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On the conservativeness and the recurrence of symmetric jump-diffusions

Jun Masamune [a], Toshihiro Uemura [b], Jian Wang [c,*]

[a] *Department of Mathematics and Statistics, Penn State, Altoona, 3000 Ivyside Park, Altoona, PA 16601, USA*
[b] *Department of Mathematics, Faculty of Engineering Science, Kansai University, Suita-shi, Osaka 564-8680, Japan*
[c] *School of Mathematics and Computer Science, Fujian Normal University, 350007, Fuzhou, PR China*

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Abstract

Sufficient conditions for a symmetric jump-diffusion process to be conservative and recurrent are given in terms of the volume of the state space and the jump kernel of the process. A number of examples are presented to illustrate the optimality of these conditions; in particular, the situation is allowed to be that the state space is topologically disconnected but the particles can jump from a connected component to the other components.

Keywords: Regular Dirichlet form; Jump process; Integral-derivation property; Conservation property; Recurrence

Contents

* Corresponding author.
E-mail addresses: jum35@psu.edu (J. Masamune), t-uemura@kansai-u.ac.jp (T. Uemura), jianwang@fjnu.edu.cn (J. Wang).

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1. Introduction and main results

Let (X, d, m) be a metric measure space. We assume that every metric ball $B(x, r) = \{z \in X\colon d(x, z) < r\}$ centered at $x \in X$ with radius $r > 0$ is pre-compact, and the measure m is a Radon measure with full support. In particular, X is locally compact and separable. Let $(\mathcal{E}, \mathcal{F})$ be a regular symmetric Dirichlet form in $L^2(X; m)$. We denote the extended Dirichlet space of $(\mathcal{E}, \mathcal{F})$ by $\mathcal{F}_e$, and a quasi-continuous version of $u \in \mathcal{F}_e$ by $\tilde{u}$. According to the Beurling–Deny theorem, see, e.g., [8, Theorem 3.2.1 and Lemma 4.5.4], we can express $(\mathcal{E}, \mathcal{F})$ as follows

$$\mathcal{E}(u, v) = \mathcal{E}^{(c)}(u, v) + \iint_{x \neq y} \big(\tilde{u}(x) - \tilde{u}(y)\big)\big(\tilde{v}(x) - \tilde{v}(y)\big)\, J(dx, dy) + \int_X \tilde{u}(x)\tilde{v}(x)\, k(dx) \quad \text{for any } u, v \in \mathcal{F}_e,$$

where $(\mathcal{E}^{(c)}, C_0(X) \cap \mathcal{F})$ is a strongly-local symmetric form and $C_0(X)$ is the space of all real-valued continuous functions on X with compact support; J is a symmetric positive Radon measure on the product space $X \times X$ off the diagonal $\{(x, x)\colon x \in X\}$; and k is a positive Radon measure on X.

Let $\mu_{\langle \cdot, \cdot \rangle}$ be a bounded signed measure, see [8, Lemma 3.2.3], such that

$$\mathcal{E}^{(c)}(u, v) = \frac{1}{2}\mu_{\langle u, v \rangle}(X) = \frac{1}{2}\int_X \mu_{\langle u, v \rangle}(dx) \quad \text{for } u, v \in \mathcal{F}_e.$$

Throughout the paper, we assume the following set (A) of conditions:

(A-1) The killing measure k does not appear; that is, the corresponding process is *no killing inside*.

(A-2) For each $u, v \in \mathcal{F}_e$, the measure $\mu_{\langle u, v \rangle}$ is absolutely continuous with respect to m. We denote the corresponding Radon–Nikodym density by $\Gamma^{(c)}(u, v)$; namely,

$$\mu_{\langle u, v \rangle}(dx) = \Gamma^{(c)}(u, v)(x)\, m(dx).$$

(A-3) The jump measure J has a symmetric kernel $j(x, dy)$ over $X \times \mathcal{B}(X)$ such that

$$J(dx, dy) = j(x, dy)\, m(dx) \big(= j(y, dx)\, m(dy) = J(dy, dx)\big).$$

For $u, v \in \mathcal{F}_e$, define

$$\Gamma^{(j)}(u,v)(x) = \int\limits_{x\neq y} \big(\tilde{u}(x) - \tilde{u}(y)\big)\big(\tilde{v}(x) - \tilde{v}(y)\big)\, j(x,dy),$$

and

$$\mathcal{E}^{(j)}(u,v) = \int \Gamma^{(j)}(u,v)(x)\, m(dx).$$

Therefore, the form $\mathcal{E}$ has the following expression for any $u, v \in \mathcal{F}_e$:

$$\begin{aligned}\mathcal{E}(u,v) &= \mathcal{E}^{(c)}(u,v) + \mathcal{E}^{(j)}(u,v)\\ &= \frac{1}{2}\int\limits_X \Gamma^{(c)}(u,v)(x)\, m(dx) + \int\limits_X \Gamma^{(j)}(u,v)(x)\, m(dx)\\ &= \frac{1}{2}\int\limits_X \Gamma^{(c)}(u,v)(x)\, m(dx) + \iint\limits_{x\neq y} \big(\tilde{u}(x) - \tilde{u}(y)\big)\big(\tilde{v}(x) - \tilde{v}(y)\big)\, j(x,dy)\, m(dx).\end{aligned}$$

Let ψ_K be the distance function from a compact set K of X, i.e., $\psi_K(\cdot) = \inf_{y\in K} d(\cdot, y)$. For every $r > 0$, we denote $B(K,r) = \{x \in X\colon\ \psi_K < r\}$ and its closure $\{x \in X\colon\ \psi_K \leqslant r\}$ by $\overline{B}(K,r)$. Clearly, $B(K,r)$ is pre-compact. Let $\mathcal{F}_{\mathrm{loc}}$ be the set of measurable functions u such that for each relatively compact open set G of X there exists $w \in \mathcal{F}$ which satisfies that $u|_G = w|_G$ m-a.e. Additionally, we assume the following set (M) of conditions so that both $\mathcal{E}^{(c)}$ and $\mathcal{E}^{(j)}$ are compatible with the distance d:

(M-1) $\psi_K \in \mathcal{F}_{\mathrm{loc}}$ for every compact set $K \subset X$,
(M-2) $M_c := \operatorname{ess\,sup}_{x\in X^{(c)}} \Gamma^{(c)}(d,d)(x) < \infty$,
(M-3) $M_j := \operatorname{ess\,sup}_{x\in X^{(j)}} \int_{x\neq y} (1 \wedge d^2(x,y))\, j(x,dy) < \infty$,

where $X^{(c)} = \{x \in X\colon\ \Gamma^{(c)} \neq 0\}$ and $X^{(j)} = \{x \in X\colon\ \Gamma^{(j)} \neq 0\}$.

There are many classical examples of symmetric diffusions or symmetric pure jump processes whose Dirichlet form satisfies conditions (A) and (M): for instance, strongly-local Dirichlet forms on a metric measure space, whose distance is the Carnot–Carathéodori distance associated with the Dirichlet form. This includes canonical Dirichlet forms on Riemannian manifolds, CR manifolds, sub-Riemannian manifolds, and weighted manifolds; divergence type operators with bounded coefficients on Euclidean spaces; the sum of squares of vector fields satisfying Hörmader's condition, the quantum graphs, and pre-fractals. Other examples are symmetric α-stable Lévy processes with $\alpha \in (0,2)$ on Euclidean spaces, and symmetric random walks on graphs.

Let A be the generator of $(\mathcal{E}, \mathcal{F})$ in $L^2(X;m)$. We denote the associated semigroup and the resolvent by $(T_t)_{t\geqslant 0} = (e^{tA})_{t\geqslant 0}$ and $G = \int_0^\infty T_t\, dt$, respectively. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *conservative* if

$$T_t 1 \equiv 1, \quad m\text{-a.e. for any } t > 0$$

and *recurrent* if

$$Gf(x) \equiv 0 \text{ or } \infty \quad \text{for any } f \in L^1_+(X;m) \text{ and } m\text{-a.e. } x \in X.$$

It is a classical result that Brownian motion on $\mathbb{R}^n$ is conservative for any $n \geqslant 1$ and is recurrent if and only if $n = 1, 2$. This result has been generalized to the Wiener process of complete Riemannian manifolds, and one of the most important discoveries is that a certain bound of the volume at infinity – rather than the dimension – implies these properties. This fact was first found by M.P. Gaffney [10] for the conservativeness, and it has been refined by various methods in [1,23,36,17,5,14]. Especially, R. Azencott [1] and A. Grigor'yan [14] demonstrated that the conservativeness may fail without a condition on the curvature or volume. On the other hand, the recurrence of the Wiener process of Riemannian manifolds or jump processes has been investigated by several authors in [4,22,38,11,12,28]. Furthermore, K.-T. Sturm [35] extended the theory to a general strongly-local regular Dirichlet form on a metric measure space equipped with the Carnot–Carathéodori distance.

