Matrix Tensor Notation
Part I. Rectilinear Orthogonal Coordinates

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Abstract—A notation for vectors (first order tensors) and tensors (second order tensors) in physical three-dimensional space is proposed that satisfies a number of requirements which are missing in the customary vector, matrix and tensor notations. It is designed particularly to distinguish between vectors and tensors and their representation as vectors and matrices in different coordinate systems. This is achieved by the introduction of the base as a tensor quantity in the fundamental equation for the relation between a tensor and its representation as a matrix. The main purpose of the new notation is that it can be used in the teaching situation, therefore, it conveys all the information explicitly in the symbols, and it can be used in handwriting. All different numerical quantities have different symbols so that in numerical and symbolic computer applications the symbols can be copied directly to similar computer names which provides for well chosen names of variables in a computer program. It is also shown that the same notation is equally useful for vectors in abstract higher dimensional space and transformations and transforms in function space.

Part I discusses the notation for orthogonal coordinates, including Cartesian coordinates for which the notation is very simple.

Part II discusses how the same notation is equally efficient for skew and curved coordinates, and how it is integrated with tensor notation for higher than second order tensors.

1. INTRODUCTION

In mathematical language, equations are the sentences, symbols are the words and grammar are the rules by which symbols are connected. The way symbols are written is called notation. It seems that the importance of notation has not been realized enough. Deficiencies in notation convey incomplete information with all the consequences that are known in communication theory.

Present vector, matrix and tensor notation lack systematic and consistent symbolic distinction between different quantities, causing difficulties in communication especially in the teaching and computer programming situation.

Another aspect that seems to have been neglected is that in the same analysis of a larger problem we may encounter physical tensors and vectors, matrices and vector arrays, complex numbers, mean values, derivatives, transforms, etc., all in the same context. We need distinctive symbols for all of them, i.e., we cannot use the same symbol for a vector and a mean value, or for an index and the imaginary unit, or for a transform of a function and differentiation of the function, etc.

We have three different algebras available for tensors, viz., vector algebra, matrix algebra and tensor algebra. Vector Algebra treats physical vectors in the sense of Gibbs vectors symbolically by special symbols, and algebraically by the rules for components. No distinction is made between row and column vectors. But it doesn’t include second order tensors and that makes it unsuitable for treatment of all but the simplest problems of mechanics. Some introductory textbooks in mechanics and fluid mechanics make pathetic reading in their effort to avoid second order tensors.
On the other hand, transformations are done by matrix algebra, so that vectors in physical problems are actually treated with two different algebras.

Tensor Algebra includes tensors of all orders but does not have the advantage of a graphical representation of the operations involved as in matrix algebra, which may be the reason that it is rarely used for lower order tensors. Present tensor notation, however, has no systematic distinction between physical tensors and their representations in several different coordinate systems, such as may typically occur in rigid body dynamics. While it is true that conservative persons with experience in tensor algebra do not need another notation, it doesn’t mean that it cannot be improved to make the derivation, teaching and programming process more efficient, just like a computer shifts the emphasis of the subject and makes calculations more efficient even for persons familiar with the slide rule.

Matrix Algebra can treat components of vectors and second order tensors which is sufficient for a large area of applications, while many higher order tensor operations can also be reduced to matrix operations. Therefore matrix algebra is for many physical problems sufficient to execute the algebraic operations given by a tensor equation. With the proposed matrix tensor notation we can write tensor equations in familiar matrix algebra, with additions to conventional matrix notation that distinguish between the physical form and the representation of any tensor by scalar quantities in any coordinate system.

One cause of the difficulties that arise in the understanding of the meaning of tensor relations is that there are no established unique terms in the scientific language that distinguish between a physical tensor and the array of quantities that describe the tensor in a particular coordinate system, which in the case of a vector may be arranged as an array or a row vector or a column vector, and in the case of a second order tensor may be arranged in a matrix in two different ways, or stretched into one long vector in row or column form. One of the worst offenders in this respect is a widely known book [1] which is called Matrix-Tensor Methods... because it claims that a matrix and a tensor are the same entities, which is exactly the cause of confusion in understanding the tensor transformation. Some authors use the term components of a tensor [2], some matrix of components [3], and others the term tensor-matrix. The term dyadic is also used [4], but this is incorrect because a dyad and dyadic are defined differently [3,5,6]. In most mathematical texts, the physical tensor is not written at all, and the array of components is termed a tensor and used for the definition of a tensor in the coordinate transformation equation. Even [7], with their elaborate introduction of terms and symbols, don't make this distinction.

The term vector is used for a first order tensor and its array of components, and unfortunately in matrix algebra for a set of algebraic quantities that may have nothing to do with a first order tensor. Sometimes the first order tensor is called a physical vector and the array of components is called an algebraic vector to make a distinction. Even the term component is ambiguous, because, in engineering and physics, a component is still a vector while for the purpose of computations it is not. The terms vector components and scalar components are widely used in engineering to make a distinction [3,8,9].

The term scalar, as a noun, is reserved by some authors [1] for the zero order tensor which is invariant under coordinate transformation, while typically in engineering any quantity that is not a tensor is called scalar, which includes the components of an array representing a vector or a tensor—from there the term scalar components—which changes very much under coordinate transformation, although in such context it is used as an adjective.

It may well be that the lack of distinctive terms and corresponding instructive symbols in present tensor notation is the reason for the accepted fact that most engineers are comfortable with matrix, but not with tensor algebra [10]. With the proposed matrix tensor notation matrix algebra blends smoothly into tensor algebra.

In our matrix tensor notation, we don’t offer new terms but we do offer symbols that distinguish between all these meanings. In the meantime, we use compound terms and suggest that the terms scalar, component, vector, tensor and matrix are default expressions (deleting some parts) of the
full compound terms which may only be used when no confusion is possible, i.e., when they have been properly defined in the context and are not used for different entities. Similarly, we may use a default notation of less elaborate symbols in a mathematical description when there is no chance of confusion. This corresponds to elaborate full names of variables in a main computer program but shorter default names in a subroutine. The result is a detailed notation conveying all necessary tensor information, consisting of a letter symbol with subscripts and superscripts with an algebra that includes vector, matrix and tensor algebra. The algebra requires consistency and it is shown that the conventional meaning of the transposed symbol of vectors within matrix algebra and the meaning of the transformation matrix in tensor analysis must be changed. With the proposed notation equations of mechanics are easier to derive, write, read and understand, especially in Cartesian coordinates.

To explain our matrix tensor notation symbols, we will first present the compound terms we are going to use to distinguish the different entities that occur in analysis and computation with tensors.

A number is a computational quantity from algebra of real or complex numbers.

A physical scalar or zero order physical tensor is an ordinary physical entity which has no directional property. This is, of course, not a pure 'number' [1,3], but a number times a physical unit or units. It may become a so-called dimensionless number by methods of dimensional analysis. Therefore, a physical scalar or zero order tensor is a physical entity expressed by the same scalar quantity independent of any orientation of axes.

A scalar quantity, on the other hand, is any quantity that consists of a number times a physical unit, but may well be a part, element or component of a higher order tensor in particular orientation of axes, i.e., not necessarily a zero order tensor. Computation is done with the numbers only, their value depending on the physical units, but these are usually not distinguished separately and also called scalars. The default for all the above terms is scalar.

A physical vector or first order physical tensor is a physical entity which has direction. Although direction can only be defined with respect to a reference frame, the physical vector as a concept is independent of any particular axes orientation. The vector components of a physical vector are the physical vectors in the direction of particular chosen axes, that add up by the law of summation of vectors to the physical vector. The scalar components are the scalar quantities that describe the physical vector components as coefficients of the direction vectors [3,8,9]. Therefore, scalar component \( x \) direction vector = vector component. When the scalar components are written as an array, this array is the vector array. In vector algebra, a vector can be written as a sum of components but not as an array, but in matrix algebra it is, and then it may be written as column vector array or as row vector array. On the other hand, a number of scalar quantities may be collected into an array which is unfortunately also called a vector in matrix algebra, and this we call an algebraic vector, or more specific algebraic column vector or algebraic row vector. The actual number array used for computations depends on the physical units used, but is also called vector array and algebraic vector, respectively. This is in accordance with the usual practice for scalars. Therefore, a physical vector is expressed in any particular system of direction vectors by an algebraic vector which is called the vector array. The default for all the above terms is vector.

A physical tensor or second order physical tensor is a physical entity which is defined by the property that it is a linear vector function that operates on a physical vector to produce another physical vector [3]. Although such an explanation is difficult to demonstrate without particular axes, the mathematical definition by the law of transformation has no reason of existence without such a physical definition, and should be seen as a consequence only, which is an important aspect in the teaching situation. The use of the mathematical definition of a tensor [1] confuses the distinction between the physical tensor and its tensor matrix. Furthermore, the mathematical definition confuses the fact that transformation is dependent on direction vectors even if no coordinates are used at all, which is often the case in rigid body dynamics. The concept of a tensor is independent of any orientation of axes. In a particular system of direction vectors, a
physical tensor is expressed by nine scalar quantities. In tensor notation there is no rule how these are arranged. We call them tensor array. They can be arranged as a matrix, which is then called a tensor matrix. (This is not possible for higher order tensors.) The actual number matrix used for computations depends on the physical units used, but is also called tensor matrix and algebraic matrix, respectively. The default for all the above terms is tensor.

If a number of scalar quantities are coupled by a system of linear equations, we call the rectangular arrangement of coefficients an algebraic matrix. Default: matrix. We strongly recommend that the term matrix should not be used as a default for tensor.

Therefore, a physical tensor is expressed in any system of direction vectors by an algebraic matrix which is called the tensor matrix. Another possible representation of a tensor in a system of direction vectors follows from our tensor matrix notation and is explained in the text.

The need to distinguish between a tensor and its representation by an array of scalar quantities would not be so important if it weren't for transformations. On the other hand these transformations help to understand the tensor concept, and without them the computation of tensor equations cannot be implemented, which is just as an important aspect because in the end in engineering we need numbers.

Even ordinary English grammar must be used correctly if a problem must be described exactly, and we must distinguish between distinguish and differentiate, and between complicated and complex algebra.

We now discuss the corresponding mathematical symbols. For mathematical equations we need single symbols to denote all of the above defined different quantities. Yet a single character as used in the literature as a mathematical symbol is unpractical for many reasons, a fact that has been recognized with incredible foresight by the designers of the FORTRAN computer language [11]. In the last three decades, the use of computer programs has taught many scientists the use of precise symbolic distinction of different quantities. A good principle is to write a tensor/vector equation as a computer program would require it.

First of all, there are notations to distinguish between scalars, vectors and tensors, to reduce the effort to read, understand or check an equation. Second, there are notations to distinguish between physical tensors and their representation by tensor arrays. Third, there are notations that distinguish between tensor arrays in different coordinate axes.

There exist many different notations in the literature to distinguish matrices and vectors from scalars. One method is to use boldface, and a further distinction between matrices and vectors can be made by using capital and small letters as in the classic work of [5]. Many texts don't make any distinction and use ordinary letters, small and capital, for scalars, vectors and matrices which make difficult reading [1,12]. On the other hand, boldface is used by many authors and journals to distinguish physical vectors and tensors from their representation by algebraic vectors and matrices [3,8,13,14].

To distinguish representations of the same vector in different coordinate axes there are many different uses of subscripts and superscripts [15,16].

