C_{r,k}}\frac{1}{|a_{\varphi(1)}\cdots a_{\varphi(r)}|} \le \alpha^r;$$

in particular, the series is convergent.

We have just seen that with a special right-hand side each coordinate of the solution of the truncated system approaches a limit as n goes to infinity, that is, by introducing yet another index,

$$\begin{aligned} x_k^{(n),r} &= (-1)^r \left(\prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{1}{1-\frac{a_k}{a_i}} \right) \left(\sum_{\varphi\in C_{r,k}^n} \frac{1}{a_{\varphi(1)}\cdots a_{\varphi(r)}} \right) \\ &\xrightarrow{n\to\infty} (-1)^r \left(\prod_{\substack{i=0 \\ i\neq k}}^{\infty} \frac{1}{1-\frac{a_k}{a_i}} \right) \left(\sum_{\varphi\in C_{r,k}} \frac{1}{a_{\varphi(1)}\cdots a_{\varphi(r)}} \right) \stackrel{\text{def}}{=} x_k^{[r]} \end{aligned} \tag{6}$$

Then we would like to know whether the coordinatewise limit solves the infinite system.

We simply write $x_k^{(n)}$ and x_k instead of $x_k^{(n),0}$ and $x_k^{[0]}$, respectively. Recall that this corresponds to the case $r = 0$, so to the right hand side consisting of the vector with one in the top position and zeros elsewhere. Recall that in this case $\sum_{\varphi\in C_{r,k}} \frac{1}{a_{\varphi(1)}\cdots a_{\varphi(r)}}$ has to be replaced by 1. Observe that if $n \geq k$, then

$$\begin{aligned} |x_k^{(n)}| &= \prod_{i=0}^{k-1} \left| \frac{1}{1-\frac{a_k}{a_i}} \right| \prod_{i=k+1}^{n} \left| \frac{1}{1-\frac{a_k}{a_i}} \right| \\ &\leq \prod_{i=0}^{k-1} \frac{1}{\left|\frac{a_k}{a_i}\right| - 1} \prod_{i=k+1}^{n} \frac{1}{1-\left|\frac{a_k}{a_i}\right|} \leq \left| \prod_{\substack{i=0 \\ i\neq k}}^{\infty} \frac{1}{1-\left|\frac{a_k}{a_i}\right|} \right| \stackrel{\text{def}}{=} b_k; \end{aligned}$$

and $|x_k| = b_k$ holds whenever the numbers a_i are positive.

We are now ready to prove Theorem 1.2.

Proof of main theorem. By the construction of $x_k^{(n),r}$,

$$\sum_{k=0}^{n} a_k^j x_k^{(n),r} = \delta_{jr}$$

for all $n \in \mathbb{N}$ and $0 \leq j, r \leq n$.

From (6), one easily sees that $|x_k^{(n),r}| \leq b_k\alpha^r$, and hence $|a_k^j x_k^{(n),r}| \leq \alpha^r |a_k^j| b_k$. So, if $n \geq k$, the dominated convergence theorem yields

$$\begin{aligned} \delta_{jr} &= \lim_{n\to\infty} \sum_{k=0}^{n} a_k^j x_k^{(n),r} = \lim_{n\to\infty} \sum_{k=0}^{\infty} \chi(k\leq n) a_k^j x_k^{(n),r} \\ &= \sum_{k=0}^{\infty} \lim_{n\to\infty} \chi(k\leq n) a_k^j x_k^{(n),r} = \sum_{k=0}^{\infty} a_k^j x_k^{[r]}. \end{aligned}$$