Recently, there has been a tremendous amount of work devoted to the conservation property of a non-local Dirichlet form; for instance, the physical Laplacian on an infinite graph [7,6,39–41,24,18–20] and non-local Dirichlet forms [26,15,33]; however, as far as the authors know, there is only one result by Z.-Q. Chen and T. Kumagai [3] for the Dirichlet form which has both the strongly-local and non-local terms. Due to its nature, the associated process is called a *jump-diffusion process*.

Our first main purpose is to investigate the conservative property of a jump-diffusion process. For any $x \in X$ and $r > 0$, the volume of $\overline{B}(x,r)$ is denoted by $V(x,r)$.

Theorem 1.1. *If*

$$\liminf_{r\to\infty} \frac{\ln V(x_0,r)}{r\ln r} < \infty, \tag{1.1}$$

for some $x_0 \in X$, then $(\mathcal{E},\mathcal{F})$ is conservative.

This result was obtained for a non-local Dirichlet form in [15, Theorem 1.1], where the left-hand side of (1.1) is required to be less than $1/2$. Let us explain the significance of removing the constant $1/2$ by comparing the uniqueness class with the conservation property. Let $\mathcal{U}$ be the set of the solutions to the Cauchy problem of the heat equation with zero initial data. If any $u \in \mathcal{U}$ is identically 0, then $\mathcal{U}$ is called a *uniqueness class*. Under an integrability assumption, determining the uniqueness class implies the conservativeness of Riemannian manifolds [13], Dirichlet forms [35], and graphs [20]. In fact, A. Grigor'yan [13] and K.-T. Sturm [35] established the sharp conservation test for complete Riemannian manifolds and strongly-local Dirichlet forms, respectively, in this way. However, X. Huang [20, Section 3.3] constructed an example of a graph, which verifies that the constant 1/2 is indeed needed for the uniqueness class. Therefore, Theorem 1.1 together with Huang's example demonstrates that the uniqueness class condition is really stronger than the conservation property for a graph.

Next, we turn to the recurrence. For any $x \in X$ and $r > 0$, the volumes of the closed ball $\overline{B}(x,r)$ intersected with $X^{(c)}$ and $X^{(j)}$ are denoted by $V^{(c)}(x,r)$ and $V^{(j)}(x,r)$, respectively. For $r > 0$, define

$$\omega(r) = \sup_{x \in X^{(j)}} \int_{x \neq y} \big(d(x,y) \wedge r\big)^2 \, j(x,dy).$$

Our second main result is

Theorem 1.2. *If*

$$\liminf_{r \to \infty} \frac{1}{r^2} \big[V^{(c)}(x_0,r) + V^{(j)}(x_0,r)\omega(r)\big] < \infty, \tag{1.2}$$

for some $x_0 \in X$, then $(\mathcal{E},\mathcal{F})$ is recurrent.

Theorem 1.2 was proven in the case of the Wiener process (namely, the process does not jump) on a complete Riemannian manifold by S.Y. Cheng and S.T. Yau [4]. Theorem 1.2 is sharp for an isotropic symmetric α-stable Lévy process on $\mathbb{R}^n$, see, e.g., [30, Corollary 37.17 and Theorem 37.18] or Example 5.2 in Section 5. Here, let us mention that [30, Corollary 37.17 and Theorem 37.18] are derived from the characteristic functions of the associated processes, see [32] for the recent development on this topic; while Theorem 1.2 is based on the theory of Dirichlet forms.

This paper is organized as follows. Section 2 is devoted to the preliminaries. Here we establish an integral-derivation type property for a Dirichlet form of jump-process type, which is a technical key to prove the conservation property. The main results, Theorems 1.1 and 1.2, are proved in Sections 3 and 4, respectively. Finally, in Section 5 we present some examples of symmetric jump-diffusions to illustrate the power of our main theorems.

2. Preliminaries: the integral-derivation property

In this section, we first prepare the preliminaries and then proceed to establish an integral-derivation type property for a Dirichlet form with jump-diffusion type. This will be used to prove the conservation property in the next section.

We begin with the following quite elementary fact.

Lemma 2.1. *If $u \in \mathcal{F}_{\mathrm{loc}} \cap L^\infty$ has compact support, where $L^\infty = L^\infty(X)$ is the space of real-valued bounded measurable functions on X, then $u \in \mathcal{F} \cap L^\infty$.*

Proof. Suppose that $\operatorname{supp} u \subset K$ with a compact set K. Let $\eta \in \mathcal{F} \cap L^\infty$ agree with u on $B(K,1)$. Because of the regularity and the fact that the constant function belongs to $\mathcal{F}_{\mathrm{loc}}$, see the remark in [8, p. 117], there is a function $\chi \in \mathcal{F} \cap L^\infty$ such that $\chi|_K = 1$ and $\operatorname{supp} \chi \subset B(K,1)$. Since $\eta\chi \in \mathcal{F}$ and $u = \eta\chi$, the statement follows. □

For the sake of simplicity, hereafter we denote $\Gamma[\cdot] = \Gamma(\cdot,\cdot)$, $\mathcal{E}[\cdot] = \mathcal{E}(\cdot,\cdot)$, etc. We say that the jump range of $\mathcal{E}$ or $\mathcal{E}^{(j)}$ is uniformly bounded, if there exists a constant $a > 0$ such that $\operatorname{supp}(j(x,\cdot)) \subset B(x,a)$ for every $x \in X$.

Lemma 2.2. *Suppose that the jump range of $\mathcal{E}$ is uniformly bounded. If $u \in \mathcal{F}_{\mathrm{loc}} \cap L^\infty$ is constant outside a compact set, then for any $v \in \mathcal{F} \cap L^\infty$, $uv \in \mathcal{F} \cap L^\infty$.*

Proof. Let $K \subset X$ be a compact set such that u is constant outside it. Consider the sequence of cut-off functions $(\chi_l)_{l\in\mathbb{N}}$, where for $l \geqslant 1$,

$$\chi_l = \big(\big(2 - l^{-1}\psi\big) \wedge 1\big)_+.$$

By Lemma 2.1, the function χ_l belongs to $\mathcal{F}$ for any $l \geqslant 1$. Obviously, $\chi_l = 1$ on $B(K,l)$ and $\mathrm{supp}(\chi_l) \subset \overline{B}(K,2l)$.

We set for any $l \geqslant 1$, $v_l = uv\chi_l$. Since $u \in \mathcal{F}_{\mathrm{loc}} \cap L^\infty$ and $v \in \mathcal{F} \cap L^\infty$, v_l belongs to $\mathcal{F}_{\mathrm{loc}} \cap L^\infty$ and has compact support. Hence, Lemma 2.1 shows that $v_l \in \mathcal{F}$ for any $l \geqslant 1$.

Next, we claim that the sequence $(v_l)_{l\geqslant 1}$ is $\mathcal{E}$-Cauchy. Set $\chi_{l,l'} = \chi_l - \chi_{l'}$ for $l, l' \geqslant 1$. Since the jump range of $\mathcal{E}$ is uniformly bounded, for large enough l and l',

$$\mathcal{E}[v_l - v_{l'}] = \mathcal{E}\big[(\chi_l - \chi_{l'})uv\big] = \kappa \cdot \mathcal{E}[\chi_{l,l'}v],$$

where $\kappa = u|_{K^c}$. By [8, Lemma 3.2.5],

$$\mathcal{E}^{(c)}[\chi_{l,l'}v] \leqslant 2\int v^2 \Gamma^{(c)}[\chi_{l,l'}]\,dm + 2\int \chi_{l,l'}^2 \Gamma^{(c)}[v]\,dm.$$

Because of (M) and the chain rule of the strongly-local Dirichlet form, see, e.g., [35, p. 190], $\Gamma^{(c)}[\chi_{l,l'}] \to 0$ as $l, l' \to \infty$. This together with the fact $\chi_{l,l'} \to 0$ as $l, l' \to \infty$ yields that $\mathcal{E}^{(c)}[\chi_{l,l'}v]$ tends to zero as $l, l' \to \infty$.