Classic tensor analysis distinguishes only between orders of tensors, and the tensor symbol represents the array of components. The distinction between different coordinate systems is usually done by a prime (x'), but this is inadequate because it cannot be used for several coordinate systems, and it is bad practice because the prime is the favourite symbol for so many other distinctions, as well as for the derivative. Schouten [17] proposes to use the same name, called kernel, and different indices for different coordinate axes, but this is in contradiction of accepted use of indices in tensor analysis, and his own equations show that it doesn't work. Borg [1] uses primes to distinguish scalar components of different coordinate axes, but uses the same symbol for the corresponding different vector arrays (equations (2-6) and (2-7)) which is unacceptable and confusing in a computer program and in the teaching situation.
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For handwriting many different symbols are used. For the physical vector mostly the arrow, either above or below the vector name, $\vec{v}$ or $\overline{v}$, but also the bar, $\bar{v}$ or $\overline{\bar{v}}$, the tilde, $\tilde{v}$ or $\overline{\tilde{v}}$, and the hat, $\hat{v}$, are used, where most of the time, it is not clear whether the physical tensor or the tensor array is meant. For matrices, these symbols are often doubled, also the square brackets are sometimes used, $[A]$.

Yet, the use of so many different notations points to the fact that none satisfies all the needs of a good notation. We propose a notation to satisfies all the needs that we are going to set out in the beginning.

The proposed notation is a matrix notation extended with familiar symbols from vector notation, but used as subscripts and superscripts similar to tensor notation, to distinguish between physical tensors and their different representations by vectors and matrices of scalar quantities in different systems of axes. From this, we deduce the name Matrix Tensor Notation. Therefore, it is limited to second order tensors which is, however, sufficient for vector analysis and most of the applications in mechanics. It also uses the distinction between column and row vectors. Although this may be a disadvantage, we employ it usefully in the new notation.

The fundamental equation is a relation between a physical vector, i.e., a first order tensor, and its representation in a system of axes, written as a matrix equation. This is similar to the dimensional equation of a scalar quantity in terms of physical units and a number, a subject which is treated in [18] and that has received great attention during the time of the conversion to the SI system of units, employing different symbols to distinguish between number, physical quantity and physical unit [19]. From the fundamental equation, follow all the rules of transformations between different coordinate axes by a consistent application of the same matrix algebra rules of the additional symbols. From this we deduce the name Matrix Tensor Algebra. Consistency also forces us to introduce some new and instructive concepts.

The proposed matrix tensor notation seems more complicated than either vector, matrix or tensor notation. However this complication is not an addition, it merely reveals the detail which is hidden in a tensor equation and often not defined by present methods, and often not realized. Therefore, the extra symbols allow much easier understanding of the equations, leaving the mind free to do more complicated problems. When no confusion can result, e.g., when only one Cartesian coordinate system is involved, the simpler defaults can be used. The notation becomes particularly simple and instructive in Cartesian coordinates, in which most equations of mechanics are written, including rotating coordinates. But it is also helpful to understand equations in skew coordinates, even equations derived by methods of Analytical Dynamics which are equations of mechanics projected on skew coordinates. We list here our requirements for a good notation for tensor quantities and operations:

1. must be easily written by hand;
2. distinguish between vector and scalar quantities;
3. distinguish between (second order) tensors and vectors;
4. distinguish between physical vectors and their representation by vector arrays, and between physical (second order) tensors and their representation by matrices;
5. distinguish between row and column vectors;
6. use the same symbol as name for the same vector or tensor in either its physical sense or its representation by a vector array or matrix in different coordinate axes;
7. distinguish between matrix/vector representations of the same vector/tensor in different coordinate axes;
8. must be equally valid in orthonormal and in skew coordinate axes;
9. must indicate all intended operations uniquely;
10. must be equally valid in all dimensions; (This means that the vector cross-product in three dimensions must be generalized to an equivalent operation for all dimensions.)
11. must be equally valid for algebraic vector/matrix algebra which has no connection to any metric space;
12. must be applicable to differentials;
13. defaults must be allowed, i.e., not all the symbols need to be written down explicitly if it is clear from the context, for the sake of simplicity to avoid repetitive elaborate symbols.

2. GENERAL RULES OF MATRIX AND ALGEBRAIC VECTOR NOTATION

Requirement 1 rules out boldface as vector/matrix symbol. We use the overbar \( \overline{\_} \) and underbar \( \underline{\_} \), making use of different combinations, which makes it easy to write by hand. In the following, we exploit this graphical placement of bars in various ways for further symbols.

We adopt the rule from [5] that vectors and elements of vectors are indicated by lowercase letters, and matrices are indicated by capital letters. (Some computer languages already distinguish between capital and lowercase letters.) Elements of matrices are sometimes written as lowercase and sometimes as capital letters, according to the way they have been derived.

The notation is developed by starting with skeleton symbols for matrix and vector quantities that are purely algebraic, i.e., have no connection to any coordinate axes, either physically or abstract. For these cases, we start by using subscripts to indicate elements. Consider the simultaneous linear equations

\[
\begin{align*}
\text{a}_{11} \text{x}_1 + \text{a}_{12} \text{x}_2 + \text{a}_{13} \text{x}_3 &= \text{r}_1, \\
\text{a}_{21} \text{x}_1 + \text{a}_{22} \text{x}_2 + \text{a}_{23} \text{x}_3 &= \text{r}_2, \\
\text{a}_{33} \text{x}_1 + \text{a}_{32} \text{x}_2 + \text{a}_{33} \text{x}_3 &= \text{r}_3.
\end{align*}
\]

The idea of a matrix is to separate the interlaced quantities into matrix \( A \), vector \( x \) and vector \( r \), where \( A \) is defined by the geometric rectangular array

\[
A = \begin{bmatrix}
\text{a}_{11} & \text{a}_{12} & \text{a}_{13} \\
\text{a}_{21} & \text{a}_{22} & \text{a}_{23} \\
\text{a}_{31} & \text{a}_{32} & \text{a}_{33}
\end{bmatrix}.
\]

The vectors \( x \) and \( r \) are written as arrays, however, their arrangement as rows or columns is actually arbitrary, as is shown by developing the general connection in three planes (see Figure 1) or on one plane in three different ways (see Figure 2).

The final choice, which is now adopted everywhere, is the one in Figure 2(b), which is the useful graphical scheme proposed by [5], which now determines the definition of matrix multiplication according to rows times columns for each scalar element in the result, which practically replaces the scalar product of vector algebra. In particular, the vectors \( x \) and \( r \) in this equation are considered column vectors, but matrix \( A \) consists of either a column of rows or a row of columns, both interpretations being equally valid and useful. These geometric properties of the arrangement of elements in matrix algebra therefore play an important and attractive role which we are exploiting in our matrix tensor notation.

So far, using only the names without matrix notation, the skeleton equation is, using the customary dot \( \cdot \) from vector algebra for scalar multiplication,

\[
A \cdot x = r,
\]

where \( A \) is the name of the matrix, \( x \) and \( r \) are the names of the vectors. The dot \( \cdot \) is not customary in matrix algebra, but we find it convenient to separate our more complicated symbols, a necessity that has also been followed by computer programming languages. The vector \( x \) can be enlarged to a matrix \( X \) with a corresponding right hand matrix \( R \) so that a matrix equation may read

\[
A \cdot X = R.
\]
We now adopt the point of view that row vectors are introduced naturally by letting matrix $A$ shrink to one row $a$, and matrix $R$ to one row $r$, as in

$$a \cdot X = r.$$ 

This consistent approach shows that a row vector is not necessarily derived from a column vector but may appear naturally in an application, for which we will give some convincing examples later. Our point of view is that if no distinction is made between a vector as a column and a single column matrix in matrix algebra then the same applies to a row. In fact, the row/column arrangement in matrix algebra is only then completely dual. Written as a numerical array, there is no difference between a matrix and a vector, yet symbolically matrix elements have two subscripts while vector elements have only one, a fact that could not be overlooked in typical symbolic computer programs, e.g., MACSYMA [20]. We propose in our notation to include vectors formally in matrix algebra, so that they be treated the same way as matrices but have a single subscript only. While this formal concept may seem practically trivial, it has an important consequence in our notation. We do not allow a scalar vector product other than a row · column, which means that the scalar dot product of vector algebra can only be performed between a row vector and a column vector, in that sequence. That means, generally, we cannot simply define a vector any more as in vector algebra, we must define it as either row or column vector. It turns out that this in itself is an instructive property.
2.1. Transpose

The accepted interpretation of the transpose of a matrix as a row/column interchange is an operation on a matrix but also a symbol on the result, i.e.,

\[ \text{Transpose of } A \equiv |A|^{\top} = A^{\top}. \]

Consistency requires that for both extremes of \( a \) being a row or a column vector

\[ \text{Transpose of column } a = \text{row } a^{\top} \quad \text{and} \quad \text{Transpose of row } b = \text{column } b^{\top}. \]

This consistent approach forces us to discard the customary notation of matrix/vector algebra that a row vector must be written as a transpose and vice versa, that a vector with a transpose symbol means a row vector. In our matrix tensor notation the operation transpose can be done on a row or a column vector and is not an indication of whether the vector is a row or a column. Just as an original matrix must be defined before it can be transposed, a vector must be defined as a row or column vector before it can be transposed. In any case, we call the given form of any matrix/vector, without the transposed symbol, the posed form to distinguish it from the transposed form.

2.2. Vector

We propose the bar — as symbol to indicate a vector, placed in two different positions that distinguish a column and a row vector, reflecting the transposed geometry of the elements,

(1) \[ \bar{a} = \text{column vector } \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \]

(2) \[ \bar{b} = \text{row vector } \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \]

where we use the conventional square brackets [ ] to indicate matrix elements. We may use curly brackets { } for the elements of a vector array to indicate the formal difference between a vector and a single row or column matrix

\[ \bar{a} = \text{column vector } \{ a_1 \} \{ a_2 \} \{ a_3 \} \]

\[ \bar{b} = \text{row vector } \{ b_1, b_2, b_3 \}. \]

On the other hand, because the transpose sign changes the column and row character of a vector, the transposed vector has also a transposed vector bar symbol

\[ \text{Transpose of } \bar{a} \equiv [\bar{a}]^{\top} = \bar{a}^{\top} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \]

\[ \text{Transpose of } \bar{b} \equiv [\bar{b}]^{\top} = \bar{b}^{\top} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} . \]

At first sight, it may seem unnecessary that the transpose symbol occurs with the transposed vector name, but it does indicate what the original posed form was, a distinction which proves to be as necessary as it is for matrices. It will be shown later that it is not required in Cartesian coordinates.
2.3. Matrix

In consequence of the dual row/column character of a matrix, its character is indicated by an underbar _ as well as an overbar __.

\[ \bar{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \]  

(3)

The fully notated algebraic matrix equations are now written as in the following self-explanatory examples:

\[ \bar{A} \cdot \bar{x} = \bar{r} \]

transposed: \[ x^T \cdot A^T = r^T \]

\[ y \cdot \bar{A} = \bar{p} \]

transposed: \[ A^T \cdot y^T = p^T \]

\[ \bar{A} \cdot \bar{S} = \bar{Q} \]

transposed: \[ S^T \cdot \bar{A}^T = Q^T \]

\[ y \cdot \bar{v} = s \] (scalar)

transposed: \[ v^T \cdot \bar{u} = s \]

2.4. Multiplication Rules

We have now the following rule for matrix multiplication:

name \((A)\) \cdot name \((x)\) = newname \((r)\),

(Rule-I)

unless the matrix is the identity matrix \( \bar{A} = \bar{I} \) when it plays the role of a unit quantity that doesn't change the name.

unit \((I)\) \cdot name \((x)\) = samename \((x)\).