Take any right-hand side $d \in l_1(\alpha)$, so $d \in \omega$ with $\sum_r |d_r|\alpha^r < \infty$. Let $y^{(n)} = (y_0^{(n)}, y_1^{(n)}, \ldots, y_n^{(n)}) \in \mathbb{C}^{n+1}$ be the solution of the truncated system $A_n y^{(n)} = P_n d$. By linear combination, $y_k^{(n)} = \sum_{r=0}^{n} d_r x_k^{(n),r}$. Since $|x_k^{(n),r}| \leq b_k\alpha^r$, one sees that

$$y_k^{(n)} \xrightarrow{n\to\infty} \sum_{r=0}^{\infty} d_r x_k^{[r]} \stackrel{\text{def}}{=} y_k$$

for all $k \geq 0$. Furthermore,

$$\sum_{k=0}^{\infty} a_k^j y_k = \sum_{k=0}^{\infty} a_k^j \sum_{r=0}^{\infty} d_r x_k^{[r]} = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} a_k^j d_r x_k^{[r]}$$
$$= \sum_{r=0}^{\infty} d_r \sum_{k=0}^{\infty} a_k^j x_k^{[r]} = \sum_{r=0}^{\infty} d_r \delta_{jr} = d_j$$

for any $j \geq 0$. It is allowed to change the order of summation because

$$\sum_{k,r} |a_k^j d_r x_k^{[r]}| \leq \sum_{k,r} |a_k^j d_r b_k \alpha^r| = \left(\sum_{r=0}^{\infty} |d_r| \alpha^r \right) \left(\sum_{k=0}^{\infty} |a_k^j| b_k \right) < \infty.$$

The latter estimate also gives $y \in D_{\text{abs}}(A)$, and in particular, with $j = 0$, $y \in l_1$. It remains to prove that $y^{(n)} \to y$ in l_1. Note that $|y_k^{(n)}| \leq \sum_{r=0}^{n} |d_r x_k^{(n),r}| \leq \sum_{r=0}^{\infty} |d_r b_k \alpha^r|$ thus $|y_k^{(n)} - y_k| \leq |y_k^{(n)}| + |y_k| \leq b_k \sum_{r=0}^{\infty} |d_r| \alpha^r + |y_k|$ and $\sum_{k=0}^{\infty} \left(b_k \sum_{r=0}^{\infty} |d_r| \alpha^r + |y_k| \right) < \infty$; therefore the dominated convergence theorem once again applies to give

$$\lim_{n\to\infty} \sum_{k=0}^{n} |y_k^{(n)} - y_k| = \lim_{n\to\infty} \sum_{k=0}^{\infty} \chi(k \leq n) |y_k^{(n)} - y_k|$$
$$= \sum_{k=0}^{\infty} \lim_{n\to\infty} \chi(k \leq n) |y_k^{(n)} - y_k| = 0.$$

It follows that

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} |y_k^{(n)} - y_k| = \lim_{n\to\infty} \sum_{k=0}^{n} |y_k^{(n)} - y_k| + \lim_{n\to\infty} \sum_{k=n+1}^{\infty} |y_k| = 0,$$

that is, $y^{(n)} \xrightarrow{l_1} y$. The proof is complete. □

Next we state a lemma that will be useful for the remainder of this section.

Lemma 2.1. *Assume that conditions (3)–(5) hold for the sequence $a_0, a_1, a_2, \ldots$. Then they also hold for the sequence $a_1, a_2, a_3, \ldots$.*

Proof. That the conditions (3) and (4) hold for the sequence with the first term removed is trivial. It remains to consider the third condition. For the sake of notation, let us denote $a'_k = a_{k+1}$, and let b'_k be defined as

$$b'_k = \left| \prod_{\substack{i=0 \\ i\neq k}}^{\infty} \frac{1}{1 - \left| \frac{a'_k}{a'_i} \right|} \right|.$$

Observe that $\frac{a'_k}{a'_i} = \frac{a_{k+1}}{a_{i+1}}$. Thus

$$b'_k = \left| \prod_{\substack{i=0 \\ i\neq k}}^{\infty} \frac{1}{1 - \left| \frac{a_{k+1}}{a_{i+1}} \right|} \right| = \frac{b_{k+1}}{\left| 1 - \left| \frac{a_{k+1}}{a_0} \right| \right|}.$$

Now we have to show that $\sum_{k=1}^{\infty} |(a_k')^j| b_k' < \infty$ for all $j = 0, 1, 2, \ldots$. To see this, consider

$$(a_k')^j b_k' = a_{k+1}^j \frac{b_{k+1}}{\left|1 - \left|\frac{a_{k+1}}{a_0}\right|\right|},$$

so

$$\sum_{k=1}^{\infty} |(a_k')^j| b_k' = \sum_k \frac{|a_{k+1}^j| b_{k+1}}{\left|1 - \left|\frac{a_{k+1}}{a_0}\right|\right|}.$$