On the other hand,

$$\begin{aligned}
\mathcal{E}^{(j)}[\chi_{l,l'}v] &\leqslant 2\int v^2(x) \int \big(\chi_{l,l'}(x) - \chi_{l,l'}(y)\big)^2 j(x,dy)\,m(dx) \\
&\quad + 2\iint \chi_{l,l'}^2(y)\big(v(x) - v(y)\big)^2 j(x,dy)\,m(dx) \\
&=: (I) + (II).
\end{aligned}$$

For any $x \in X$,

$$\begin{aligned}
&\int \big(\chi_{l,l'}(x) - \chi_{l,l'}(y)\big)^2 j(x,dy) \\
&\quad = \int \big(\big(\chi_l(x) - \chi_l(y)\big) - \big(\chi_{l'}(x) - \chi_{l'}(y)\big)\big)^2 j(x,dy) \\
&\quad \leqslant 2\int \big(\chi_l(x) - \chi_l(y)\big)^2 j(x,dy) + 2\int \big(\chi_{l'}(x) - \chi_{l'}(y)\big)^2 j(x,dy) \\
&\quad \leqslant 2\big(l^{-2} + l'^{-2}\big)\int d(x,y)^2 j(x,dy).
\end{aligned}$$

Combining the fact that $\mathrm{supp}(j(x,dy)) \subset B(x,a)$ for all $x \in X$ and some $a > 0$ with the assumption (M), the last term in the right-hand side of the equation above is dominated by

$$2\big(1 + a^2\big)M_j\big(l^{-2} + l'^{-2}\big),$$

which tends to 0 as $l, l' \to \infty$. Hence $(I) \to 0$ as $l, l' \to \infty$. Since $\chi_{l,l'} \to 0$, m-a.e. as $l, l' \to \infty$, $(II) \to 0$ as $l, l' \to \infty$. Thus, $\mathcal{E}^{(j)}[\chi_{l,l'} v] \to 0$ as $l, l' \to \infty$, and so the desired claim follows.

Finally, since $v_l \to uv$, m-a.e. as $l \to \infty$, $uv \in \mathcal{F}_e$. This together with the fact $uv \in L^2$ and [8, Theorem 1.5.2(iii)] yields that $uv \in \mathcal{F}$. □

The following is the integral-derivation property for our Dirichlet form.

Lemma 2.3. *Suppose that the jump range of $\mathcal{E}$ is uniformly bounded. If $u \in \mathcal{F} \cap L^\infty$ and $\phi \in \mathcal{F}_{\mathrm{loc}} \cap L^\infty$ is constant outside a compact set, then*

$$\mathcal{E}(u, u\phi) = \int u\Gamma(u, \phi)\, dm + \int \phi\Gamma[u]\, dm, \tag{2.3}$$

where $\Gamma = \frac{1}{2}(\Gamma^{(c)} + \Gamma^{(j)})$.

Proof. According to Lemma 2.2, $u\phi \in \mathcal{F}$. By the derivation property of $\mathcal{E}^{(c)}$, see, e.g., [8, Lemma 3.2.5 and the note on p. 117],

$$\int \Gamma^{(c)}(u, u\phi)\, dm = \int u\Gamma^{(c)}(u, \phi)\, dm + \int \phi\Gamma^{(c)}[u]\, dm.$$

Next, by the integral property of a non-local Dirichlet form, see [27, Proposition 2.2], we have

$$\int \Gamma^{(j)}(u, u\phi)\, dm = \int u\Gamma^{(j)}(u, \phi)\, dm + \int \phi\Gamma^{(j)}[u]\, dm.$$

Combining the two identities, we obtain (2.3). □

3. Proof of Theorem 1.1: the conservation property

The aim of this section is to prove Theorem 1.1. For any $a > 0$, consider a symmetric form $(\mathcal{E}^{(j,a)}, \mathcal{F})$ defined by

$$\mathcal{E}^{(j,a)}[u] = \iint \bigl(u(x) - u(y)\bigr)^2 \mathbb{1}_{\{d(x,y) \leqslant a\}}\, j(x, dy)\, m(dx) \quad \text{for } u \in \mathcal{F}.$$

Under the condition (M), $(\mathcal{E}^{(j,a)} + \mathcal{E}^{(c)}, \mathcal{F})$ is a regular Dirichlet form, and it is conservative if and only if so is $(\mathcal{E}, \mathcal{F})$, see [31, Section 4] and [26, Section 3]. Clearly, $(\mathcal{E}^{(j,a)}, \mathcal{F})$ has uniformly bounded range. Therefore, in order to prove the conservation property, we may and do assume that $\mathcal{E}$ has uniformly bounded jump range. More precisely, we suppose that there exists a constant $a > 0$ such that

$$j(x, dy) = \mathbb{1}_{B(x,a)}(y)\, j(x, dy) \quad \text{for all } x \in X.$$

Our proof is basically the Davies method [5], which was used also in [15]; however, we are able to get a better result because of the choice of a. In this section, the constant a will be

$$a = a(x_0, m) := \left[8 \liminf_{r \to \infty} \frac{\log V(x_0, r)}{r \log r} + 9\right]^{-1}, \tag{3.4}$$

where $x_0 \in X$ is the reference point in Theorem 1.1. For $f \in C_0(X)$ with $f \geqslant 0$, set

$$\psi(x) = d\big(x, \operatorname{supp}(f)\big)$$

and

$$\phi(x) = e^{\alpha\psi(x)},$$

where $\alpha > 0$ is a constant determined later. Note that if $n \geqslant 1$ and $x \in X$ satisfy

$$n \geqslant a^{-1}\big[4a + 2d\big(x_0, \operatorname{supp}(f)\big)\big] \quad \text{and} \quad (n-2)a \leqslant d(x, x_0) \leqslant (n+1)a,$$

then

$$\psi(x) \geqslant d(x, x_0) - d\big(x_0, \operatorname{supp}(f)\big) \geqslant (n-2)a - d\big(x_0, \operatorname{supp}(f)\big) \geqslant an/2,$$

and so

$$\phi(x) = e^{\alpha\psi(x)} \geqslant e^{a\alpha n/2}. \tag{3.5}$$

For the function f above and any $t \geqslant 0$, we denote $u_t = T_t f$. Since $(T_t)_{t\geqslant 0}$ is analytic, u_t belongs to the domain of the L^2-generator A of $(\mathcal{E}, \mathcal{F})$; in particular, $u_t \in \mathcal{F} \cap L^\infty$ for any $t > 0$.

The following lemma provides the key estimate.

Lemma 3.1. *Using the notations above, for any $t \geqslant 0$,*

$$\int_0^t \int \phi\Gamma[u_s]\, dm\, ds \leqslant 2e^{\gamma t}\big\|\phi^{1/2} f\big\|_2^2, \tag{3.6}$$

where $\gamma = \alpha^2(e^{2\alpha a} + 1)M/2$ and $M = M_c \vee M_j$.

Proof. In the following, we denote the norm and the inner product of $L^2(X; m)$ by $\|\cdot\|_2$ and $\langle\cdot,\cdot\rangle$, respectively. For any $n \geqslant 1$, set

$$\phi_n(x) = e^{\alpha(\psi(x)\wedge n)}.$$

Since $\psi \in \mathcal{F}_{\mathrm{loc}}$, we may apply an argument in [8, pp. 116–117] to deduce that $\phi_n \in \mathcal{F}_{\mathrm{loc}}$ for every $n \geqslant 1$. Taking into account that $\psi \in L^\infty$ is constant outside a compact set, Lemma 2.2 shows that for every $t > 0$ and $n \geqslant 1$, $u_t\phi_n \in \mathcal{F}$. Therefore, by Lemma 2.3, for all $t > 0$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\big\|\phi_n^{1/2}u_t\big\|_2^2 &= \langle \dot{u}_t, \phi_n u_t\rangle \\
&= -\mathcal{E}(u_t, \phi_n u_t) \\
&= -\int \phi_n\,\Gamma[u_t]\,dm - \int u_t\,\Gamma(u_t, \phi_n)\,dm \\
&\leqslant -\int \phi_n\,\Gamma[u_t]\,dm + \left|\int u_t\,\Gamma(u_t, \phi_n)\,dm\right|,
\end{aligned}$$

where $\dot{u}_t = \frac{d}{dt}u_t$. This is,

$$\int \phi_n \Gamma[u_t]\,dm \leqslant \left|\int u_t \Gamma(u_t,\phi_n)\,dm\right| - \frac{1}{2}\frac{d}{dt}\big\|\phi_n^{1/2}u_t\big\|_2^2. \tag{3.7}$$

Next, we estimate the first term on the right side of this equation. For every $x \in X$, according to the Cauchy–Schwarz inequality,

$$\begin{aligned}\big|\Gamma^{(j)}(u_t,\phi_n)(x)\big| &= \left|\int \big(u_t(x)-u_t(y)\big)\big(\phi_n(x)-\phi_n(y)\big)\,j(x,dy)\right| \\ &\leqslant \sqrt{\int \big(u_t(x)-u_t(y)\big)^2\,j(x,dy)}\sqrt{\int \big(\phi_n(x)-\phi_n(y)\big)^2\,j(x,dy)} \\ &= \sqrt{\Gamma^{(j)}[u_t](x)}\sqrt{\Gamma^{(j)}[\phi_n](x)}.\end{aligned}$$