(Rule-II)

Vector bar symbols cancel diagonally across the \(\cdot\) sign:

\[ \bar{A} \cdot \bar{x} = \bar{r} \]

(Rule-III)

so that there are rules for names as well as symbols, which are a combination of ordinary algebra rules for quotients and the non-commutative rule of matrix multiplication. We have supplemented the matrix algebra by a symbols algebra. In every matrix equation, the remaining vector bar symbols after cancellation must be equal on both sides, i.e.,

\[ \bar{A} \cdot \bar{X} \cdot \bar{v} = \bar{r} \]

\[ y \cdot \bar{A} \cdot \bar{B} \cdot \bar{u} = s. \]

Of course, the sequence in which the multiplications are carried out is arbitrary. All possible sequences of the last equation above are shown in Figure 3, which is an extended scheme of [5].
The cancellation of vector bar symbols can be interpreted as a check on a physical equation of dimensions or units on both sides of the equation, i.e., vector symbols are dimensions that must check on each side of a matrix equation. We use the modern left division by a matrix indicated by a backslash \ and right division by a matrix indicated by a slash /

\[
\overrightarrow{A} \backslash \overrightarrow{r} = \overrightarrow{x} \quad \text{check: } \overrightarrow{A} \backslash \overrightarrow{r} = \overrightarrow{x} \\
\overrightarrow{s} / \overrightarrow{A} = \overrightarrow{y} \quad \text{check: } \overrightarrow{s} / \overrightarrow{A} = \overrightarrow{y}.
\]

The cancellation of vector bar symbols is inverted by the division sign, which makes it more difficult to check but is a consistent consequence of our algebra of symbols. Note that the matrix division indicates a different matrix operation than the multiplication with the inverse.

If the bar symbols in a product of vectors are diagonally the other way round they cannot cancel and we consider the operation an outer multiplication producing a dyad

\[
\overrightarrow{a} \cdot \overrightarrow{b} = \overrightarrow{ab} = \text{new matrix } \overrightarrow{C},
\]

although we also use the different multiplication sign \( \circ \) to emphasize the outer product

\[
\overrightarrow{a} \circ \overrightarrow{b} = \overrightarrow{ab} = \overrightarrow{C} = \begin{bmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{bmatrix}, \tag{4}
\]

such that the column and row forms of both vectors are preserved. The sign \( \circ \) may be necessary in cases where we cannot put the vectors in their proper matrix algebra order, as in

\[
\overrightarrow{b} \circ \overrightarrow{a} = \overrightarrow{ba} = \overrightarrow{C}^\top = \begin{bmatrix} b_x a_x & b_x a_y & b_x a_z \\ b_y a_x & b_y a_y & b_y a_z \\ b_z a_x & b_z a_y & b_z a_z \end{bmatrix}, \tag{5}
\]
2.5. The Inverse

If the inverse of a matrix exists, we denote the new matrix by the customary notation

\[ [A]^{-1} = A^{-1}, \]

where the \(-1\) on the left hand side is an operation, but on the right hand side is part of the name of the new matrix.

2.6. Diagonal Matrices

To distinguish diagonal matrices we use a small letter indicating that it has only as many elements as a vector,

\[ \bar{b} = \begin{bmatrix} b_x & 0 & 0 \\ 0 & b_y & 0 \\ 0 & 0 & b_z \end{bmatrix}. \]

The customary diagonal line on the matrix symbol can always be added.

3. PHYSICAL VECTORS

Our notation is equally valid for any dimensional space, therefore we will use two-dimensional or three-dimensional examples as convenient to explain the notation for physical vectors in space. We propose, for the notation of a physical vector, the customary arrow symbol \( \rightarrow \) written above the vector name

physical vector \( \vec{v} = \vec{v} \).

The physical vector is not a column vector or a row vector, nevertheless writing the arrow above the vector name means that we regard the physical vector as written above as a column vector consistent with our notation. We accept this on the grounds that we are going to write physical tensor-vector equations the same as matrix equations, obeying matrix laws, simulating actual computations in matrix algebra, where we must distinguish between row and column vectors.

Quantitative computations with physical vectors are similar to quantitative computations with other physical quantities, where

physical quantity = physical unit \( \cdot \) numerical quantity,

which for physical vectors becomes

physical vector = physical unit vectors \( \cdot \) scalar physical quantities.

3.1. Orthogonal Coordinates

Starting with Cartesian coordinates, in Figure 4(a) are shown three orthogonal coordinate axes \( x, y, z \) and three orthogonal unit direction vectors \( \hat{i}, \hat{j}, \hat{k} \) with which any physical vector in 3-D space can be described by the customary expansion in components

\[ \vec{v} = \hat{i} v_x + \hat{j} v_y + \hat{k} v_z. \]  

(6)

The direction vectors have the physical dimension of direction only, an absolute value of numerical 1, while the ordinary physical dimension units are included in the components of the vector. When the three direction vectors are used to describe a physical vector, they are called base vectors.
4. BASE

The three base vectors form a physical (geometrical in this case) base (also called basis by some authors) for any vector in 3-D space, but we introduce a more specific mathematical definition of a base. To write equation (6) as a matrix equation, we define the row vector of base vectors as a new single quantity

\[ \overrightarrow{E} \equiv \begin{bmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \end{bmatrix} . \]  

We call this new quantity, which is neither a tensor nor a matrix, a base and write equation (6)

\[ \overrightarrow{v} = \overrightarrow{E} \cdot \overrightarrow{v} . \]  

Our new interpretation of base in matrix tensor notation is: a base is a single quantity, not merely the set of base vectors, just like a tensor is interpreted as a single quantity rather than a set of nine scalar quantities. The transpose of this quantity is also called triad [16], but it was never interpreted as a single entity, and used as a vector rather than the interpretation that we are using here.

Note also that in (8) no coordinates are used, only direction vectors. Direction vectors are quantities that need axes for their definition or vice versa, but their numeric direction value is not necessarily 1, generally. Therefore, for one single set of axes there may be different direction vectors and different bases. Even for orthogonal axes we may have non-unit direction vectors; impractical as this may be, it must be included in a complete notation. We define a Cartesian base as the base constructed from three orthonormal base vectors. Note that we don’t need coordinates as length variables along the axes to define direction vectors. In many applications, coordinates are not used.

As soon as different orientation of axes are used there are different sets of direction vectors, therefore different bases, which must be distinguished from each other by a name which may be a number or a letter. In Figure 4(a), a base with no name is shown, which can be used as a default if there is no other base from which it has to be distinguished. In Figure 4(b), a base with the name “1” is shown, and in Figure 4(c), a base with the name “2” (in anticipation of stationary). Another base may be called “r” (in anticipation of rotating). The possible alternatives of Figure 4 are that (b) uses a number for a name and x, y, z as counters and (c) uses a letter for a name and numbers as counters, which is closer to computational application. Names and counters in the two alternatives are interchanged because it is convenient to have the number as last character. Both methods have advantages and disadvantages that make them more suitable to particular applications.

In the following, we use the method of using a letter for the name of a basis. In each coordinate system the coordinates get the name subscripted by the coordinate counting number 1, 2, 3.

Figure 4. Coordinate axes and base vectors.

(a) 

(b) 

(c)
as in tensor notation, which can be continued for any number of coordinates. Using the symbol \( e \) for a direction vector, whether of unit magnitude or not, the three (or \( n \)) direction vectors are subscripted with the same coordinate name as shown in Figure 4(c). We also distinguish the base with the same name by adding the name to the base underbar, and define the base \( s \)

\[
\overrightarrow{E}_s \equiv \begin{bmatrix} e_{s1} & e_{s2} & e_{s3} \end{bmatrix}.
\]

**4.1. Orthonormal Base**

In orthonormal, or Cartesian, base all direction vectors have unit magnitude. The components of a vector measured in a base are subscripted the same way as the direction vectors and the base name added to the vector overbar, so that we define

\[
\overrightarrow{v}^s \equiv \begin{bmatrix} v_{s1} \\ v_{s2} \\ v_{s3} \end{bmatrix}.
\]

Therefore, the physical vector is expressed in base \( s \) by the equation

\[
\overrightarrow{v} = e_{s1} v_{s1} + e_{s2} v_{s2} + e_{s3} v_{s3}
\]

\[
= \begin{bmatrix} e_{s1} & e_{s2} & e_{s3} \end{bmatrix} \cdot \begin{bmatrix} v_{s1} \\ v_{s2} \\ v_{s3} \end{bmatrix},
\]

which is, by the previous definitions, the matrix equation

\[
\overrightarrow{v} = \overrightarrow{E}_s \cdot \overrightarrow{v}^s,
\]

which we call the *Fundamental Equation* of the Matrix Tensor Notation because, from now on, every further rule follows from this definition. Equation (11) follows the Rule III of cancellation of equal symbols diagonally across the dot \( \cdot \), including now the base name

\[
\overrightarrow{v} = \overrightarrow{E} \cdot \overrightarrow{v},
\]

(Rule-IV)

and the Rule II of names

\[
\text{name } v = \text{Unit } E \times \text{name } v,
\]

realizing that \( E \) is acting as a unit of some sort, viz., with respect to the vector name. We call the notation \( \overrightarrow{v}^s \) *base notation* of a vector or tensor. If quantities in multiplication follow Rule III, we call them *conformable* from the similar meaning in matrix multiplication. If there is only one single set of coordinate axes used within a scope of a problem, we need not use a base name, i.e., we leave the base name out as a default option, which we also call *baseless notation*. Equation (8) is the default of (11) if there is no need to distinguish the base \( s \) from any other. Writing the tensors with the arrow \( \overrightarrow{\cdot} \) we call physical space notation or just *space notation*, noting the difference between base and space, also physical notation. The arrowhead can actually be taken as a special base symbol.

**4.2. The Inverse Base**

We have chosen the arrow above \( \overrightarrow{\cdot} \) for the physical vector symbol in the same position as the overbar \( \overrightarrow{\cdot} \) for an algebraic column vector, of which there also exists a transposed form which
is the underbar _ for an algebraic row vector. For the bars we have a symbolic algebra, and
the bars and the arrow appear in the same equation (4). Therefore, to make the matrix tensor
notation complete, we introduce the arrow below _ for the same physical vector, but regarding
it in this form as a row vector

physical vector \( \vec{v} = v \),

yet it is a mathematical quantity which is not the same as \( \vec{v} \), but rather

\[
\left[ \begin{array}{c}
\vec{v}
\end{array} \right]^\top = \vec{v} \quad \text{and} \quad \left[ \begin{array}{c}
\vec{v}
\end{array} \right]^\top = \vec{v}.
\]

Note that we have two symbols for the one and the same physical vector apparently as a result of
the dual nature of matrix algebra. It is not possible in our notation to write the physical vector
in a sort of neutral form once we have committed ourselves to include physical vectors in matrix
algebra of tensors. With this admittedly very artificial concept, we can combine physical vectors
and tensors and their algebraic representations into one single matrix algebra by letting these
physical vector symbols obey the same algebra as the algebraic ones, viz., they cancel diagonally
across the \( \cdot \) sign

\[
\vec{v} \cdot \vec{v} = v^2, \\
\vec{v} \cdot \vec{v} = uv \cos \theta.
\]

We may think of this operation being possible without coordinate axes by knowing the absolute
value of the vectors and the angle \( \theta \) between them. With this concept we can invert Equation (11)

\[
\vec{v}^s = \vec{E}_s \cdot \vec{v},
\]

where

\[
\vec{E}_s = \left[ \frac{e_1}{e_2} \right]^{-1}
\]

and the inverse is defined by the equation

\[
\vec{E}_s \cdot \vec{E}_s = \vec{l},
\]

where \( \vec{l} \) is the numerical unit matrix (Identity matrix). We use the row form of the unit direction
vectors to define this inverse as a column of direction vectors

\[
\vec{E}_s = \left[ \begin{array}{c} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{array} \right]
\]

where the row direction vectors of orthonormal base are physically the same as the column
direction vectors, but mathematically defined as transposed

\[
\vec{e}_1 = \left[ \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} \right]^\top, \quad \vec{e}_2 = \left[ \begin{array}{c} e_2 \\ e_3 \end{array} \right]^\top, \quad \vec{e}_3 = \left[ \begin{array}{c} e_3 \end{array} \right]^\top.
\]

The quantity \( \vec{E}_s \) satisfies equation (15) as shown in the layout of this multiplication by Falk's
scheme below.
Note how in (14) \([E]^{-1}\) also obeys the rule of a unit, \([\text{Unit}]^{-1} = \text{Unit}\). Therefore, the \(-1\) symbol as part of the name \(E\) is superfluous, \(E^{-1} = E\). Also, according to the same rules

\[
\vec{E}_s \cdot \vec{E}_s = \vec{E}_s',
\]

and therefore,

\[
\vec{E}_s'^{s} = \vec{I}.
\]

We also find from our definition of the transpose of a physical vector that the quantity in (16) is the transpose of the base. Therefore, we obtain the important result for orthonormal base

\[
\left[\vec{E}_s\right]^{-1} \equiv \vec{E}_s' = \left[\vec{E}_s\right]^T.
\]

We can interpret the mathematical operation in (11) as the physical operation of composing the vector \(\vec{v}\) from the components \(\vec{v}^s\), and equation (13) as extracting the components \(\vec{v}^s\) from the vector \(\vec{v}\). Applying the same principle to the base vectors, we find that equation (17) means extracting the components from the base vectors and therefore that

\[
\vec{E}_s = \begin{bmatrix} -s_{e1} & -s_{e2} & -s_{e3} \\ e_{s1} & e_{s2} & e_{s3} \end{bmatrix},
\]

which are the base vectors expressed in their own base, resulting in the familiar unit matrix, but demonstrating how this result is embedded in our notation.