Since $|a_k| \to \infty$ there is a k_0 such that for $k \geq k_0$ we have $|\frac{a_{k+1}}{a_0}| - 1 = \left|1 - |\frac{a_{k+1}}{a_0}|\right| \geq 1$, so

$$\frac{1}{\left|1 - \left|\frac{a_{k+1}}{a_0}\right|\right|} \leq 1,$$

and hence for $k \geq k_0$ $(a_k')^j b_k' \leq a_{k+1}^j b_{k+1}$. Then by (5)

$$\sum_{k=1}^{\infty} |(a_k')^j| b_k' = \sum_{k=1}^{k_0-1} |(a_k')^j| b_k' + \sum_{k=k_0}^{\infty} |(a_k')^j| b_k'$$
$$\leq \sum_{k=1}^{k_0-1} |(a_k')^j| b_k' + \sum_{k=k_0}^{\infty} |a_{k+1}^j| b_{k+1} < \infty.$$

This proves the lemma. □

It is important to note that in general l_1 is not fully contained in $D_{\max}(A)$, and that we also do not claim that the finite section method holds for the situation $A : l_1 \mapsto l_1(\alpha)$. In fact, if the finite section method would hold for the situation $A : l_1 \mapsto l_1(\alpha)$ then A would be invertible. Indeed, the following proposition holds.

Proposition 2.2. *Let Y be a Banach space of sequences such that for every n the coordinate map $x \to x_n$ from Y to $\mathbb{C}$ is continuous. Suppose $A : l_1 \mapsto Y$ is a matrix mapping of l_1 into Y. If the finite section method is applicable to the equation $Ax = y$ for all $y \in Y$, then A is (continuously) invertible.*

Proof. It is straightforward from the definition of applicability that A is surjective. For any $y \in Y$, the sequence $x_n = A_n^{-1} P_n y$ is convergent; hence, by the Banach–Steinhaus theorem, $\{A_n^{-1} P_n\}$ is equicontinuous. Take any $x \in X$ with $Ax = 0$ and observe that

$$P_n x = A_n^{-1} A_n P_n x = (A_n^{-1} P_n) A P_n x.$$

Now $P_n x \to x$ and hence $A P_n x \to Ax = 0$ by Theorem 4.1.5 in [6], which, when applied to the case at hand, states that every matrix mapping from l_1 into Y is continuous. The equicontinuity of $A_n^{-1} P_n$ then yields $(A_n^{-1} P_n) A P_n x \to 0$; therefore $x = 0$ and A is injective. □

If the conditions (3)–(5) are satisfied, then the operator A cannot be a map from the whole of l_1 into $l_1(\alpha)$. Indeed, if that were the case, A would be invertible by the proposition above. This is not the case; we shall show that A is not invertible.

To see that A is not injective viewed as a map from l_1 onto $l_1(\alpha)$, observe that by the Lemma 2.1 Theorem 1.2 also holds for the sequence $a_1, a_2, a_3, \ldots$. That is, if we consider the operator A_1 formed by deleting the first column of A, then this operator, viewed as a linear map from a domain in l_1 to $l_1(\alpha')$, where $\alpha' = \sum_{k=1}^{\infty} \frac{1}{a_k}$ is onto as well. Observe that we can view A_1 as $A_1 = AS$, where S is the forward shift, but formally that is a different operator as it still maps into $l_1(\alpha)$. (Compare also the argument presented in the last paragraph of the introduction.)

Note that $\alpha' < \alpha$. It follows that $l_1(\alpha) \subset l_1(\alpha')$. Now take $d \in l_1(\alpha)$; then, by our previous theorem, the finite section method gives us an $x \in l_1$ such that $Ax = d$. Since $d \in l_1(\alpha')$ as well, by the finite section method we obtain an $x_1 \in l_1$ such that $A_1 x_1 = d$. But this implies that $A(Sx_1) = d$. We would be done if the first coordinate of x is non-zero, but that may not be the case. So we continue. In fact, we can repeat the argument above, and show that for any $j = 1, 2, \ldots$ there is an x_j such that $AS^j x_j = d$. Now take j_0 so large that the j_0'th coordinate of x is not 0. Then for $j > j_0$ we have that $S^j x_j \neq x$.