By the Cauchy–Schwarz inequality again,

$$\begin{aligned}\left|\int u_t\Gamma^{(j)}(u_t,\phi_n)\,dm\right| &\leqslant \int \phi_n^{1/2}\sqrt{\Gamma^{(j)}[u_t]}\phi_n^{-1/2}\sqrt{u_t^2\Gamma^{(j)}[\phi_n]}\,dm \\ &\leqslant \sqrt{\int \phi_n\Gamma^{(j)}[u_t]\,dm}\sqrt{\int \phi_n^{-1}\,u_t^2\,\Gamma^{(j)}[\phi_n]\,dm}.\end{aligned}$$

Since

$$\big|e^{\alpha r}-1\big| \leqslant \alpha\, e^{\alpha a}|r| \quad \text{for any } r\in(0,a],$$

it follows that

$$\big|\phi_n(x)-\phi_n(y)\big| \leqslant \alpha e^{\alpha a}\,\phi_n(x)\,d(x,y) \quad \text{for any } x,y\in X \text{ with } d(x,y)\leqslant a,$$

and so

$$\Gamma^{(j)}[\phi_n](x) \leqslant \big(\alpha e^{\alpha a}\phi(x)\big)^2\int d^2(x,y)\,j(x,dy) \quad \text{for every } x\in X.$$

Since $\operatorname{supp}(j(x,dy)) \subset B(x,a)$ for any $x \in X$ and some constant $a \in (0,1)$, we get

$$\begin{aligned}\int \phi_n^{-1}\,u_t^2\Gamma^{(j)}[\phi_n]\,dm &\leqslant \alpha^2e^{2\alpha a}\int \phi_n(x)\,u_t^2(x)\int d(x,y)^2\,j(x,dy)\,m(dx) \\ &\leqslant \alpha^2e^{2\alpha a}\int \phi_n(x)\,u_t^2(x)\int \big(d(x,y)\wedge a\big)^2\,j(x,dy)\,m(dx) \\ &\leqslant M_j\alpha^2e^{2\alpha a}\int \phi_n\,u_t^2\,dm.\end{aligned}$$

Therefore, for any $\lambda > 0$,

$$\left|\int u_t \Gamma^{(j)}(u_t,\phi_n)\,dm\right| \leqslant \sqrt{M_j\int \phi_n\,\Gamma^{(j)}[u_t]\,dm}\sqrt{\alpha^2 e^{2\alpha a}\int \phi_n\,u_t^2\,dm}$$
$$\leqslant \frac{M_j}{2\lambda}\int \phi_n\,\Gamma^{(j)}[u_t]\,dm + \frac{\lambda\alpha^2 e^{2\alpha a}}{2}\int \phi_n\,u_t^2\,dm$$
$$= \frac{M_j}{2\lambda}\int \phi_n\,\Gamma^{(j)}[u_t]\,dm + \frac{\lambda\alpha^2 e^{2\alpha a}}{2}\big\|\phi_n^{1/2}u_t\big\|_2^2,$$

where in the last inequality we have used the fact that $2\xi\eta \leqslant \lambda^{-1}\xi^2 + \lambda\eta^2$ for any $\xi,\eta \geqslant 0$ and $\lambda > 0$.

On the other hand, we apply the argument above for the local term to get that

$$\left|\int u_t \Gamma^{(c)}(u_t,\phi_n)\,dm\right| \leqslant \sqrt{\int \phi_n \Gamma^{(c)}[u_t]\,dm}\sqrt{\int \phi_n^{-1}u_t^2\Gamma^{(c)}[\phi_n]\,dm}.$$

According to the chain rule for a strongly-local Dirichlet form, see, e.g., [35, p. 190],

$$\int \phi_n^{-1}u_t^2\Gamma^{(c)}[\phi_n]\,dm \leqslant \alpha^2\int u_t^2\phi_n\Gamma^{(c)}[d]\,dm,$$

which along with the assumption (M) gives us

$$\int \phi_n^{-1}\,u_t^2\,\Gamma^{(c)}[\phi_n]\,dm \leqslant M_c\alpha^2\int u_t^2\phi_n\,dm.$$

We again follow the argument above to obtain the estimate:

$$\left|\int u_t \Gamma^{(c)}(u_t,\phi_n)\,dm\right| \leqslant \frac{M_c}{2\lambda}\int \phi_n\,\Gamma^{(c)}[u_t]\,dm + \frac{\lambda\alpha^2}{2}\big\|\phi_n^{1/2}u_t\big\|_2^2 \quad \text{for any } \lambda > 0.$$

Combining the estimates for the non-local and strongly-local terms, we get that

$$\left|\int u_t \Gamma(u_t,\phi_n)\,dm\right| \leqslant \frac{M}{2\lambda}\int \phi_n\Gamma[u_t]\,dm + \frac{\lambda\alpha^2(e^{2\alpha a}+1)}{2}\big\|\phi_n^{1/2}u_t\big\|_2^2.$$

By applying this inequality for (3.7), we have

$$\left(2-\frac{M}{\lambda}\right)\int \phi_n\Gamma[u_s]\,dm \leqslant \lambda\alpha^2\big(e^{2\alpha a}+1\big)\big\|\phi_n^{1/2}u_s\big\|_2^2 - \frac{d}{ds}\big\|\phi_n^{1/2}u_s\big\|_2^2. \tag{3.8}$$

If we integrate this with respect to s over $[0,t]$, then

$$\left(2-\frac{M}{\lambda}\right)\int_0^t\!\!\int \phi_n\Gamma[u_s]\,dm$$
$$\leqslant \lambda\alpha^2\big(e^{2\alpha a}+1\big)\int_0^t \big\|\phi_n^{1/2}u_s\big\|_2^2\,ds - \big(\big\|\phi_n^{1/2}u_t\big\|_2^2 - \big\|\phi_n^{1/2}f\big\|_2^2\big). \tag{3.9}$$

We estimate $\|\phi_n^{1/2}u_s\|_2^2$ for any $s \leqslant t$ by first letting $\lambda = M/2$ in (3.8),

$$\frac{d}{ds}\big\|\phi_n^{1/2}u_s\big\|_2^2 \leqslant \frac{M\alpha^2(e^{2\alpha a}+1)}{2}\big\|\phi_n^{1/2}u_s\big\|_2^2,$$

and then, by applying the Gronwall inequality:

$$\big\|\phi_n^{1/2}u_s\big\|_2^2 \leqslant \exp\left(\frac{M\alpha^2(e^{2\alpha a}+1)s}{2}\right)\big\|\phi_n^{1/2}f\big\|_2^2.$$

Substituting this into (3.9), we have

$$\left(2-\frac{M}{\lambda}\right)\int_0^t\int \phi_n\Gamma[u_s]\,dm\,ds$$
$$\leqslant \big\|\phi_n^{1/2}f\big\|_2^2 + \frac{2\lambda}{M}\big[\exp\big(M\alpha^2\big(e^{2\alpha a}+1\big)t/2\big)-1\big]\big\|\phi_n^{1/2}f\big\|_2^2.$$

Setting $\lambda = M$, this becomes

$$\int_0^t\int \phi_n\Gamma[u_s]\,dm\,ds \leqslant 2\exp\big(M\alpha^2\big(e^{2\alpha a}+1\big)t/2\big)\big\|\phi_n^{1/2}f\big\|_2^2.$$

The required assertion (3.6) follows by letting $n \to \infty$. □

We are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We adopt the notations in the proof of Lemma 3.1. Define a cut-off function g_n for any $n \geqslant 1$ as follows

$$g_n(x) := \big(\big(n - a^{-1}d(x,x_0)\big)\wedge 1\big)_+.$$

By Lemma 2.1, g_n belongs to $\mathcal{F}$. To the end of the proof, we show that there exists a sequence $(n_k)_{k\geqslant 0}$ such that $n_k \to \infty$ as $k \to \infty$, and for every $t > 0$,

$$\int_0^t \langle \dot{u}_s, g_{n_k}\rangle\,ds \to 0 \quad \text{as } k \to \infty.$$

Indeed, we can deduce from this and the dominated convergence theorem that

$$\langle T_t f, 1\rangle = \lim_{k\to\infty}\langle u_t, g_{n_k}\rangle = \lim_{k\to\infty}\langle f, g_{n_k}\rangle + \lim_{k\to\infty}\int_0^t \langle \dot{u}_s, g_{n_k}\rangle\,ds = \langle f, 1\rangle,$$

which immediately implies the conservation property.