We can also extract the components in base \(s\) of a vector by an operation using the row form of the physical vector

\[
\vec{v} \cdot \vec{e}_{s1} = v_{s1}, \quad \vec{v} \cdot \vec{e}_{s2} = v_{s2}, \quad \vec{v} \cdot \vec{e}_{s3} = v_{s3},
\]

which is the same as

\[
\vec{v} \cdot \vec{E}_s = \vec{v}',
\]

where

\[
\vec{v}' = \begin{bmatrix} v_{s1} & v_{s2} & v_{s3} \end{bmatrix}.
\]

Therefore, we find that \(\vec{v}'\) means the row vector of the same components as \(\vec{v}^s\). This means that in orthonormal base we have the important and convenient relation

\[
\left[\vec{v}^s\right]^T \equiv \vec{v}'^s = \vec{v}' \quad \text{and} \quad \left[\vec{v}_s\right]^T \equiv \vec{v}'^s = \vec{v}'.
\]
equations (12) and (22) shows that we consider physical space as orthonormal. We will find later that defining the same physical vector as two vectors \( \vec{v} \) and \( \vec{v} \) is a redundancy due to the orthonormal space. Another special property of the orthonormal base is that after transposing we don’t need to attach the transposed sign to it. It is superfluous because of the orthonormal property of base \( s \).

\[
\left[ \vec{E}_s \right]^T = \vec{E}_s^T = \vec{E}_s^s \quad \text{and} \quad \left[ \vec{E}_s^* \right]^T = \vec{E}_s^{*T} = \vec{E}_s^s.
\]

Inverting equation (21), we obtain

\[
\vec{v} = v_s \cdot \vec{E}_s^s.
\]

All relations between components and physical vectors can be summarized in the single equation

\[
\vec{v} \cdot \vec{v} = v_s \cdot \vec{E}_s^s \cdot \vec{v}^s = v_s \cdot \vec{v}^s = v^2.
\]

One important feature of our notation is that for orthonormal bases and vectors in orthonormal base, we never need the transpose sign.

At this stage, we have shown that in our notation the four equations (11), (13), (21) and (24) are transformation equations of vectors between physical space and base \( s \).

### 5. TENSORS

Let there be a tensor (second order) by the name of \( \mathbf{K} \) that operates on a vector \( \vec{u} \) to produce a vector \( \vec{f} \). In base \( s \), the corresponding equation would be a matrix equation, writing the component vectors as column vectors. We want to simulate this equation in space notation

\[
\overrightarrow{\mathbf{K}} \cdot \vec{u} = \vec{f},
\]

introducing the tensor symbol \( \overrightarrow{\mathbf{K}} \) analog to the matrix notation. Transforming both vectors from base \( s \),

\[
\overrightarrow{\mathbf{K}} \cdot \vec{E}_s^s \cdot \vec{u}^s = \vec{E}_s^s \cdot \vec{f}^s.
\]

Since both sides of the equation above are vectors, we may premultiply with the inverse of \( \vec{E}_s^s \), with the result

\[
\vec{E}_s^s \cdot \overrightarrow{\mathbf{K}} \cdot \vec{E}_s^s \cdot \vec{u}^s = \vec{f}^s.
\]

However, by our notation rules (II), (III) and (IV), we must write the left-hand quantity

\[
\vec{E}_s^s \cdot \overrightarrow{\mathbf{K}} \cdot \vec{E}_s^s = \mathbf{K}_s^s
\]

to obtain the matrix equation

\[
\overrightarrow{\mathbf{K}}_s \cdot \vec{u}^s = \vec{f}^s.
\]

Equation (28) is the transformation law for tensors between physical space and base \( s \), written in matrix tensor notation, and equation (29) shows the physical tensor equation (26) transformed in consistent notation as a matrix equation in base \( s \). We express this in the words:
$\overrightarrow{K}_s^s$ is the tensor matrix that represents the physical tensor $\overrightarrow{K}$ in base $s$, or simply, $\overrightarrow{K}_s^s$ is the tensor $K$ in base $s$. Because of the fundamental equation we have a mathematical equation relating the physical tensor and its representation by a matrix by using the base as a well-defined single quantity. Another observation is that there should be nothing to prevent us from leaving the transformation incomplete as in (27), which, written in consistent notation, becomes

$$\overrightarrow{K}_s^s \cdot \overrightarrow{u}^s = \overrightarrow{f}.$$

In this equation the tensor $K$ is in a mixed form, partly in physical space, partly in base $s$. Therefore, we may have all of the following forms of the same tensor $K$

$$\overrightarrow{K}_s^s, \overrightarrow{K}_s^s, \overrightarrow{K}_s^s, \overrightarrow{K}_s^s,$$

each of which can be transformed into any other by the base. These are new forms not encountered in tensor analysis. They are necessary concepts in our matrix tensor notation to make the matrix tensor algebra complete. The transpose of (26) in space notation is

$$\underline{u} \cdot \overrightarrow{K}^s = \overrightarrow{f} \quad \text{if the tensor is symmetric,}$$

$$\underline{u} \cdot \overrightarrow{K}^s = \overrightarrow{f} \quad \text{if the tensor is not symmetric.}$$

The transformation law for the transposed tensor is the same as that for the posed tensor

$$\overrightarrow{E}_s^s, \overrightarrow{E}_s^s, \overrightarrow{E}_s^s = \overrightarrow{K}_s^s.$$

The transformations of all space, base or mixed space base forms of tensors and vectors follow from the only possible conformable way that multiplication with the base is possible. We apply now the same transformation algebra to the base

$$\overrightarrow{E}_s^s, \overrightarrow{E}_s^s, \overrightarrow{E}_s^s = \overrightarrow{E}_s^s,$$

and to the transformation equation (11)

$$\overrightarrow{E}_s^s \cdot \overrightarrow{v} = \overrightarrow{E}_s^s \cdot \overrightarrow{v}^s$$

$$= \overrightarrow{E}_s^s \cdot \overrightarrow{v}^s$$

$$= \overrightarrow{v}.$$

This equation shows that $\overrightarrow{E}_s^s$ is the Unit space tensor and the base is merely a mixed form of the unit tensor, i.e., the base is actually the space tensor, which is truly appropriate because we use it to describe space in a particular biased form. This space tensor is the Unit Quantity in our matrix-tensor notation that is required in every algebra. The tensor transformation equations can be regarded as mixed forms of

$$\overrightarrow{E}_s^s \cdot \overrightarrow{v} = \overrightarrow{v}$$

$$\underline{v} \cdot \overrightarrow{E} = \overrightarrow{v}$$

$$\overrightarrow{E}_s^s \cdot \overrightarrow{E}_s^s = \overrightarrow{E}_s^s,$$

emphasizing that the physical tensor quantity in all these equations remains the same but the base may change. This space tensor is also called the Fundamental Tensor, denoted $G$ [14], where
the letter $G$ is used with the metric $g_{ij}$ of tensor analysis in mind. This aspect will be discussed in Part II.

Theoretically, it is satisfactory to be able to write vector and tensor equations in space form, i.e., without reference to a particular base, and to bring two customary notations into one algebra. Computationally, it is just a nicety, as we can always choose one orthonormal base to write the equations, even without specifying the name, and use the default notation of bars without name because the transformations show that in all vector/tensor equations the physical space notation can be replaced by base notation of a fixed orthonormal base. Space notation emphasizes the physical and base notation emphasizes the computational aspect. Computationally important, however, are transformations between different bases.

6. TRANSFORMATIONS

Consider two (different) orthonormal bases $s$ and $r$ as shown in 2-D space in Figure 5, with notation valid for any dimension. Each base is individually defined, base $s$ as before by (9), and base $r$ similarly by

$$E_r = \left[ \frac{\alpha_1}{e_{r1}} \frac{\alpha_2}{e_{r2}} \frac{\alpha_3}{e_{r3}} \right].$$

The direction vectors of base $r$ are described in base $s$ by the column vectors

$$E_r = \left[ \frac{\alpha_1}{e_{r1}} \frac{\alpha_2}{e_{r2}} \frac{\alpha_3}{e_{r3}} \right],$$

which are obtained formally from the physical vectors by applying equation (13) to each one,

$$= \left[ E_s \cdot e_{r1} \quad E_s \cdot e_{r2} \quad E_s \cdot e_{r3} \right]$$

$$= \left[ \frac{\alpha_1}{e_{r1}} \frac{\alpha_2}{e_{r2}} \frac{\alpha_3}{e_{r3}} \right]$$

$$= \frac{\alpha_1}{e_{r1}} \quad \frac{\alpha_2}{e_{r2}} \quad \frac{\alpha_3}{e_{r3}}$$

$$= \frac{\alpha_1}{e_{r1}} \quad \frac{\alpha_2}{e_{r2}} \quad \frac{\alpha_3}{e_{r3}}$$

$$= \frac{\alpha_1}{e_{r1}} \quad \frac{\alpha_2}{e_{r2}} \quad \frac{\alpha_3}{e_{r3}}$$

$$= \frac{\alpha_1}{e_{r1}} \quad \frac{\alpha_2}{e_{r2}} \quad \frac{\alpha_3}{e_{r3}}$$

according to our notation rules. The new symbol has been produced

$$E_r^s = \left[ \frac{\alpha_1}{e_{r1}} \frac{\alpha_2}{e_{r2}} \frac{\alpha_3}{e_{r3}} \right].$$
which is the matrix of direction cosines. The quantity $\overrightarrow{E}_r^s$ is the collection of column direction vectors of base $r$ measured in base $s$. Similarly, we can express the direction vectors of base $s$ in base $r$ by the operation

$$\overrightarrow{E}_r^s \cdot \begin{bmatrix} e_{s1} & e_{s2} & e_{s3} \end{bmatrix} = \overrightarrow{E}_r^s \cdot \overrightarrow{E}_s^r \equiv \overrightarrow{E}_r^r,$$

where now

$$\overrightarrow{E}_s^r = \begin{bmatrix} \overrightarrow{e}_r^s & \overrightarrow{e}_r^s & \overrightarrow{e}_r^s \end{bmatrix} \equiv \begin{bmatrix} e_{rs1} & e_{rs2} & e_{rs3} \\ e_{rs2} & e_{rs2} & e_{rs3} \\
 e_{rs3} & e_{rs3} & e_{rs3} \end{bmatrix} = \begin{bmatrix} \cos \alpha_{11} & \cos \alpha_{12} & \cos \alpha_{13} \\ \cos \alpha_{21} & \cos \alpha_{22} & \cos \alpha_{23} \\ \cos \alpha_{31} & \cos \alpha_{32} & \cos \alpha_{33} \end{bmatrix}

= \left[ \overrightarrow{E}_r^s \right]^T = \left[ \overrightarrow{E}_r^s \right]^{-1}.