Next, we discuss the following idea. We do not know whether or not $A(l_1 \to l_1(\alpha))$ is closed, but suppose for the sake of argument that it is. Denote for the moment the domain of this operator by $X = D_{l_1, l_1(\alpha)}$, and equip this domain with the graph norm $|||x||| := \|x\|_{l_1} + \|Ax\|_{l_1(\alpha)}$. This norm makes $X, |||\cdot|||$ into a Banach space, and we can view $A : X \to l_1(\alpha)$ as a bounded linear map between Banach spaces. Obviously, one could hope to apply the finite section method (in the sense of [3,1]) to this more standard situation, thus obtaining the main theorem this way. However, as already observed, A is not one-to-one on the vector space X, and it would have to be to apply the finite section method in the sense of [3,1].

3. At least quadratic growth

If the a_k-s are given by some formula, it may be possible to derive a closed form of the product defining b_k. This also makes it easier to check that the condition of Theorem 1.2 is fulfilled.

In this and in the next section it is more convenient to use the indices $1, 2, 3, \ldots$ rather than $0, 1, 2, \ldots$ as we did so far; thus our matrix A and sequence $b_1, b_2, \ldots$ are now built from the numbers $a_1, a_2, \ldots$ in the way above. We take $a_k = k^p$, where p is an integer and $p \geq 2$.

First we consider the case where $p = 2$.

Lemma 3.1. *For $a_k = k^2$ we have for all k:*

$$b_k = \left| \prod_{\substack{i=0 \\ i \neq k}}^{\infty} \frac{1}{1 - \left|\frac{k^2}{i^2}\right|} \right| = 2.$$

Proof. We compute with the reciprocal of b_k:

$$\frac{1}{b_k} = \prod_{\substack{n=1 \\ n \neq k}}^{\infty} \left|1 - \frac{k^2}{n^2}\right| = \lim_{z \to k} \prod_{\substack{n=1 \\ n \neq k}}^{\infty} \left|1 - \frac{z^2}{n^2}\right| = \left| \lim_{z \to k} \frac{1}{1 - \left(\frac{z}{k}\right)^2} \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^2\right) \right|.$$

Since

$$1 - \left(\frac{z}{n}\right)^2 = \left(1 + \frac{z}{n}\right)\left(1 - \frac{z}{n}\right)$$

we can write the reciprocal of b_k as

$$\frac{1}{b_k} = \left| \lim_{z \to k} \frac{1}{1 - \left(\frac{z}{k}\right)^2} \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^2\right) \right|$$

$$= \left| \lim_{z\to k} \frac{1}{\left(1+\frac{z}{k}\right)\left(1-\frac{z}{k}\right)} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)\left(1-\frac{z}{n}\right) \right|.$$

Now apply Theorem 5, Chapter 2 of [4] to see that this is equal to

$$\left| \lim_{z\to k} \frac{1}{\left(1+\frac{z}{k}\right)\left(1-\frac{z}{k}\right)} \cdot \frac{1}{\Gamma(1+z)\Gamma(1-z)} \right| = \frac{1}{2} \cdot \frac{1}{\Gamma(1+k)} \cdot \left| \lim_{z\to k} \frac{1}{\left(1-\frac{z}{k}\right)\Gamma(1-z)} \right|.$$

Now apply some well-known facts concerning the gamma function (see e.g., [8]) to see that this is equal to

$$\frac{1}{2} \cdot \frac{1}{k!} \cdot \left| (-1)^{k+1} k! \right| = \frac{1}{2}.$$

This proves the lemma. □

Next, we shall prove that for $p > 2$ the b_k's are exponentially decaying. This is done in several steps. We start with the following lemma.