Since $(u_s)_{s>0}$ solves the heat equation and $g_n \in \mathcal{F}$,

$$\int_0^t \langle \dot{u}_s, g_n\rangle\, ds = -\int_0^t \mathcal{E}(u_s, g_n)\, ds = -\int_0^t \bigl(\mathcal{E}^{(c)}(u_s, g_n) + \mathcal{E}^{(j)}(u_s, g_n)\bigr)\, ds. \tag{3.10}$$

First, we estimate the second term, the harder one, on the right side. For any $t > 0$,

$$\begin{aligned}
\left|\int_0^t \mathcal{E}^{(j)}(u_s, g_n)\, ds\right| &\leqslant \int_0^t \left|\int \Gamma^{(j)}(u_s, g_n)\, dm\right| ds \\
&\leqslant \int_0^t \left[\int \sqrt{\Gamma^{(j)}[u_s]}\sqrt{\Gamma^{(j)}[g_n]}\, dm\right] ds \\
&= \int_0^t \left[\int \sqrt{\phi \Gamma^{(j)}[u_s]}\sqrt{\phi^{-1}\Gamma^{(j)}[g_n]}\, dm\right] ds \\
&\leqslant \int_0^t \sqrt{\int \phi \Gamma^{(j)}[u_s]\, dm}\sqrt{\int \phi^{-1}\Gamma^{(j)}[g_n]\, dm}\, ds \\
&\leqslant \sqrt{\int_0^t\int \phi \Gamma^{(j)}[u_s]\, dm\, ds}\sqrt{\int_0^t\int \phi^{-1}\Gamma^{(j)}[g_n]\, dm\, ds} \\
&= \sqrt{\int_0^t\int \phi \Gamma^{(j)}[u_s]\, dm\, ds}\sqrt{t\int \phi^{-1}\Gamma^{(j)}[g_n]\, dm},
\end{aligned} \tag{3.11}$$

where all the inequalities above follow from the Cauchy–Schwarz inequality. For any $n > 0$, let A_n denote the following annulus associated with the constant a

$$A_n = A_n(a) = \overline{B}\bigl(x_0, (n+1)a\bigr) \setminus B\bigl(x_0, (n-2)a\bigr).$$

Since $\operatorname{supp}(g_n) \subset B(x_0, na)$ and $\operatorname{supp}(j(x, dy)) \subset B(x, a)$ for all $x \in X$, it holds that if $x \notin A_n$,

$$\Gamma^{(j)}[g_n](x) = \int \bigl(g_n(x) - g_n(y)\bigr)^2\, j(x, dy) = 0;$$

if $x \in A_n$,

$$\begin{aligned}
\Gamma^{(j)}[g_n](x) &\leqslant a^{-2}\int d(x, y)^2\, j(x, dy) \\
&\leqslant a^{-2}\int \bigl(d(x, y) \wedge a\bigr)^2\, j(x, dy) \\
&\leqslant a^{-2} M_j,
\end{aligned}$$

where in the last inequality we have used the fact that $0 < a < 1$. Choosing n large enough so that $n \geqslant a^{-1}[4a + 2d(x_0, \operatorname{supp}(f))]$, we get from (3.5) that

$$\int \phi^{-1} \Gamma^{(j)}[g_n]\, dm = \int_{A_n} \phi^{-1} \Gamma^{(j)}[g_n]\, dm \leqslant a^{-2} M_j e^{-\alpha a n/2} m(A_n).$$

Therefore, by (3.11),

$$\left| \int_0^t \mathcal{E}^{(j)}(u_s, g_n)\, ds \right|^2 \leqslant a^{-2} t M_j e^{-\alpha a n/2} m(A_n) \int_0^t \int \phi \Gamma^{(j)}[u_s]\, dm\, ds.$$

In a similar way, we can prove that

$$\left| \int_0^t \mathcal{E}^{(c)}(u_s, g_n)\, ds \right|^2 \leqslant a^{-2} t M_c e^{-\alpha a n/2} m(A_n) \int_0^t \int \phi \Gamma^{(c)}[u_s]\, dm\, ds.$$

Therefore,

$$\left| \int_0^t \mathcal{E}(u_s, g_n)\, ds \right|^2 \leqslant 2a^{-2} t M e^{-\alpha a n/2} m(A_n) \int_0^t \int \phi \Gamma[u_s]\, dm\, ds. \tag{3.12}$$

We now apply (3.12) and Lemma 3.1 for (3.10) to get that

$$\begin{aligned} &\left| \int_0^t \langle \dot{u}_s, g_n \rangle\, ds \right|^2 \\ &\quad \leqslant 2a^{-2} t M e^{-\alpha a n/2} m(A_n) \int_0^t \int \phi \Gamma[u_s]\, dm\, ds \\ &\quad \leqslant 4a^{-2} t M \left\| \phi^{1/2} f \right\|_2^2 \exp\left(\frac{M\alpha^2 (e^{2\alpha a} + 1)t}{2} - \frac{\alpha a n}{2} + \log m(A_n) \right). \end{aligned} \tag{3.13}$$

Finally, we estimate (3.13) by applying the volume assumption (1.1). Indeed, according to (1.1), there exists a sequence $(n_k)_{k\geqslant 1}$ such that $n_k \to \infty$ as $k \to \infty$, and for a large enough $k \geqslant 1$,

$$\begin{aligned} \log m(A_{n_k}) &\leqslant \log V\big(x_0, (n_k+1)a\big) \\ &\leqslant (c_3 - 1/2)\big((n_k+1)a\big) \log\big((n_k+1)a\big) \\ &\leqslant a c_3 n_k \log n_k, \end{aligned}$$

where

$$c_3 = \liminf_{r\to\infty} \frac{\log V(x_0, r)}{r \log r} + 1.$$

Taking $\alpha = 4c_3 \log n_k$ and k large enough such that $n_k \geqslant a^{-1}[4a + 2d(x_0, \operatorname{supp}(f))]$, we estimate the right side of (3.13) to get

$$\left| \int_0^t \langle \dot{u}_s, g_{n_k} \rangle \, ds \right|^2 \leqslant 4a^{-2} t M \left\| \phi^{1/2} f \right\|_2^2 \times \exp\left(\frac{M\alpha^2 (e^{2\alpha a} + 1)t}{2} - 2ac_3 n_k \log n_k + ac_3 n_k \log n_k \right)$$
$$= 4a^{-2} t M \left\| \phi^{1/2} f \right\|_2^2 \exp\left(\frac{M\alpha^2 (e^{2\alpha a} + 1)t}{2} - ac_3 n_k \log n_k \right).$$

Since $e^{2\alpha a} = n_k^{8ac_3}$ and $8ac_3 < 1$, the inequality above implies that for any $t > 0$

$$\lim_{k \to \infty} \int_0^t \langle \dot{u}_s, g_{n_k} \rangle \, ds = 0.$$

This completes the proof. □

4. Proof of Theorem 1.2: the recurrence

This section is devoted to the proof of the recurrence test, Theorem 1.2.

Proof of Theorem 1.2. Let $x_0 \in X$ be the reference point in Theorem 1.2. For $R > 2$, set

$$\theta_R(x) = \left(\left(\frac{R - d(x, x_0)}{R - 1} \right) \wedge 1 \right)_+ .$$

Since θ_R belongs to $\mathcal{F}_{\mathrm{loc}} \cap L^\infty$ and has compact support, by Lemma 2.1, θ_R belongs to $\mathcal{F}$. According to the condition (M) and the chain-rule for a strongly-local Dirichlet form,

$$\mathcal{E}^{(c)}[\theta_R] = \int_X \Gamma^{(c)}[\theta_R] \, dm$$
$$= \left(\frac{1}{R-1} \right)^2 \int_{\overline{B}(x_0, R)} \Gamma^{(c)}[d] \, dm$$
$$\leqslant M_c \left(\frac{1}{R-1} \right)^2 V^{(c)}(x_0, R)$$
$$\leqslant \frac{4 M_c V^{(c)}(x_0, R)}{R^2}.$$

On the other hand, we find that for any $c_1 > 2$

$$
\begin{aligned}
\mathcal{E}^{(j)}[\theta_R] &= \iint \big(\theta_R(x)-\theta_R(y)\big)^2 \, j(x,dy)\, m(dx) \\
&\leqslant \frac{2}{(R-1)^2} \int\limits_{B(x_0,R)} \int\limits_{B(x_0,c_1 R)} d(x,y)^2 \, j(x,dy)\, m(dx) \\
&\quad + 2 \int\limits_{B(x_0,R)} \int\limits_{B(x_0,c_1 R)^c} j(x,dy)\, m(dx) \\
&\leqslant \frac{2}{(R-1)^2} \int\limits_{B(x_0,R)} \int\limits_{d(x,y)\leqslant 2c_1 R} d(x,y)^2 \, j(x,dy)\, m(dx) \\
&\quad + 2 \int\limits_{B(x_0,R)} \int\limits_{d(x,y)\geqslant (c_1-1)R} j(x,dy)\, m(dx),
\end{aligned}
$$

where we used the facts that $d(x,y) \leqslant R + c_1 R \leqslant 2c_1 R$ if $x \in B(x_0,R)$ and $y \in B(x_0,c_1 R)$; $d(x,y) \geqslant c_1 R - R \geqslant R_1$ if $x \in B(x_0,R)$ and $y \notin B(x_0,c_1 R)$. The last expression is bounded from above by

$$
\begin{aligned}
&\leqslant \frac{8c_1^2}{(R-1)^2} \int\limits_{B(x_0,R)} \int \big(d(x,y)\wedge R\big)^2 \, j(x,dy)\, m(dx) \\
&\quad + \frac{2}{R^2} \int\limits_{B(x_0,R)} \int \big(d(x,y)\wedge R\big)^2 \, j(x,dy)\, m(dx) \\
&\leqslant \frac{33\, c_1^2}{R^2} \int\limits_{B(x_0,R)} \int \big(d(x,y)\wedge R\big)^2 \, j(x,dy)\, m(dx).
\end{aligned}
$$