The quantity $\overrightarrow{E}_s^r$ is the collection of column direction vectors of base $s$ measured in base $r$. Alternatively, we could have used the same algebra in transposed form to find that $\overrightarrow{E}_s^r$ is the collection of row direction vectors of base $s$ measured in base $r$ and $\overrightarrow{E}_s^r$ is the collection of row direction vectors of base $r$ measured in base $s$. All the possible combinations of subscripts have their particular meaning, but because of the orthonormality of the bases all $e_{r_i s_j} = e_{s_j r_i}$, $i, j = 1, 2, 3$, so that we have a slight overnotation in the orthonormal base; this is a necessary consequence of the matrix tensor notation being complete. In a computer application we need to store one matrix only, but if we want to store the transposed form then the matrix tensor notation provides the proper name. For the corresponding variable names in a computer program, we use the convention from matrix algebra that the first index indicates the row and the second the column, which correspond to the upper and lower subscripts of the tensor in tensor matrix notation, respectively. Therefore, the computer name for $\overrightarrow{E}_s^r$ is $E_\sigma r$, and for $\overrightarrow{E}_s^r$ is $E_\sigma s$, and the element $e_{r1s2} = E_{rs}(1, 2)$. All of the following are the different forms that the same space tensor $\overrightarrow{E}_s$ may assume in two different orthonormal bases $s$ and $r$

$$\overrightarrow{E}_s, \overrightarrow{E}_s, \overrightarrow{E}_s, \overrightarrow{E}_s, \overrightarrow{E}_s, \overrightarrow{E}_s, \overrightarrow{E}_s, \overrightarrow{E}_s, \overrightarrow{E}_s, \overrightarrow{E}_r, \overrightarrow{E}_r, \overrightarrow{E}_r, \overrightarrow{E}_r, \overrightarrow{E}_r.$$

The relations between them are completely determined by the algebraic rules of our matrix tensor notation. All the relations derived for base $s$ are equally valid in base $r$, particularly equation (18)

$$\overrightarrow{E}_r^r = I.$$

Note also that the derived equations are equally valid for a stationary or rotating base.
6.1. Transformation of Vectors

The same vector \( \mathbf{v} \) can be described in two different bases by the fundamental equation with components as shown in Figure 5(b), in base \( s \) by (11) and in base \( r \) by

\[
\mathbf{v} = \mathbf{E}_r \cdot \mathbf{v}^r,
\]

where

\[
\mathbf{v} = \begin{bmatrix} v_{1r} \\ v_{2r} \\ v_{3r} \end{bmatrix},
\]

and therefore,

\[
\mathbf{E}_s \cdot \mathbf{v}^s = \mathbf{E}_r \cdot \mathbf{v}^r.
\]

Premultiplying by \( \mathbf{E}_s \) gives

\[
\mathbf{v}^s = \mathbf{E}_r \cdot \mathbf{E}_s \cdot \mathbf{v}^r
\]

from which we recognize that \( \mathbf{E}_r \) is the appropriate transformation matrix. By a similar derivation

\[
\mathbf{v}^r = \mathbf{E}_s \cdot \mathbf{v}^s.
\]

The components of \( \mathbf{v}_s \) and \( \mathbf{v}_r \) are the same as the components of \( \mathbf{v}^s \) and \( \mathbf{v}^r \), respectively, in orthonormal base. Therefore, by transposing equations (33) and (34), their transformations are

\[
\mathbf{v}_s = \mathbf{v}_r \cdot \mathbf{E}_r^s
\]

\[
\mathbf{v}_r = \mathbf{v}_s \cdot \mathbf{E}_s^r.
\]

All the vector transformations (33)–(36) are completely defined by the algebra of the notation, and the vectors and their base can be recognized in this notation.

6.2. Transformation of Tensors

Using the definition of a (second order) tensor as a linear vector operator, we transform the vectors in (29) from base \( s \) to base \( r \) according to (33)

\[
\mathbf{K}_s \cdot \mathbf{E}_s \cdot \mathbf{u}^r = \mathbf{E}_r \cdot \mathbf{f}^r.
\]

Premultiplying by \( \mathbf{E}_s \) gives

\[
\mathbf{E}_s \cdot \mathbf{K}_s \cdot \mathbf{E}_r \cdot \mathbf{u}^r = \mathbf{f}^r
\]

We find that the tensor transformation as derived from the vector transformation is

\[
\mathbf{K}_r = \mathbf{E}_s \cdot \mathbf{K}_s \cdot \mathbf{E}_r.
\]
The transformation of the inverses follows from inverting the two transformation equations (37) and (38), i.e.,

\[
\left[ \mathbf{K}_r^s \right]^{-1} = \frac{1}{\mathbf{E}_r^s \cdot \mathbf{K}_r^s \cdot \mathbf{E}_r^s}
\]

\[
\begin{align*}
\left[ \mathbf{K}_r^s \right]^{-1} &= \left[ \mathbf{E}_r^s \right]^{-1} \cdot \left[ \mathbf{K}_r^s \right]^{-1} \cdot \left[ \mathbf{E}_r^s \right]^{-1} \\
&= \mathbf{K}_r^s \cdot \mathbf{K}_r^s \cdot \mathbf{E}_r^s \\
&= \mathbf{K}_r^s \cdot \mathbf{K}_r^s \cdot \mathbf{E}_r^s \\
&= \mathbf{K}_r^s \cdot \mathbf{K}_r^s \cdot \mathbf{E}_r^s
\end{align*}
\]

(39)

and

\[
\mathbf{K}_r^s = \mathbf{E}_r^s \cdot \mathbf{K}_r^s \cdot \mathbf{E}_r^s.
\]

(40)

The inverse of a tensor must get the $-1$ symbol as part of its name or else get a new name, which may often be practical in view of a new physical meaning. All tensor transformations are defined by the algebra of the notation. Their names are the same in any base, and the bases are recognizable. We can see the limitation to second order tensors: there are only two sides of a matrix accessible to multiplication by the transformation matrix.

6.3. Mixed Bases

Tensor representation in mixed bases are transformations from the space notation by simply doing multiplications with different bases, which is a perfectly legitimate operation in tensor matrix algebra if the quantities are conformable, i.e.,

\[
\mathbf{E}_r^s \cdot \mathbf{K}_r^s \cdot \mathbf{E}_r^s = \mathbf{K}_r^s,
\]

(41)

which can also be obtained by multiplication of the tensor in one base with only one transformation matrix, i.e.,

\[
\mathbf{E}_r^s \cdot \mathbf{K}_r^s = \mathbf{K}_r^s.
\]

(42)

We call this a half transformation. The result is again a recognizable form of the tensor, i.e., it is merely another state of transformation of the same physical tensor. But specifically applying this result to the unit space tensor means that the base $\mathbf{E}_s$ is itself a mixed space base form of the unit space tensor, and we identify the transformation matrix $\mathbf{E}_r^s$ as a mixed base representation of the unit space tensor. Therefore, the transformation matrix that identifies a tensor of any order in tensor algebra is itself a second order tensor, contrary to the customary view in tensor calculus.

Actually, we must qualify the reference to tensor calculus at this stage. There the transformation matrix is derived from a coordinate transformation, while here we consider direction vectors independent of coordinates. In Cartesian coordinates this doesn’t make a difference, but in other coordinates, customary tensor algebra has no symbol for the tensor matrix expressed in unit direction vectors.

The notation for the tensor elements in the most general form is accordingly

\[
\mathbf{K}_r^s = \begin{bmatrix} K_{r1s1} & K_{r1s2} & K_{r1s3} \\ K_{r2s1} & K_{r2s2} & K_{r2s3} \\ K_{r3s1} & K_{r3s2} & K_{r3s3} \end{bmatrix}.
\]
The result of our matrix tensor notation is that multiplications can be made in any order that is conformable, i.e., vector symbols diagonally across the \( \cdot \) sign must be of equal bases to cancel, and all others must remain and must be equal on both sides of an equation, like any other physical dimensions. The names of the physical quantities remain independent of their representation in any base, which is part of the total symbol. The following are possible examples

\[
\begin{align*}
\vec{E}_r \cdot \frac{v}{r} &= \vec{v} \\
\vec{K}_s \cdot \frac{v}{s} &= \vec{f} \\
\gamma \cdot \vec{E}_r \cdot \vec{K}_s \cdot \frac{v}{r} &= \vec{w} \\
\gamma \cdot \vec{K}_s \cdot \frac{v}{s} &= \vec{u}^2 \\
\vec{A}_r \cdot \vec{E}_s \cdot \vec{C}_s &= \vec{D}_r
\end{align*}
\]

The extension to any number of bases is obvious, although the choice of names for many different bases must be made carefully.

Computations in mixed bases can be represented geometrically in the same extended Falk's scheme with the conformable base names indicated as dimensions as shown in Figure 6. Different dimensions means simply that same bases may span subspaces embedded in one larger dimensional space.

Finally, we stress that the algebra that has emerged from matrix tensor notation in orthogonal base is dependent on the property of the base that the inverse is equal to the transpose. We have actually a number of superfluous laws inasmuch as every equation can be written in posed and transposed form. For orthonormal base we always prefer the column vector as the posed form before the row vector.
6.4. Alternative Base Names

Using numbers rather than symbols for base names makes it easier to accommodate a larger number of different bases and is easy to read because the \( \vec{i}, \vec{j}, \vec{k} \) axes and direction vectors \( \vec{i}, \vec{j}, \vec{k} \) are familiar, but it is strictly limited to real three-dimensional Cartesian coordinates. Corresponding symbols are, e.g., for bases 4 and 5

\[
\vec{E}_4 = \begin{bmatrix}
\vec{i}_4 \\
\vec{j}_4 \\
\vec{k}_4
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_4 \\
y_4 \\
z_4
\end{bmatrix}, \quad
\begin{bmatrix}
v_4
\end{bmatrix}
\]

\[
\begin{bmatrix}
K_{x4y5} \\
K_{x4y5} \\
K_{x4y5}
\end{bmatrix}
\]

The notation of elements looks unfamiliar because the numbers are not counters while the letters are. A typical equation would be

\[
\vec{w} = \vec{E}_1 \cdot \vec{w}_1 + \vec{E}_2 \cdot \vec{w}_2 + \vec{E}_3 \cdot \vec{w}_3 + \vec{E}_4 \cdot \vec{w}_4 + \vec{E}_5 \cdot \vec{w}_5,
\]

where \( \vec{w}_1, \ldots, \vec{w}_5 \) are names of vectors. Another possibility is to combine the familiar axes notations with a letter name of a base—the base may then, of course, not have the name \( x, y \) or \( z \), e.g., for bases \( s \) and \( r \)

\[
\vec{E}_s = \begin{bmatrix}
\vec{i}_r \\
\vec{j}_r \\
\vec{k}_r
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_r \\
y_r \\
z_r
\end{bmatrix}, \quad
\begin{bmatrix}
v_r
\end{bmatrix}
\]

\[
\begin{bmatrix}
K_{xysz} \\
K_{yhrs} \\
K_{zrsz}
\end{bmatrix}
\]

6.5. Alternative Symbols

The same idea of our matrix tensor notation may be expressed in alternative symbols according to preference, e.g.,

\[
\vec{v}^s = \{v\}^s = \{v\}^s
\]

\[
\vec{v}^s = \{v\}^s = \{v\}^s
\]

\[
\{A\}^s = \{A\}^s
\]

\[
\vdots
\]

7. THE CROSS PRODUCT

The computational rule for the cross product is usually given in the form, taking the angular velocity pseudo-vector as an example,

\[
\omega \times \vec{r} = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
\omega_x & \omega_y & \omega_z \\
\vec{r}_x & \vec{r}_y & \vec{r}_z
\end{vmatrix}
\]
This equation is not satisfactory inasmuch as the left-hand side is a form independent of a base while the right-hand side is for one particular base. Therefore to be consistent, the rule in space notation must also be able to be stated in orthonormal base notation, which in this case is

\[
\vec{\omega} \times \vec{r} = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
\omega_x & \omega_y & \omega_z \\
r_x & r_y & r_z 
\end{vmatrix},
\]

where formally

\[
\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]

The equivalent matrix equation is valid for any dimensional space, yet to emphasize the equivalence in the three-dimensional case we write the corresponding matrix with the symbol \(\vec{\omega}\), so that

\[
\vec{\omega} \times \vec{r} = \vec{\omega} \cdot \vec{r} = \vec{W} \cdot \vec{r},
\]

and in space notation,

\[
\vec{\omega} \times \vec{r} = \vec{\omega}_r \cdot \vec{r} = \vec{W}_r \cdot \vec{r}.
\]

Here, \(\vec{\omega}_r\) is the angular velocity tensor that is often denoted by \(\vec{\omega}\) in printed text. This symbol is derived from the origin of the tensor from a vector, but the new tensor name is \(\vec{W}_r\).