Lemma 3.2. *Let $p > 2$ be a real number. Then there exist $c > 1$ and an integer k_0 such that*

$$\left(\frac{k}{j}\right)^p - 1 \geq c\left(\left(\frac{k}{j}\right)^2 - 1\right),$$

for all integers $k \geq k_0$ and $1 \leq j < k$.

Proof. We put

$$f_k(j) = \frac{\left(\frac{k}{j}\right)^p - 1}{\left(\frac{k}{j}\right)^2 - 1} = \frac{j^{2-p}(k^p - j^p)}{k^2 - j^2}.$$

We do not suppose here that j is an integer. By L'Hospital's rule,

$$\lim_{k\to\infty} f_k(k-1) = \lim_{k\to\infty} \frac{\left(1+\frac{1}{k-1}\right)^p - 1}{\left(1+\frac{1}{k-1}\right)^2 - 1} = \lim_{k\to\infty} \frac{p\left(1+\frac{1}{k-1}\right)^{p-1}}{2\left(1+\frac{1}{k-1}\right)}$$

$$= \frac{p}{2} \lim_{k\to\infty} \left(1+\frac{1}{k-1}\right)^{p-2} = \frac{p}{2} > 1,$$

so the assertion holds for $j = k-1$. We complete the proof by showing that $f_k(j)$ is monotonically decreasing in $1 \leq j \leq k-1$. If we denote $g_k(j) = (2-p)k^{p+2} - (2-p)k^p j^2 - 2j^p k^2 + 2k^p j^2$, then the derivative of $f_k(j)$ can be written in the form

$$f_k'(j) = \frac{g_k(j)}{j^{p-1}(k^2 - j^2)^2},$$

thus it suffices to see that $g_k(j) \leq 0$ $(1 \leq j \leq k)$. As is easily verified, $g_k(0) < 0$, $g_k(k) = 0$ and $g_k'(j) \geq 0$ $(0 \leq j \leq k)$. □

Further on, let $p > 2$ and suppose $a_k = k^p$. Let k_0 and $c > 1$ be given by the lemma above. Then for $k \geq k_0$ one obtains by multiplication

$$\prod_{j=1}^{k-1}\left(\left(\frac{k}{j}\right)^p - 1\right) \geq c^{k-1}\prod_{j=1}^{k-1}\left(\left(\frac{k}{j}\right)^2 - 1\right),$$

and therefore

$$\frac{1}{b_k} = \prod_{\substack{j=1\\ j\neq k}}^{\infty}\left|1 - \frac{k^p}{j^p}\right| = \prod_{j=1}^{k-1}\left(\left(\frac{k}{j}\right)^p - 1\right)\prod_{j=k+1}^{\infty}\left(1 - \left(\frac{k}{j}\right)^p\right)$$
$$\geq c^{k-1}\prod_{j=1}^{k-1}\left(\left(\frac{k}{j}\right)^2 - 1\right)\prod_{j=k+1}^{\infty}\left(1 - \left(\frac{k}{j}\right)^2\right) = \frac{1}{2}c^{k-1},$$

where the last equality uses Lemma 3.1. This yields the desired exponential decay of the sequence b_k:

$$b_k \leq 2c^{1-k}. \tag{7}$$

The estimation easily extends to any sequence a_k growing rapidly enough.

Lemma 3.3. *Let e_k, f_k $(k \geq 1)$ be strictly increasing sequences of positive numbers, and suppose that there exists a positive integer k_0 such that*

$$\frac{f_k}{e_k} \geq \max_{1\leq j<k}\frac{f_j}{e_j}$$

for all $k \geq k_0$. Then

$$\left|\prod_{\substack{j=1\\ j\neq k}}^{\infty}\frac{1}{1-\frac{f_k}{f_j}}\right| \leq \left|\prod_{\substack{j=1\\ j\neq k}}^{\infty}\frac{1}{1-\frac{e_k}{e_j}}\right|$$

whenever $k \geq k_0$.

We summarize these results in the next proposition.