Therefore, under the assumption (M), we have that for $c_2 = 4M_c + 33c_1^2$

$$
\begin{aligned}
\mathcal{E}[\theta_R] &\leqslant \frac{1}{R^2}\bigg[4M_c V^{(c)}(x_0,R) + 33c_1^2 V^{(j)}(x_0,R) \sup_{x\in X^{(j)}} \int \big(d(x,y)\wedge R\big)^2 \, j(x,dy)\bigg] \\
&\leqslant \frac{c_2}{R^2}\bigg[V^{(c)}(x_0,R) + V^{(j)}(x_0,R) \sup_{x\in X^{(j)}} \int \big(d(x,y)\wedge R\big)^2 \, j(x,dy)\bigg].
\end{aligned}
$$

According to the volume condition (1.2), there exists a sequence $(n_k)_{k\geqslant 0}$ such that $n_k \to \infty$ as $k \to \infty$, and

$$
\liminf_{k\to\infty} \mathcal{E}[\theta_{R_{n_k}}] < \infty.
$$

Applying [8, Theorem 1.6.3] and [34, (1.6.1) and (1.6.1′)], this completes the proof. □

5. Examples

In this section we present some examples to illustrate the power of Theorems 1.1 and 1.2. Throughout the section, we denote the space of real-valued Lipschitz continuous functions with compact support on a metric space X by $C_0^{\mathrm{Lip}}(X)$. For a measure space (X,m) and a quadratic form $\mathcal{E}$ defined in $L^2(X;m)$, we denote

$$\mathcal{E}_1[u] = \|u\|_{L^2}^2 + \mathcal{E}[u],$$

whenever the right side makes sense. We start with the following remark for the volume test in Theorem 1.1.

Remark 5.1. Let (X,d,m) be a complete metric measure space such that m is a Radon measure with full support. Assume that there is a point $x_0 \in X$ such that

$$\sup_{r>0} \frac{V(x_0,2r)}{V(x_0,r)} < \infty,$$

where $V(x_0,r)$ denotes the volume of the closed ball centered at x_0 with radius $r>0$. This assumption is called the *volume doubling condition* at point x_0, and it implies that there is a constant $\kappa>0$ such that

$$\sup_{r>0} \frac{V(x_0,r)}{r^\kappa} < \infty.$$

In particular, condition (1.1) in Theorem 1.1 is satisfied. A typical example which fulfills the volume doubling condition is a Riemannian manifold with non-negative Ricci curvature.

5.1. Sharpness examples

In the following example, we consider two classes of symmetric jump processes on the so called *κ-set*.

Example 5.2. Let $(X,|\cdot|,m)$ be a closed κ-set in $\mathbb{R}^n$ with $0<\kappa\leqslant n$, i.e., $|\cdot|$ is the Euclidean distance, and for all $x\in X$ and $r>0$,

$$m\big(B(x,r)\big) \asymp r^\kappa.$$

Here, the symbol $\asymp$ means that the ratio of the left and the right-hand sides is pinched by two positive constants. Assume that the jump kernel $j(x,dy)$ has a density $j(x,y)$ with respect to the measure $m(dy)$ such that one of the following two conditions is satisfied with a constant $\alpha\in(0,2)$:

$$\text{(i)} \qquad j(x,y) \asymp \frac{1}{|x-y|^{\kappa+\alpha}} \mathbb{1}_{\{|x-y|\leqslant 1\}} + \frac{1}{|x-y|^{\kappa+\beta}} \mathbb{1}_{\{|x-y|>1\}}, \quad \text{where } 0<\beta<\infty;$$

$$\text{(ii)} \qquad j(x,y) \asymp \frac{1}{|x-y|^{\kappa+\alpha}} \mathbb{1}_{\{|x-y|\leqslant 1\}} + \frac{e^{-c|x-y|}}{|x-y|^{\kappa+\alpha}} \mathbb{1}_{\{|x-y|>1\}}, \quad \text{where } c>0.$$

For $u,v\in C_0^{\mathrm{Lip}}(X)$, define

$$\mathcal{E}(u,v) = \iint_{x\neq y} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big) j(x,y)\, m(dx)\, m(dy).$$

Let $\mathcal{F}$ be the closure of $C_0^{\mathrm{Lip}}(X)$ with respect to the $\sqrt{\mathcal{E}_1}$-norm. The symmetric form $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form in $L^2(X, m)$, see, e.g., [37]. According to Theorems 1.1 and 1.2, the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is conservative, and it is recurrent if additionally $0 < \kappa \leqslant \beta \wedge 2$ and $0 < \kappa \leqslant 2$ for the cases (i) and (ii), respectively.

Remark 5.3. Example 5.2 is motivated by recent developments for layered stable processes [16] and tempering stable processes [29]. In particular, in case (i) if $\beta = \alpha$, then the associated Hunt process is called a *stable-like process* [2].

5.2. *Disconnected space*

The following example shows that the state space may be topologically disconnected, and the particles jump between different connected components and it behaves as a jump-diffusion inside a connected component.

Example 5.4. Let $X = \bigcup_{i\in\mathbb{Z}} X_i$, where for each $i \in \mathbb{Z}$, $X_i = \{(x_i, i) \in \mathbb{R}^{n+1}\colon\ x_i \in \mathbb{R}^n\}$. Any point x in X can be expressed uniquely as $x = (x_i, i)$ with $x_i \in \mathbb{R}^n$ and $i \in \mathbb{Z}$, and we denote the associated projections by $p : X \to \mathbb{R}^n$ and $q : X \to \mathbb{Z}$. For any $x, y \in X$, the distance d is given by

$$d(x, y) = \big|p(x) - p(y)\big| + \big|q(x) - q(y)\big|,$$

where $|\cdot|$ is the Euclidean distance. Let $m(dx) = \sum_{i\in\mathbb{Z}} m_i(dx_i)$ be a measure on X such that for each $i \geqslant 1$, $m_i(dx_i) = \Psi(x_i)\,dx_i$ is a measure on X_i, where $\Psi \in C(\mathbb{R}^n)$ is a positive function, and dx_i is the n-dimensional Lebesgue measure. Clearly, m is a Radon measure on X. The state space is the triple (X, d, m).

For any $u \in C_0^{\mathrm{Lip}}(X)$, define

$$\mathcal{E}[u] = \mathcal{E}^{(c)}[u] + \mathcal{E}^{(j)}[u],$$

where

$$\mathcal{E}^{(c)}[u] = \int_X |\nabla u|^2\,dm,$$

$$\mathcal{E}^{(j)}[u] = \int_X \int_{x\neq y} \big(u(x) - u(y)\big)^2 j(x, y)\,m(dx)\,m(dy),$$

and

$$j(x, y) \asymp \frac{d(x, y)^{-(n+\alpha)}\mathbb{1}_{\{d(x,y)<1\}} + d(x, y)^{-(n+\beta+1)}\mathbb{1}_{\{d(x,y)\geqslant 1\}}}{\Psi(p(x)) + \Psi(p(y))}, \quad x, y \in X$$

with some constants $0 < \alpha < 2$ and $\beta > 0$. Let $\mathcal{F}$ be the closure of $C_0^{\mathrm{Lip}}(X)$ with respect to the $\sqrt{\mathcal{E}_1}$-norm. Since for any $x \in X$

$$\int_{x\neq y} \left(1\wedge d(x,y)^2\right) j(x,y)\, m(dy)$$

$$\leqslant \int_{0<d(x,y)<1} \frac{d(x,y)^{-(n+\alpha-2)}\Psi(p(y))\,dp(y)}{\Psi(p(x))+\Psi(p(y))} + \int_{d(x,y)\geqslant 1} \frac{d(x,y)^{-(n+\beta+1)}\Psi(p(y))\,dp(y)}{\Psi(p(x))+\Psi(p(y))}$$

$$\leqslant \int_{0<d(x,y)<1} d(x,y)^{-(n+\alpha-2)}\,dp(y) + 2\sum_{k\geqslant 0} \int_{|p(x)-p(y)|\geqslant k+1} \left|p(x)-p(y)\right|^{-(n+\beta+1)}\,dp(y),$$

which is bounded from above by some absolute constant $c>0$, it follows form the proof of [37] that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form in $L^2(X,m)$.