8. THE ROTATION TENSOR

In (33) and (34), we have written relations between vectors in different bases with the application to an identical vector \(v\) in mind. Consider now a rotated vector as a new vector in space, obtained from the original vector by a tensor operation

\[
\vec{v}_R = \vec{R} \cdot \vec{v}_S,
\]

where the subscripts now refer not to bases but to the different positions \(R\) and \(S\), i.e., \(v_R\) and \(v_S\) are two different vector names. As a special case, we consider the rigid rotation of all direction vectors of the base \(\vec{E}_s\) to \(\vec{E}_r\)

\[
\vec{E}_r = \vec{R} \cdot \vec{E}_s.
\]

In this equation, we have apparently violated Rule IV of equal bases on both sides of the equation. This is made up by the factor \(\vec{R}\) which cannot be regarded as a physical unit tensor. The difficulty arises because we have used the base names as a collective name for the three vectors that we have rotated, i.e., in (44) \(\vec{E}\) is not regarded as tensor. The subscripts \(s\) and \(r\) in both (43) and (44) indicate different vectors on both sides of the equations. It is a result of the dual way rigid body rotations can be treated, either by attaching a rotating base to them or to rotate them by a rotation tensor. We cannot do without the rotation tensor because we must be able to express rotation in space notation without reference to a particular base. The rotation of the base shows the relation between the two different physical concepts of a rotation tensor and a rotated base.
Transforming equation (44) to base $s$ gives
\[ \overline{E}'_r = \overline{R}'^s \cdot \overline{E}'_s = \overline{R}'^s. \] (45)

In this equation, the rotation tensor in one base is equated to another quantity in mixed bases. Normally this would not be possible for a tensor because a simple transformation to base $r$ on the right shows that
\[ \overline{E}'_r \cdot \overline{E}'^r = \overline{E}'_s \cdot \overline{R}'^s \cdot \overline{E}'_s \]
\[ \overline{E}'_r = \overline{R}'^r. \] (46)

Therefore, with (45) we have
\[ \overline{R}'^r = \overline{R}'^s, \] (47)

which means that the elements of the tensor $R$ never change in any orthonormal base transformation. This property is true for the one and only one class of tensors which is the rotation tensor (of which the identity tensor may be considered a special case). We emphasize again that
\[ \overrightarrow{R} \neq \overrightarrow{E}, \]

which may be shown by transforming equation (46) by postmultiplication
\[ \overline{E}'_r \cdot \overline{E}'^r = \overrightarrow{R}, \] (48)

where again the nonconforming of base symbols results in a new name. At the same time, the equations that lead to (48) are transformations that prove that $R$ is a tensor. Our notation shows clearly that although the rotation tensor $\overline{R}'^s$ and the transformation matrix $\overline{E}'^s$ are algebraically the same matrices, they are different tensors, i.e., expressions of different concepts.

We can use the tensor $R$ to show that any non-conformable mixed base definition of tensor is a tensor, e.g., define
\[
\text{new } \overline{B}'^s = \text{given tensor } \overline{A}'^r,
\]
\[ \overline{B}'_s = \overline{E}'_r \cdot \overline{B}'_s = \overline{E}'_r \cdot \overline{A}'^r \cdot \overline{E}'_r = \overline{R}'^s \cdot \overline{A}'^s \cdot \overline{R}'^s, \]

which is a tensor matrix, which written in space notation is
\[ \overrightarrow{B} = \overrightarrow{R} \cdot \overrightarrow{A} \cdot \overrightarrow{R}. \]

9. THE POSITION VECTOR

The position vector is extraordinary because it joins two different positions in space while other vectors are a function of one position only. The components of the position vector in orthonormal base are Cartesian coordinates. This sets the position vector apart from other vectors. Cartesian coordinates are therefore always implied in orthonormal base. We define two position vectors
\[ \overrightarrow{o} = \text{Euler position vector of fixed space}, \]
\[ \overrightarrow{r} = \text{Lagrange position vector of an identical point or particle}. \]
We find this distinction very useful in both real and abstract space. Although \( \vec{s} \) and \( \vec{r} \) are expressed in the same Cartesian coordinates,

\[
\vec{s} = i x + j y + k z = \vec{E}_s \cdot \vec{s},
\]

we express

\[
\vec{r} = i x + j y + k z = \vec{E}_s \cdot \vec{r}
\]

only if no confusion can result, e.g., for a single particle, but in a continuum we write

\[
\vec{r} = i r_x + j r_y + k r_z
\]

to avoid confusion. The different symbols point to the difference that \( \vec{r} \) is a function of time \( t \) and \( \vec{s} \) is not. In continuum mechanics \( s \) and \( t \) are the independent variables and \( \vec{r} \) is a function of an initial value which may be \( s \) and \( t \). Denoting the time derivative of an identical particle by \( \dot{} \), we write

\[
\dot{\vec{s}} = \vec{v}, \quad \text{velocity of the particle},
\]

\[
\dot{\vec{r}} = \vec{v} = \vec{a}, \quad \text{acceleration of the particle},
\]

while \( \vec{s} \) and \( \vec{s} \) don’t exist or are zero. Similarly, a displacement should be called \( \vec{r} \), not \( \vec{s} \), even in simple cases, so that

\[
\vec{r} = \vec{s} + \vec{u}.
\]

In fluid mechanics, the difference between \( \vec{r} \) and \( \vec{s} \) is difficult to distinguish, and the Lagrange time derivative is usually written as the Stokes Derivative \( \vec{r} = \frac{D\vec{s}}{dt} \) [21] (unlike the convention in fluid mechanics, we use the capital \( D \) only for the identical differential of \( \vec{s} \) because \( dt \) is the independent total differential). In finite displacement (non-linear) elasticity, however, this is no longer possible.

### 10. Rotating Base

Let \( \vec{E}_r \) be a time dependent base, i.e., a rotating base. The time derivative of the rotating base is given by the formula from mechanics, written in tensor matrix notation

\[
\frac{d}{dt} \vec{E}_r \equiv \dot{\vec{E}}_r = \vec{\omega} \times \vec{E}_r
\]

\[
= \vec{W} \cdot \vec{E}_r
\]

\[
= \vec{W} \cdot \vec{E}_r
\]

\( \vec{W} \) is the angular velocity tensor \( \vec{\omega} \). The corresponding angular velocity vector \( \vec{\omega} \) exists only in three-dimensional space while the tensor \( \vec{W} \) exists in any dimension. Transformed to a fixed base \( \vec{E}_s \), equation (49) becomes
\[ \hat{E}_r = \hat{W}_s \cdot \hat{E}_r \]  
\[ = \hat{W}_s \]  
\[ = \hat{E}_r \cdot \hat{W}_r, \]  
where \( \hat{W}_s \) and \( \hat{W}_r \) is an anti-symmetric tensor matrix in base \( s \) and \( r \), respectively, and has only \( n(n-1)/2 \) different elements in any particular base, which makes it possible to write a corresponding pseudo-vector in three dimensions, and which is singular in every odd dimension, so that the cross product cannot be inverted. Therefore, we have

\[
\begin{align*}
\left[ \hat{W}_s \right]^T &= -\hat{W}_s \\
\hat{E}_s &= -\hat{W}_s \\
&= -\hat{E}_s \cdot \hat{W}_s \\
&= -\hat{W}_r \cdot \hat{E}_s.
\end{align*}
\]

Equations (50) and (52) show how the matrix tensor notation produces the angular velocity tensor in mixed bases which leads to the product forms. The form usually used in continuum mechanics,

\[ \hat{R} = \hat{W} \cdot \hat{R}, \]

is obtained by substituting the rotation tensor \( R \) from (45) and (46) into (51) and (53)

\[
\begin{align*}
\hat{R}_s &= \hat{W}_s \cdot \hat{R}_s \\
\hat{R}_r &= \hat{R}_r \cdot \hat{W}_r.
\end{align*}
\]

10.1. Relative and Apparent Vectors in Rotating Base

If \( C \) is a fixed or moving reference point in space, a common notation of its vector of position, velocity, acceleration, etc., is \( \vec{r}_C, \vec{v}_C, \vec{a}_C, \) etc. If \( P \) is the position of a point or particle of interest, the corresponding vectors are \( \vec{r}_P, \vec{v}_P, \vec{a}_P, \) etc. The relative vector of position, velocity, acceleration, etc., of \( P \) relative to \( C \) is defined and denoted by

\[
\begin{align*}
\vec{r}_{P/C} &= \vec{r}_P - \vec{r}_C \\
\vec{v}_{P/C} &= \vec{v}_P - \vec{v}_C \\
\vec{a}_{P/C} &= \vec{a}_P - \vec{a}_C
\end{align*}
\]

If instead of \( P \) we refer to a general point, the notation may be defaulted by \( \vec{r}_{/C}, \vec{v}_{/C}, \vec{a}_{/C}, \) etc. We call a vector produced by a time derivative in rotating base an apparent vector, which distinguishes it from a relative vector. The relation between kinematic vectors in fixed base \( s \) and
rotating base $r$ becomes, in matrix tensor notation, by straightforward differentiation of the Lagrange position vector and application of (51),

$$
\begin{align*}
\dot{r}^- &= \overrightarrow{E} \cdot r^- \\
\ddot{r}^- &= \overrightarrow{E} \cdot \ddot{r}^- \\
&+ \overrightarrow{W} \cdot \overrightarrow{E} \cdot \ddot{r}^- \\
&+ \overrightarrow{W} \cdot \overrightarrow{W} \cdot \overrightarrow{E} \cdot \ddot{r}^- \\
&+ \overrightarrow{W} \cdot \overrightarrow{W} \cdot \overrightarrow{W} \cdot \overrightarrow{E} \cdot \ddot{r}^- \\
&+ \overrightarrow{W} \cdot \overrightarrow{W} \cdot \overrightarrow{W} \cdot \overrightarrow{W} \cdot \overrightarrow{E} \cdot \ddot{r}^- \\
&+ \overrightarrow{W} \cdot \overrightarrow{W} \cdot \overrightarrow{W} \cdot \overrightarrow{W} \cdot \overrightarrow{W} \cdot \overrightarrow{E} \cdot \ddot{r}^-
\end{align*}
$$

apparent velocity

relative velocity

base velocity
due to rotation
apparent acceleration
Coriolis acceleration
base rotational acceleration
base centripetal acceleration.

The time derivatives of vectors in base $s$ are apparent time derivatives which produce the apparent velocity and acceleration vectors. While the apparent velocity is not identical to the relative velocity, referred to the same position fixed on the rotating base, the apparent acceleration is not equal to the relative acceleration, which according to definition is equal to apparent acceleration plus Coriolis acceleration. In some modern texts on mechanics, relative acceleration is precisely defined, but then the apparent acceleration is denoted relative acceleration \[20\], which is an outright misleading term. Seldom does a student check that the acceleration is not relative according to definition. A proper notation of these terms is found in \[21\].