Proposition 3.4. *Suppose that there exist a real number $p > 2$ and a positive integer k_0 such that $\frac{a_k}{k^p} \geq \max_{1\leq j<k}\frac{a_j}{j^p}$ for all $k \geq k_0$. Then there is a real number $c > 0$ and a positive integer k_1 such that $b_k \leq e^{-ck}$, whenever $k \geq k_1$.*

We note that the awkward-looking condition $\frac{a_k}{k^p} \geq \max_{1\leq j<k}\frac{a_j}{j^p}$ $(k \geq k_0)$ is satisfied either if the sequence $\frac{a_k}{k^p}$ is monotonically increasing, or $\frac{a_k}{k^p} \to \infty$ increasingly from some k_1 on.

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. Assume that $|a_k| \sim k^p$ for some $p > 2$, and that $0 < |a_0| < |a_1| < \cdots$. Then (3) and (4) are satisfied. To see that also (5) is satisfied we use the previous proposition: $|a_k^j|b_k \sim k^{jp}b_k$. Since $b_k \leq e^{-ck}$ we have that $\sum_k |a_k^j|b_k$ converges. $\square$

Remark 3.5. Let us denote $u = (1, 0, 0, \ldots)$. In the case of $a_k = k^2$, according to our definition, the finite section method is not applicable to the system $Ax = u$, simply because it yields

$x_k = 2(-1)^{k+1}$, and $(x_1, x_2, \ldots) = x \notin D_{\max}(A)$; even $\sum_k x_k$ is divergent. The situation is similar for the system (1) we mentioned in the introduction, which is given by the numbers $a_k = (2k-1)^2$. As is calculated in [5], and also an easy consequence of our treatment, the result of the finite section method here is $x_k = \frac{4(-1)^{k+1}}{\pi(2k-1)}$, so $\sum_k a_k x_k = \sum_k \frac{4}{\pi}(-1)^{k+1}(2k-1)$ is divergent.

However, it is possible to interpret the system $Ax = d$ in a wider sense: for the respective series, we substitute the usual concept of convergence by that of the Abel convergence. Generally speaking, given a matrix

$$C = \begin{pmatrix} c_{00} & c_{01} & c_{02} & \cdots \\ c_{10} & c_{11} & c_{12} & \cdots \\ c_{20} & c_{21} & c_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix},$$

we put $D_{\mathrm{Ab}}(C) = \{x \in \omega | \lim_{r\to 1} \sum_k c_{jk} x_k r^k \ \text{existsforall} j\}$, and write $Cx \overset{\mathrm{Ab}}{=} d$, if $\lim_{r\to 1} \sum_{k=0}^{\infty} c_{jk} x_k r^k = d_j$ for all j. Indeed, this is an extension; $D_{\max}(C) \subset D_{\mathrm{Ab}}(C)$, and if $Cx = d$, then $Cx \overset{\mathrm{Ab}}{=} d$. Moreover, it is proven in [5] that for the system $Ax = u$ with $a_k = (2k-1)^2$, the result x of the finite section method satisfies $x \in D_{\mathrm{Ab}}(A)$ and $Ax \overset{\mathrm{Ab}}{=} u$.

4. The exponential case

In this section, we consider the infinite matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & a & a^2 & a^3 & \cdots \\ 1 & a^2 & a^4 & a^6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

that is, the entries of the matrix A are given by $a_{kj} = a^{kj}$ with some a in $\mathbb{C}$, $|a| > 1$. Note that this is a Vandermonde matrix with $a_j = a^j$ for $j = 0, 1, 2, \ldots$. We shall show in this section that for this matrix A the finite section method is applicable in the sense of l_1 convergence for every right hand side which is in l_∞. Since l_∞ is a larger set than $l_1(\alpha)$ (in this case $\alpha = \frac{|a|}{|a|-1} > 1$), this result can obviously not be obtained as a consequence of Theorem 1.2.