According to the arguments above, we can easily claim that the condition (M) is satisfied. Therefore, by Theorem 1.1, if there is a constant $c>0$ such that for $r>0$ large enough

$$\sum_{0\leqslant k\leqslant [r]} \int_{B(0,[r]-k)} \Psi(z)\,dz \leqslant r^{cr}, \tag{5.14}$$

where dz is the n-dimensional Euclidean measure and $[r]$ is the least integer such that $[r]\geqslant r$, then the Dirichlet form $(\mathcal{E},\mathcal{F})$ is conservative. For instance, (5.14) is satisfied, if $\Psi(x)\leqslant |x|^{|x|}\ln|x|$ for $|x|$ large enough.

For the recurrence, we additionally assume that there are two constants $c_0, c_1>0$ such that

$$j(x,y)\leqslant \frac{\mathbb{1}_{\{d(x,y)\leqslant c_0\}}}{d(x,y)^{1+\alpha}} \tag{5.15}$$

and

$$\Psi(x)\leqslant c_1|x|^{1-n} \quad \text{for } |x| \text{ large enough.} \tag{5.16}$$

Condition (5.16) will imply that for any point $x_0\in X$,

$$\liminf_{r\to\infty}\frac{V(x_0,r)}{r^2}\leqslant 2\liminf_{r\to\infty}\frac{1}{r^2}\sum_{0\leqslant k\leqslant [r]}\int_{B(x_0,[r]-k)}\Psi(x)\,dx<\infty.$$

Next, by (5.15), there is a constant $c_2>0$ depending only on the dimension such that

$$\begin{aligned}
\omega(r) &\leqslant \sup_{x\in X}\int_X d(x,y)^2 j(x,y)\Psi\big(p(y)\big)\,dp(y)\\
&\leqslant c_1\sup_{x\in X}\int_{d(x,y)\leqslant c_0} d(x,y)^{1-\alpha}\left|p(y)\right|^{1-n}\,dp(y)\\
&\leqslant 2c_1c_2\sum_{0\leqslant k\leqslant [c_0]}\int_0^{[c_0]-k} r^{1-\alpha}\,dr<\infty.
\end{aligned}$$

Therefore, $(\mathcal{E},\mathcal{F})$ is recurrent by Theorem 1.2.

5.3. Volume tests

The first volume test for non-local Dirichlet forms to be conservative was obtained in [26, Main Result], and then refined in [15, Theorem 1.1]. It is easy to construct an example, which is not covered by these tests but by Theorem 1.1. Here, we illustrate this by using a weighted Euclidean space as well as a model manifold.

Example 5.5. Let $(\mathbb{R}, |\cdot|, m)$ be a weighted Euclidean space, where $|\cdot|$ is the Euclidean distance and the measure is $m(dx) = e^{2\lambda|x|}\,dx$ for some $\lambda > 0$. For $u \in C_0^{\mathrm{Lip}}(\mathbb{R})$, define

$$\mathcal{E}[u] = \iint\limits_{x\neq y} \big(u(x) - u(y)\big)^2 j(x, y)\, m(dx)\, m(dy),$$

where

$$j(x, y) = \big(e^{-\lambda(|x|+|y|)}\big)\mathbb{1}_{\{|x-y|\leqslant 1\}}.$$

Let $\mathcal{F}$ be the closure of $C_0^{\mathrm{Lip}}(\mathbb{R})$ with respect to the $\sqrt{\mathcal{E}_1}$-norm. The symmetric form $(\mathcal{E}, \mathcal{F})$ becomes a regular Dirichlet form in $L^2(\mathbb{R}, m)$, see, e.g., [37]. Let $j(x, dy) = j(x, y)\, m(dy)$. It holds that

$$\begin{aligned}\sup_{x\in\mathbb{R}} \int \big(1 \wedge |x-y|^2\big)\, j(x, dy) &= \sup_{x\in\mathbb{R}} \int\limits_{\{|y-x|\leqslant 1\}} |x-y|^2 j(x, y)\, m(dy)\\ &= \sup_{x\in\mathbb{R}} e^{-\lambda|x|} \int\limits_{\{|z|\leqslant 1\}} z^2 e^{\lambda|x-z|}\, dz\\ &\leqslant \int\limits_{\{|z|\leqslant 1\}} z^2 e^{\lambda|z|}\, dz < \infty.\end{aligned}$$

On the other hand, it is easy to see that in this example (1.1) is also satisfied. Therefore, according to Theorem 1.1, the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is conservative.

However, since $x \mapsto e^{-r|x|} \notin L^1(\mathbb{R}, m)$ for any $r \leqslant 2\lambda$, this example is not covered by [26, Main Result].

Example 5.6 *(Model manifolds).* (See, e.g., [14].) Let $(\mathbb{S}^n, g)$ be the n-dimensional unit sphere with $n \geqslant 1$. A *model manifold* $M = (0, +\infty) \times \mathbb{S}^n$ is a Riemannian manifold with Riemannian tensor

$$dr^2 + \sigma^2(r) g$$

where σ is a locally-Lipschitz continuous positive function on $[0, +\infty)$ such that $\sigma(0) = 0$ and $\sigma'(+0) = 0$. Thanks to these two conditions, the manifold M is geodesically complete, and so it satisfies the assumption for the state space as explained in Introduction. Let $dm = \omega_n \sigma^n(r)\, dr$ be a measure on M, where ω_n is the volume of $\mathbb{S}^n$.

For any $u \in C_0^{\mathrm{Lip}}(M)$, define

$$\mathcal{E}[u] = \mathcal{E}^{(c)}[u] + \mathcal{E}^{(j)}[u],$$

where

$$\mathcal{E}^{(c)}[u] = \int_M |\nabla u|^2 \, dm,$$

$$\mathcal{E}^{(j)}[u] := \iint_{M \times M \setminus \mathrm{diag}} \big(u(x) - u(y)\big)^2 \, j(x, y) \, m(dy) \, m(dx)$$

and

$$j(x, y) = \left[\frac{\mathbb{1}_{\{d(x,y)<1\}}}{\sigma(r(x))\sigma(r(y))} \right]^n.$$

Let $\mathcal{F}$ be the closure of $C_0^{\mathrm{Lip}}(M)$ with respect to the $\sqrt{\mathcal{E}_1}$-norm. It is easy to check that the symmetric form $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form in $L^2(M, m)$.

By [9], it is known that (M-2) is satisfied. On the other hand, since

$$\sup_{x,y \in M} j(x, y) \sigma^n\big(r(y)\big) \leqslant 1,$$

we obtain that

$$\begin{aligned} M_j &= \sup_{x \in M} \int_M \big(1 \wedge d(x, y)^2\big) \, j(x, dy) \\ &\leqslant \sup_{x \in M} \int_M d(x, y)^2 \, j(x, y) \, m(dy) \\ &\leqslant \sup_{x \in M} \int_{d(x,y)<1} d(x, y)^2 \omega_n \, dy \\ &\leqslant \omega_n. \end{aligned}$$

Therefore, (M-3) is also satisfied. Since (M-1) clearly follows, we can apply our main theorem. For example, if σ satisfies

$$\sigma(r) \asymp \big[r^r (1 + \ln r) \vee 1\big]^{1/n},$$

then for any fixed $x_0 \in M$,

$$r^{r/2} < V(x_0, r) < 2r^r \quad \text{for large } r > 0.$$

Therefore, $(\mathcal{E}, \mathcal{F})$ is conservative by Theorem 1.1. We note that this model manifold M does not satisfy the volume tests in [26,15].

5.4. A mixed-type Laplacian on graphs

A graph admits natural different "Laplacians"; namely, a physical Laplacian, a combinatorial Laplacian, and a quantum Laplacian. The former two are non-local operators, and the last one is a local operator. The combinatorial Laplacian is bounded, and so the corresponding process always is conservative. The conservativeness of the process associated with the physical Laplacian was studied in [6,7,39,40,15]. The conservativeness and recurrence of the process generated by the quantum Laplacian was studied in [35]. In the following example, we consider the sum of a physical Laplacian and a quantum Laplacian, and study its conservativeness.

Let $X = (V, E)$ be a locally finite graph, where V and E are the sets of vertices and edges, respectively. Let μ be a positive function on X, and $\omega : X \times X \to [0, \infty)$ be a symmetric non-negative function, such that $\omega(x, y) = 0$ whenever $x = y$ for $x, y \in X$ or at least one of x and y does not belong to V. Now, we recall the *standard adapted distance* d in [15]. For any $x, y \in X$, $x \sim y$ means that x, y are neighbors; that is, $(x, y) \in E$. For all $x, y \in V$ with $x \sim y$, define

$$\sigma(x, y) = \min\left\{\frac{1}{\sqrt{\deg(x)}}, \frac{1}{\sqrt{\deg(y)}}, 1\right\},$$

where

$$\deg(x) = \frac{1}{\mu(x)} \sum_{y:\ y\sim x} \omega(x, y).$$

It naturally induces a metric d on V as

$$d(x, y) = \inf\left\{\sum_{i=0}^{n-1} \sigma(x_i, x_i + 1):\ x_0, \dots, x_n \text{ is a chain connecting } x \text{ and } y\right\}.$$

The metric d can be extended to X by linear interpolation. We assume that the lengths of all edges $e \in E$ are uniformly bounded from below by a positive constant. This implies that (X, d) is a metrically complete space; in particular, our assumption on the space is satisfied.