The differentiation using rotating base produces vectors expressed in both bases which express their meaning exactly. They can be transformed easily to rotating base by using the symbols $v$ and $a$ for absolute velocity and acceleration

$$
\begin{align*}
\dot{v}^- &= \overrightarrow{E} \cdot \dot{v}^- = \overrightarrow{E} \cdot \ddot{v}^- \\
\ddot{a}^- &= \overrightarrow{E} \cdot \ddot{a}^- = \overrightarrow{E} \cdot \ddot{a}^-.
\end{align*}
$$

Since apparent derivatives are difficult to write in fixed base or space notation, we use the symbol

$$
\begin{align*}
\text{apparent velocity} &= \frac{\partial \dot{r}}{\partial t} = \dot{r}^- = \overrightarrow{E} \cdot \ddot{r}^- = v_{\text{app}} = v_{\text{rel}} \\
\text{apparent acceleration} &= \frac{\partial^2 \dot{r}}{\partial t^2} = \ddot{r}^- = \overrightarrow{E} \cdot \dddot{r}^- = a_{\text{app}} \neq a_{\text{rel}}.
\end{align*}
$$

However, this rather formal space notation transformation is not necessary for computation. The notation of the derived equations already allows for the identification of each term. Similarly, the motion in a rotating base in another rotating base, etc., can be described resulting in simple descriptive symbols of quite complicated motions.

10.2. Euler Equations of Motion of the Rigid Body

This is an example to show how equations of motion in rotating base are derived using matrix tensor notation. Let

$$
\overrightarrow{I} = \text{moment of inertia of rigid body about a fixed point or center of mass}$$
\( \omega \) = angular velocity of the rigid body

\( q \) = torque at a fixed point or center of mass acting on the rigid body,

then the equation of rotation is

\[
\dot{q} = \frac{d}{dt} \left[ \mathbf{J} \cdot \omega \right].
\]

(54)

Let \( s \) be a fixed base and \( r \) a rotating base fixed to the rigid body and transform Equation (54) from base \( r \), and using equation (52),

\[
\mathbf{E}^s \cdot \dot{q}^r = \frac{d}{dt} \mathbf{E}^s \cdot \left[ \mathbf{J}^r \cdot \dot{\omega}^r \right]
\]

\[
= \mathbf{W}^s \cdot \mathbf{J}^r \cdot \dot{\omega}^r + \mathbf{E}^s \cdot \mathbf{J}^r \cdot \ddot{\omega}^r.
\]

Therefore, the Euler equation in matrix tensor notation in base \( r \) is

\[
\dot{q}^r = \mathbf{W}^r \cdot \mathbf{J}^r \cdot \dot{\omega}^r + \mathbf{J}^r \cdot \ddot{\omega}^r.
\]

To be useful, it must also be stated that \( \dot{\omega}^r \) transforms like a vector to fixed base

\[
\dot{\omega}^s = \mathbf{E}^s \cdot \dot{\omega}^r,
\]

because the product \( \mathbf{W}^s \cdot \dot{\omega}^r \) vanishes, \( \mathbf{W} \cdot \dot{\omega} \) corresponding to the crossproduct \( \dot{\omega} \times \ddot{\omega} \).

11. DIFFERENTIAL EXPRESSIONS IN A CONTINUUM

Any continuous scalar function \( \phi \) of Cartesian coordinates \( x, y, z \) can be expressed in terms of the position vector \( s = \mathbf{E}^s \cdot \mathbf{s} \) in an orthonormal base \( s \). Consider a total differential of position \( \delta s \), whose components are given by the total differentials of the coordinates by

\[
\delta s = \mathbf{E}^s \cdot \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}.
\]

The corresponding total differential of \( \phi \) will be \( d\phi \). We define the derivative

\[
\frac{d\phi}{ds} \quad \text{is such that} \quad \frac{d\phi}{ds} \cdot \delta s = d\phi.
\]

Comparing with the total differential of \( \phi(x, y, z) \), which consists of the partial differentials

\[
d\phi = \partial_x \phi + \partial_y \phi + \partial_z \phi,
\]

we find that

\[
\frac{d\phi}{ds} = \left[ \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} \frac{\partial \phi}{\partial z} \right] \cdot \mathbf{E}^s
\]

\[
= \frac{\partial \phi}{\delta s} \cdot \mathbf{E}^s
\]

\[
= \nabla \phi.
\]

\[\text{We use a revised partial differential notation, showing where a partial increment is divided by a total increment.}\]
Therefore, if \( \vec{s} \) is considered as a column vector, then the gradient of a scalar is a row vector, and the overbar \( \overline{\cdot} \) in the divisor \( \overline{d} \vec{s} \) is equivalent to an underbar \( \underline{\cdot} \) of the resulting quotient, indicating a row vector, which is now an additional derived rule of matrix tensor notation. The rule can be stated: division by a differential column vector produces a row vector. Similarly, we define the derivative of a vector function \( \vec{v}(s) \) as

\[
\frac{\vec{dv}}{ds} \text{ is such that } \frac{\vec{dv}}{ds} \cdot \vec{ds} = \vec{dv}.
\]

Comparing with the total differential \( \vec{dv} \), which consists of the partial differentials

\[
\vec{dv} = \partial_x v + \partial_y v + \partial_z v,
\]

we find that

\[
\frac{\vec{dv}}{ds} = \frac{\partial u}{dx} \cdot \frac{\partial v}{dy} + \frac{\partial u}{dy} \cdot \frac{\partial v}{dz} + \frac{\partial w}{dx} \cdot \frac{\partial v}{dy} + \frac{\partial w}{dz} \cdot \frac{\partial v}{dz}.
\]

\[
\equiv \overline{\vec{E}}^s \cdot \frac{\vec{dv}}{ds} \cdot \overline{\vec{E}}^s.
\]

\[
\equiv \text{grad} \vec{v}.
\]

Therefore, if \( \vec{ds} \) is considered a column vector, then the gradient of a column vector is a tensor with columns and rows. The corresponding tensor matrix \( \frac{\vec{dv}}{ds} \) is the Jacobian matrix.

The combined rules of differentials and vectors are that a total differential cancels diagonally across the \( \cdot \) sign if the name and the base are the same,

\[
\vec{dv} = \frac{\vec{dv}}{ds} \cdot \overline{\vec{E}}^s.
\]

We can transform to any other base or mixed bases

\[
\frac{\vec{dv}}{ds} = \overline{\vec{E}}^r \cdot \frac{\vec{dv}}{ds}.
\]

\[
\frac{\vec{dv}}{ds} = \frac{\vec{dv}}{ds} \cdot \overline{\vec{E}}^r.
\]

\[
\frac{\vec{dv}}{ds} = \overline{\vec{E}}^s \cdot \frac{\vec{dv}}{ds} \cdot \overline{\vec{E}}^s.
\]

which demonstrates the consistency of the notation.
12. ORTHOGONAL NON-UNIT BASE

Consider the affine transformation from affine coordinates $\alpha, \beta$ to Cartesian coordinates $x, y$ in two-dimensional space given by the linear equations

\[
\begin{align*}
x &= h_\alpha \, \alpha \\
y &= h_\beta \, \beta.
\end{align*}
\]

The coordinates can be considered purely algebraic variables which are collected into algebraic column vectors

\[
\begin{align*}
\overline{a} &= \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \\
\overline{s} &= \begin{bmatrix} x \\ y \end{bmatrix},
\end{align*}
\]

which are not vector arrays of a physical vector. Thus, they have no base name. At the same time, we can consider $x, y$ as components of the position vector $s$

\[
\overline{s} = \begin{bmatrix} x \\ y \end{bmatrix},
\]

from which follows

\[
\overline{s}' = \overline{s}, \quad (55)
\]

which is an extraordinary equation in our notation because the bases don't match. Equation (55) is a special equation of the position vector only, and only in Cartesian coordinates, that relates coordinate variables and position vector components, which can also be expressed

\[
\overrightarrow{E}_s = \frac{d\overline{s}}{ds}. \quad (56)
\]

By a similar relation we can define a base $a$ (following the notation of unitary direction vectors in [23], but in our notation the coordinates also have the name $a$)

\[
\overrightarrow{E}_a = \frac{d\overline{s}}{da} = \frac{d\overline{a}}{ds} \cdot \frac{d\overline{a}}{da} = \overrightarrow{E}_s \cdot \begin{bmatrix} h_\alpha \\ h_\beta \end{bmatrix} = \overrightarrow{E}_s \cdot \overline{h}. \quad (58)
\]

The relation between bases $a$ and $s$ is defined by

\[
\overrightarrow{E}_a = \overrightarrow{E}_s \cdot \overrightarrow{E}_a'.
\]

Therefore, the transformation matrix is

\[
\overrightarrow{E}_a' = \frac{d\overline{s}}{da}. \quad (59)
\]
Figure 7(a) shows the nonunit direction vectors of base $a$

$$\overrightarrow{E}_a = \begin{bmatrix} e_{a1} \\ e_{a2} \end{bmatrix}.$$  

We define the inverse base $a$ as in (18) also for nonunit bases

$$\left[ \overrightarrow{E}_a \right]^{-1} \equiv \overrightarrow{E}_a^T = \begin{bmatrix} e_{a1} \\ e_{a2} \end{bmatrix},$$  

which satisfies equation (15) from which we see that the vectors $e_{ai}$ are physically different from $e_{ai}$, viz.,

$$|e_{ai}| = \left| \frac{1}{e_{ai}} \right|.$$  

The inverse base is therefore different from the transpose

$$\left[ \overrightarrow{E}_a \right]^T \equiv \overrightarrow{E}_a^T \neq \overrightarrow{E}_a^a,$$

and the transpose of the inverse base is different from the posed base $a$

$$\left[ \overrightarrow{E}_a^a \right]^T \equiv \overrightarrow{E}_a^T \neq \overrightarrow{E}_a.$$  

Because of the different inverse, we have two physically different bases and because of the transpose we have two additional mathematically different bases. The transpose symbol $^T$ cannot be left out of the base symbol any more. However, the inverse symbol $^{-1}$ doesn't appear
on the base symbol even for non-unit base vectors. From the relation

\[ \vec{E}_s^\alpha \cdot \vec{E}_a^\beta = \left[ \frac{ds}{da} \right]^T \cdot \left[ \frac{ds}{da} \right] = \hbar^2, \]

the inverse can be computed by

\[ \left[ \vec{E}_s^\alpha \right]^{-1} = \vec{E}_a^\alpha = \frac{1}{\hbar^2} \cdot \vec{E}_s^\alpha, \quad (61) \]

which is the simple multiplication of the transpose by a diagonal by which means also in orthogonal base which is not normal, the inverse of the transformation matrix does not have to be calculated by inversion.