We take a closer look at Ker A. Put $e(z) = \prod_{j=0}^{\infty}(1 - \frac{z}{a^j})$. Since $\sum_j |\frac{1}{a^j}| < \infty$, it follows from a theorem of Weierstrass that $e(z)$ is an entire function admitting a simple root at each a^k, and no other roots. Let us write $e(z) = \sum_{j=0}^{\infty} e_j z^j$ and with a slight abuse of notation, which we shall adhere to further on, let e denote the sequence $e = (e_0, e_1, e_2, \ldots) \in l_1$. Clearly, $e \in \operatorname{Ker} A$. Take any $f = (f_0, f_1, f_2, \ldots) \in \operatorname{Ker} A$. Then the power series $f(z) = \sum_{j=0}^{\infty} f_j z^j$ converges for all a^k (in fact, $f(a^k) = 0$), so it converges for every $z \in \mathbb{C}$. Thus $f(z)$ is an entire function, and so is $h(z) = \frac{f(z)}{e(z)}$. Therefore $f \in \operatorname{Ker} A$ if and only if $f(z) = e(z)h(z)$ with some entire function h (since the "if" direction is even more obvious).

In order to deal with applicability we establish a lemma.

Lemma 4.1. *Let* $f_n(z) = \sum_{j=0}^{\infty} c_j^{(n)} z^j$ *be a sequence of entire complex functions such that* $f_n \to f$ *locally uniformly on the complex plane. Then* f *is entire and if* $f(z) = \sum_{j=0}^{\infty} c_j z^j$, $c^{(n)} = (c_0^{(n)}, c_1^{(n)}, \ldots)$, $c = (c_0, c_1, \ldots)$, *then* $c^{(n)} \xrightarrow{l_1} c$.

Proof. As is well known, f is entire. Consider the space of all entire functions, on which locally uniform convergence induces a completely metrizable vector space topology (see [7, Section 1.45]). Let E denote the space of Taylor coefficients of entire functions, that is, $E = \{(c_0, c_1, \ldots) \in \omega | \sum_j c_j z^j \text{ is entire}\}$. The one-to-one linear mapping $(c_0, c_1, \ldots) \mapsto \sum_j c_j z^j$ equips E with a completely metrizable topology τ, in a way that for $c^{(n)}, c$ in E, we have $c^{(n)} \xrightarrow{\tau} c$ if and only if $\sum_j c_j^{(n)} z^j \to \sum_j c_j z^j$ locally uniformly in $\mathbb{C}$. Furthermore, as a consequence of Theorem 4.1.5 or Theorem 3.2.1 in [6], the inclusion from E into l_1 is continuous. Hence τ convergence implies l_1 convergence, as asserted. □

Let us now consider the inhomogeneous system $Ag = d$,

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & a & a^2 & a^3 & \cdots \\ 1 & a^2 & a^4 & a^6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \end{pmatrix}$$

and its truncated version $A_n g^{(n)} = P_n d$. To solve this system we have to find a polynomial $q_n(z) = \sum_{k=0}^n g_k^{(n)} z^k$ such that $q_n(a^j) = d_j$ for $0 \le j \le n$; that is, the unique solution of the truncated equation is given by the coefficients of the Lagrange interpolating polynomial

$$q_n(z) = \sum_{k=0}^n \left(d_k \prod_{\substack{j=0 \\ j \neq k}}^n \frac{z - a^j}{a^k - a^j} \right).$$

If $p_n(z) = \prod_{j=0}^n \left(1 - \frac{z}{a^j}\right)$, then $p_n(z) \to e(z) = \prod_{j=0}^\infty (1 - \frac{z}{a^j}) = \sum_{j=0}^\infty e_j z^j$ locally uniformly on the complex plane, and q_n can be written as

$$\begin{aligned} q_n(z) &= \sum_{k=0}^n d_k \left(\frac{p_n(z)}{1 - \frac{z}{a^k}} \right) \frac{1}{\lim_{w \to a^k} \frac{p_n(w)}{1 - \frac{w}{a^k}}} \\ &= \sum_{k=0}^n d_k \frac{p_n(z)}{(z - a^k) p_n'(a^k)} = p_n(z) \sum_{k=0}^n \frac{d_k}{(z - a^k) p_n'(a^k)}. \end{aligned}$$