We further assume that each edge $e \in E$ is isometric to an interval of $\mathbb{R}$, which yields the measure dx on e. The space (X, d) is a *metric graph*. Consider the following measure m on X:

$$m := \delta_E \phi\, dx + \delta_V \mu,$$

where ϕ is a continuous positive function on E.

For $u \in C_0^{\mathrm{Lip}}(X)$, define

$$\mathcal{E}[u] := \mathcal{E}^{(c)}[u] + \mathcal{E}^{(j)}[u],$$

where

$$\mathcal{E}^{(c)}[u] = \int_E \left(\frac{\partial u}{\partial x}\right)^2 dm,$$

and

$$\mathcal{E}^{(j)}[u] = \sum_{x,y \in V} \big(u(x) - u(y)\big)^2 \omega(x, y).$$

The generators associated with $\mathcal{E}^{(c)}$ and $\mathcal{E}^{(j)}$ are called the *quantum graph*, see, e.g. [25] and the physical Laplacian, respectively. Let $\mathcal{F}$ be the closure of $C_0^{\mathrm{Lip}}(X)$ with respect to the $\sqrt{\mathcal{E}_1}$-norm. We have

Lemma 5.7. *The form $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form.*

Proof. First, we claim that $C_0^{\mathrm{Lip}}(X)$ is dense in $L^2(X; m)$. Let x_0 be a fixed point in V. For any $u \in L^2(X; m)$ and any $\epsilon > 0$, choose $R > 0$ so large that there is a function $v_\epsilon \in C_0^\infty(B(R) \cap E)$ which satisfies

$$\|v_\epsilon - u|_E\|_{L^2(E;dx)} < \epsilon,$$

and that the function $w_\epsilon = \mathbb{1}_{B(R)} u$ satisfies that

$$\|w_\epsilon - u|_V\|_{L^2(V;\mu)} < \epsilon,$$

where $B(R) := B(x_0, R)$. Set $\tilde{u}_\epsilon = \delta_E v_\epsilon + \delta_V w_\epsilon$. For any $x \in B(R)$ and $e \in E$ with $x \sim e$ (i.e., $x \in e$), let $\delta = \delta(x, e)$ be a positive number such that $\delta < |e|/2$, and modify $\tilde{u}_\epsilon$ on $e \cap B(x, \delta)$ so that $\tilde{u}_\epsilon$ is linear and continuous on $e \cap B(x, \delta)$. Furthermore, since $B(R) \cap V$ is finite, by the Hopf–Rinow type property of locally finite graphs [21], we are able to do this modification for any $x \in B(R) \cap V$ and any $e \in E$ with $x \sim e$. Consequently, we obtain a sequence of functions $u_\epsilon^\delta \in C_0^{\mathrm{Lip}}(B(R))$ which converges to u in $L^2(X; m)$ as $\delta, \epsilon \to 0$. The required claim is proved.

Next, we verify that $(\mathcal{E}, C_0^{\mathrm{Lip}}(X))$ is closable. Let $(u_n)_{n \geqslant 1} \subset C_0^{\mathrm{Lip}}(X)$ be an $\mathcal{E}_1$-Cauchy sequence such that $u_n \to 0$ in $L^2(X; m)$ as $n \to \infty$. One can easily prove that $\mathcal{E}^{(c)}[u_n|_E] \to 0$ as $n \to \infty$, since $\mathcal{E}^{(c)}$ is equivalent to the Dirichlet integral of an open interval. Moreover, if $v \in C_0^{\mathrm{Lip}}(X)$, then

$$\mathcal{E}^{(j)}(u_n|_V, v|_V) = \sum_{x,y \in V} \big(u_n(x) - u_n(y)\big)\big(v(x) - v(y)\big)\omega(x, y) \to 0 \quad \text{as } n \to \infty.$$

Therefore, the desired claim follows and we denote the closure of $(\mathcal{E}, C_0^{\mathrm{Lip}}(X))$ by $(\mathcal{E}, \mathcal{F})$.

The Markov property of $(\mathcal{E}, \mathcal{F})$ follows immediately from the definition of $\mathcal{E}$. Finally, since $C_0 \cap \mathcal{F}$ is both dense in C_0 and $\mathcal{F}$ with respect to the sup-norm and the $\mathcal{E}_1$-norm, respectively, $(\mathcal{E}, \mathcal{F})$ is regular. □

It is easy to see that the conditions (M-1) and (M-2) are satisfied since $X^{(c)} = E$. Moreover, since $\mathcal{E}^{(j)}$ can be expressed as

$$\mathcal{E}^{(j)}[u] = \iint_{X \times X} \big(u(x) - u(y)\big)^2 \frac{\omega(x, y)}{\mu(x)\mu(y)} m(dy)\, m(dx),$$

the associated jump kernel j and Γ_j have the forms

$$j(x, dy) = \frac{\omega(x, y)}{\mu(x)\mu(y)} m(dy)$$

and

$$\Gamma_j[u](x) = \int_X \big(u(x) - u(y)\big)^2 \frac{\omega(x, y)}{\mu(x)\mu(y)} m(dy) \quad \text{for any } x \in X.$$

Clearly, (M-3) is satisfied. Therefore the Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfies the condition (M).

To state our main result in this subsection, we need some notations. Denote by ρ the graph distance extended to X, and by $B_\rho(x_0, R)$ the associated ball at $x_0 \in V$ with radius $R > 0$. For any $n \in \mathbb{N}$, let $S_\rho(x_0, n)$ be the "boundary" $B_\rho(x_0, n) \setminus B_\rho(x_0, n-1)$.

Proposition 5.8. *If μ is the counting measure and there are a point $x_0 \in V$ and a constant $C > 0$ such that*

$$m\big(S_\rho(x_0, n)\big) \leqslant Cn^2 \quad \textit{for all large enough } n \in \mathbb{N}, \tag{5.17}$$

then $(\mathcal{E}, \mathcal{F})$ is conservative.

Proof. The condition (5.17) implies that for any $x \in V$,

$$d(x_0, x) \geqslant \delta \log \rho(x_0, x), \tag{5.18}$$

where $\delta > 0$ is a constant depending only on C in (5.17) (see [15]). Let $\overline{xx'}$ be the edge with boundary $\{x, x'\}$. Let $y \in X$ and $x, x' \in V$ such that $y \in \overline{xx'}$. Without loss of generality, we assume that $\rho(x_0, y) \leqslant \rho(x_0, x')$. By using (5.18), the triangle inequality and the fact that $d(x, x') \leqslant \rho(x, x') = 1$, we find that

$$\rho(x_0, y) \leqslant e^{d(x_0, x')/\delta} \leqslant e^{1/\delta} e^{d(x_0, x)/\delta}.$$

Since $d(x_0, y) \geqslant d(x_0, x) \wedge d(x_0, x')$, we obtain that there is a constant $c > 0$ such that

$$\rho(x_0, y) \leqslant c e^{d(x_0, y)/\delta} \quad \text{for any } y \in X.$$

It follows that there exists a constant $b > 0$ such that

$$m\big(B_d(x_0, r)\big) \leqslant m\big(B_\rho(x_0, ce^{r/\delta})\big) \leqslant \exp(br) \quad \text{for all large enough } r > 0.$$

Therefore, $(\mathcal{E}, \mathcal{F})$ is conservative by Theorem 1. □

Remark 5.9. By an example of R. Wojciechowski [41], the boundary volume growth of quadratic rate (5.17) is sharp. The second part of Proposition 5.8 was obtained in [15] for a physical Laplacian on a graph.

On the other hand, it is easy to check that the condition (5.17) is satisfied, if there is a constant $C > 0$ such that

(1) $\mu(S_\rho(x_0, n)) \leqslant Cn^2$ for all large enough $n \in \mathbb{N}$,
(2) $\phi(x) \leqslant C\rho(x_0, x)^{-2}$ for every $x \in X$.

Indeed, the first condition implies that there are at most $(Cn^2)^2$-many edges in $S_\rho(x_0, n)$ connecting vertices in $S_\rho(n)$ and $S_\rho(n-1)$. The second condition then implies that there is a constant $c > 0$ such that

$$m\big(S_\rho(x_0, n) \cap E\big) \leqslant \frac{C^3 n^4}{(n-1)^2} \leqslant cn^2 \quad \text{for all large enough } n.$$

This together with the first condition yields (5.17).

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