Now $p_{n+1}(z) = \left(1 - \frac{z}{a^{n+1}}\right) p_n(z)$; hence for the derivatives of the polynomials $p_n(z)$ the recursion formula is $p_{n+1}'(z) = p_n'(z) - \frac{1}{a^{n+1}}(p_n'(z) z + p_n(z))$, which implies $p_{n+1}'(a^k) = p_n'(a^k) \left(1 - \frac{a^k}{a^{n+1}}\right)$ $(0 \le k \le n)$. Therefore

$$|p_n'(a^k)| \ge |p_k'(a^k)| \prod_{j=1}^{n-k} \left(1 - \frac{1}{|a|^j}\right) \ge |p_k'(a^k)| \prod_{j=1}^{\infty} \left(1 - \frac{1}{|a|^j}\right) = C |p_k'(a^k)|$$

when $n \ge k + 1$. The constant C only depends on a. Suppose $|z - a^k| > \delta$ for all k, then for $k \le n - 1$

$$\left| \frac{d_k}{(z - a^k) p_n'(a^k)} \right| \le \frac{|d_k|}{\delta C |p_k'(a^k)|} = \frac{|d_k| |a^k|}{\delta C |p_{k-1}(a^k)|}.$$

Moreover, suppose that

$$\sum_k \left| \frac{d_k a^k}{p_{k-1}(a^k)} \right| < \infty.$$

(Note that $|p_{k-1}(a^k)| \to \infty$ very fast, so this condition is likely to be satisfied for a large class of d_k-s.) Locally uniform convergence of a sequence of holomorphic functions implies the (locally uniform) convergence of the derivatives, thus

$$\begin{aligned}\lim_{n\to\infty} q_n(z) &= \lim_{n\to\infty} \left(p_n(z) \sum_{k=0}^{\infty} \chi(k \le n) \frac{d_k}{(z-a^k)p_n'(a^k)} \right) \\ &= e(z) \sum_{k=0}^{\infty} \frac{d_k}{(z-a^k)e'(a^k)} \stackrel{\text{def}}{=} g(z)\end{aligned}$$

locally uniformly on $\mathbb{C} \setminus \{1, a, a^2, \ldots\}$. As can easily be verified, $g(z)$ is an entire function and $g(a^j) = d_j$ $(j \ge 0)$, so $g_n \to g$ locally uniformly on $\mathbb{C}$. One is then lead to write $g(z) = \sum_{k=0}^{\infty} g_k z^k$ to obtain a solution $g = (g_0, g_1, \ldots)$ of the original system. Indeed, by the previous lemma, $g^{(n)} \xrightarrow{l_1} g$.

Theorem 1.4 is a corollary of this.

Proof of Theorem 1.4. In view of the previous remarks, it remains to prove that for any $d \in l_\infty$ we have

$$\sum_k \left| \frac{d_k a^k}{p_{k-1}(a^k)} \right| < \infty,$$

thus it suffices to see that

$$\sum_k \left| \frac{a^k}{p_{k-1}(a^k)} \right| < \infty.$$

For k great enough, $|p_{k-1}(a_k)| = \left|\prod_{j=1}^k (1-a^j)\right| \ge \prod_{j=1}^k (|a|^j - 1) \ge (|a|^{k-1}-1)(|a|^k-1)$. So $\left|\frac{a^k}{p_{k-1}(a^k)}\right| \le \frac{1}{(|a|^{k-1}-1)} \frac{1}{(1-\frac{1}{|a|^k})} \le \frac{2}{(|a|^{k-1}-1)}$, and $\sum_k \frac{1}{|a|^{k-1}-1}$ is convergent. □

We note that there are right-hand vectors $d \in \omega$ for which the finite section method is not applicable (though, there is a solution). Let us recall that the solutions of the truncated systems were given by the coefficients of

$$q_n(z) = \sum_{k=0}^{n} \left(d_k \prod_{\substack{j=0 \\ j\ne k}}^{n} \frac{z-a^j}{a^k-a^j} \right).$$

For $z = 0$ this gives

$$q_n(0) = \sum_{k=0}^{n-1} \left(d_k \prod_{\substack{j=0 \\ j\ne k}}^{n} \frac{a^j}{a^j-a^k} \right) + d_n \prod_{j=0}^{n-1} \frac{a^j}{a^j-a^n}.$$

Now, it is clear that the numbers d_n can be chosen inductively to obtain any prescribed sequence of $q_n(0)$-s.

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