The vector transformations between space and base \( \alpha \), its inverse and transpose follow from the definition of components

\[ \vec{v} = \vec{E}_a \cdot \vec{v}_a \quad (62) \]

\[ \vec{v} = \vec{v}_a \cdot \vec{E}_a^\alpha. \quad (63) \]

Transposing equations (62) and (63) gives

\[ \vec{v} = \vec{v}_a \cdot \vec{E}_a^\alpha \quad (64) \]

\[ \vec{v} = \vec{E}_a^\alpha \cdot \vec{v}_a \quad (65) \]

From these equations, we see that Rule II still applies, that \( E \) acts as a unit on the name, i.e., even though the base \( \alpha \) consists of nonunit direction vectors, \( \vec{E}_a^\alpha \) is still a representation of the unit space tensor. From (62) and (63), we have

\[ v^2 = \vec{v} \cdot \vec{v} = \vec{v}_a \cdot \vec{v} \quad (66) \]

From these equations, we make several conclusions purely as a result of consistency. In nonorthonormal base \( \alpha \), the transposed column vectors do not have the same components as the row vectors and vice versa,

\[ \left[ \vec{v}_a \right]^T = \vec{v}_a^\alpha \neq \vec{v}_a \]

\[ \left[ \vec{v}_a \right]^T = \vec{v}^\alpha \neq \vec{v} \quad (67) \]

The transpose actually becomes part of the base name of that base which is not orthonormal so we can actually incorporate it in the base name symbol in vectors and bases

\[ \left[ \vec{v}_a \right]^T = \vec{v}_a \neq \vec{v}_a \]

\[ \left[ \vec{v}_a \right]^T = \vec{v}^\alpha \neq \vec{v} \]
Similar to the transposed vector $v^i$, the transposed base is physically the same as the posed, but is a mathematical requirement for the completeness of the matrix tensor algebra where typical transformations in nonorthonormal base become

$$
\vec{v}^a = \vec{E}^a \cdot \vec{v}_{\tau a}
$$

$$
\underline{v}^a = \underline{v}_{\tau a} \cdot \underline{E}^a
$$

$$
\underline{K}^a_{\tau a} = \underline{E}^a \cdot \underline{K}^a
$$

$$
\underline{K}^a_{\tau a} = \underline{E}^a \cdot \underline{K}^a \cdot \underline{E}^a_{\tau a}
$$

$$
\underline{K}^a = \underline{E}^a \cdot \underline{E}^a \cdot \underline{E}^a_{\tau a}
$$

This last set of equations shows how complicated the matrix tensor notation may become because of the transpose. However, in Euclidean space we hardly ever need to apply the transpose symbol $a^T$ will be shown in Part II. The equations with the transpose merely demonstrate the completeness of matrix tensor notation, that every compatible multiplication is possible and, more important, every different quantity has a different symbol and different symbols never indicate the same entity. It will also be shown in Part II that in abstract spaces we need these transpose symbols to interpret equations properly.

The two vectors $\vec{v}^a$ and $\underline{v}^a$ are not transposes of each other but different vectors. It is now clear that matrix tensor notation blends into tensor notation, where column vectors $\underline{v}^a$ in matrix tensor notation are contravariant vectors $v^i$ in tensor analysis and row vectors $\underline{v}^a$ in matrix tensor notation are covariant vectors $v_i$ in tensor analysis. The transformation of a contravariant vector to a covariant vector in tensor analysis is given by

$$
v_i = g_{ij} v^j,
$$

where $g_{ij}$ is the metric tensor. (In that respect, the notation $g$ for unitary direction vectors in [13] is more appropriate.) In matrix tensor notation

$$
\vec{E}^a_{\tau a} = \underline{E}^a \cdot \underline{E}^a_{\tau a}
$$

(67)

is the mixed base unit tensor equal to $g_{ij}$, and the transformation between column and row vectors in nonorthonormal base in matrix tensor notation is, in two steps,

$$
\underline{v}^a = \vec{E}^a \cdot \underline{v} \quad \underline{v}^a = \left[ \underline{\vec{v}}^a \right].
$$

(68)
In Part II it is shown how the matrix $E^{T_a}$ can be replaced by a stretched array $E_{a \cdot a}$, with which the transpose symbol can be avoided completely.

This base notation of matrix tensor notation corresponds to the index notation of tensor analysis for second order tensors, but with a different meaning of the indices. In Part II it will be shown that the same notation is equally applicable for skew bases.

### 12.1. Orthogonal Functions

Consider a periodic function of time $f_\ell(t)$ sampled at $N + 1$ points $t_i$ at equal time intervals in the interval $t = 0$ to $t = 2\pi$, where the values are $f_{it}$, $i = 0, 1, \ldots, N$, and collect the last $N$ values into a column vector $f$. If we introduce a measure of length $f$ such that

$$f^2 \equiv |f|^2 = \sum_{i=1}^{N} f_{it}^2,$$

then we have in fact arbitrarily defined an orthogonal function space with orthonormal base. We call this base $t$ and denote the time function sample values as vector $\overrightarrow{f^t}$, and the row vector

$$f_t = \left\{ f^t \right\}^T,$$

consistent with orthonormal base. We choose the name $F$ for a function space tensor, and therefore,

$$\overrightarrow{F^t} = [I].$$

Note that in this function space $\overrightarrow{f^t}$ is a position vector, and the coordinates $f_{it}$ are the components of $\overrightarrow{f^t}$. Now equation (25) can be written

$$f^2 = f_t \cdot \overrightarrow{f^t}. \tag{69}$$

In this function space consider the harmonic functions

$$F_\ell = 1, \cos \ell t, \sin \ell t, \cos 2t, \sin 2t, \ldots, \cos \ell t,$$

where $\ell = N/2 + 1$ and $\omega$ is an index counter for the different functions. Each of these functions sampled at $t_i$ becomes a column vector of function values $\overrightarrow{F_\omega^t}$ and all of these vectors may be used as another base in function space. We call this base $\omega$ and express it in terms of base $t$ by the matrix of sampled function values

$$\overrightarrow{F_\omega^t} = \left[ \overrightarrow{F_\omega^t} \overrightarrow{F_\omega^t} \overrightarrow{F_\omega^t} \overrightarrow{F_\omega^t} \overrightarrow{F_\omega^t} \ldots \overrightarrow{F_\omega^t} \right].$$

Now we write the Fourier coefficients which are obtained by Harmonic Analysis of the discrete function values (DDFT: Discrete Discrete Fourier Transform) as vector $\overrightarrow{F^\omega}$, and the relation between sampled function values and coefficients, the synthesis, as the matrix equation

$$\overrightarrow{f^t} = \overrightarrow{F_\omega^t} \cdot \overrightarrow{F^\omega}. \tag{71}$$
and the analysis

$$\overline{f}^\omega = \overline{F}_t^\omega \cdot \overline{f}^t,$$  \hspace{1cm} (72)

where

$$\overline{F}_t^\omega = \left[ \overline{F}_t^\omega \right]^{-1}. \hspace{1cm} (73)$$

Although the harmonic functions are not orthonormal, they are orthogonal so that the inverse is simply calculated as

$$\overline{F}_t^\omega = \left[ \overline{F}_t^\omega \right]^{-1}. \hspace{1cm} (74)$$

Any other function will have another name, e.g., for the sampled values of a function \( g_t(t) \) and its DDFT \( g_\omega(\omega) \), the DDFT is expressed as

$$\overline{g}^t = \overline{F}_t^\omega \cdot \overline{g}^\omega, \hspace{1cm} \overline{g}^\omega = \overline{F}_t^\omega \cdot \overline{g}^t.$$

Therefore, matrix tensor notation provides a convenient way of expressing function transforms in function space. The functions are expressed exactly like vectors in three-dimensional Euclidean space. The transpose sign falls away in base \( t \) because it is orthonormal, but is required in base \( \omega \) because it is not orthonormal, i.e., \( F_t^\omega \) and \( F_\omega^t \) are not the same. Therefore we have a convenient method to express in matrix algebra that \( F_t^\omega \cdot \overline{f}^\omega = F_t^\omega \cdot \overline{f}^t = F_t^\omega \cdot \overline{f}^t \), but in Parseval’s theorem \( F_t^\omega \cdot \overline{f}^\omega \neq F_t^\omega \cdot \overline{f}^t \). Note that the equations without the transpose symbol are similar to the same equations in tensor notation, now applied to function space. On the other hand, we see the need of the transpose symbol to distinguish between \( F_t^\omega \) and \( F_\omega^t \).

The function has the same name in any base: the function “in the time domain” becomes the function “in base \( t \),” which we may call appropriately time base, and the function “in the frequency domain” becomes the function “in base \( \omega \)” which we may call frequency base. \( F_t^\omega \) and \( F_\omega^t \) become the base transformation matrices. This notation has been found very useful in practical computations involving DDFT.

To complete the analogy of abstract function space and real Euclidean space we introduce the abstract function vector \( \overline{f} \), the time base \( \overline{F}_t \) and the frequency base \( \overline{F}_\omega \) with the abstract relations

$$\overline{f} = \overline{F}_t \cdot \overline{f}^t,$$ \hspace{1cm} (74)

$$\overline{f} = \overline{F}_\omega \cdot \overline{f}^\omega,$$ \hspace{1cm} (75)

$$\left[ \overline{F}_t \right]^{-1} = \overline{F}_t,$$ \hspace{1cm} (76)

$$\overline{F}_t \cdot \overline{F}_\omega = \overline{F}_\omega,$$ \hspace{1cm} (77)
where the function space is \( \mathcal{F}^t \). By these abstract definitions we can distinguish between function space and base, and we can denote a function like a vector without bias to one or other base. Artificial and impractical numerically as it may be, conceptually it is satisfying. In less scientific language we might say that \( f^t \) and \( f^\omega \) are but two different perspectives of the same function \( f \). (An example of bias is that an experienced electronics engineer may regard the spectrum of a function as a true perspective of the function and the time function merely a transform).

We extend the same notation to series and integral transforms of continuous functions in appropriate intervals, viz.,

**DCFT:** Discrete Continuous Fourier Transform, which is the synthesis of a function by Fourier series, and the inverse;

**CDFT:** Continuous Discrete Fourier Transform, which is the Fourier Analysis;

**CCFT:** Continuous Continuous Fourier Transform, which is the integral Fourier transform.

Fourier series in this notation is described by (71), where \( \mathcal{F}^t \) stands for a continuous time function, considered as an 'infinite' number of sampled values, with period \( T = \frac{2\pi}{\omega} \), \( \mathcal{F}^\omega \) for the infinite sequence of Fourier coefficients, and

\[
\mathcal{F}^t \cdot \mathcal{F}^\omega = \sum_{n=0}^{\infty} \frac{\cos (n\omega t)}{\sin (n\omega t)} \cdot f_n ,
\]

with the sequence of coefficients \( f_n \), and Fourier analysis is described by (72), where

\[
\mathcal{F}^\omega \cdot \mathcal{F}^t = \frac{1}{\pi} \int_{0}^{2\pi} \frac{\cos (n\omega t)}{\sin (n\omega t)} \cdot f_t(t) \, dt .
\]

The base transformation 'matrix'

\[
\mathcal{F}^t_{\omega} = \mathcal{F}_{\omega}(t) = \begin{bmatrix} 1 & \cos t & \sin t & \cos 2t & \sin 2t & \cdots & \text{inf} \, t \end{bmatrix}
\]

consists of a countable infinite number of columns of continuous functions regarded as an uncountable number of time-sampled values. Therefore, this matrix cannot be written numerically.

The Fourier transform and inverse Fourier transform in function space is expressed by (72) and (71) respectively, where

\[
\mathcal{F}^\omega \cdot \mathcal{F}^t = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} f_t(t) \, dt \\
\mathcal{F}^t \cdot \mathcal{F}^\omega = \int_{-\infty}^{\infty} e^{i\omega t} f_\omega(\omega) \, d\omega ,
\]

where we use the symbol \( i = \sqrt{-1} \), which leaves \( i \) free for normal use. We note that because of the column vector definition of functions the customary order of functions and coefficients is reversed to comply with matrix algebra, but again this is in line with the customary order of kernel and function in the integral transform. The kernel of an integral transform becomes the base transformation matrix but this \( \infty \times \infty \) matrix cannot be written numerically. This is consistent with the customary interpretation of an integral transform in function space and its discrete approximation by matrix algebra. Note that in the matrix tensor notation of the Fourier Transform the names of the functions are the same on both sides of the equation while in the customary the Fourier transform different names are used, which we have done by subscripts.
13. CONCLUSION

A Matrix Tensor Notation is developed to satisfy a number of requirements to distinguish quantities in vector analysis and mechanics in a consistent and unique manner. The proposed notation extends matrix notation to include physical tensors which leads to an extension of matrix algebra to include the additional symbols that completely define linear transformations. It is a complete and closed algebra in the sense that all transformations are indicated by unique symbols, and that all symbolic operations define a unique mathematical tensor operation of physical vectors and tensors and their representations in any orthogonal (and skew base as shown in Part II). The same notation is suitable for transformations in Euclidean space, abstract higher dimensional space and transformations and transforms.

This matrix-tensor notation satisfies the requirements listed in the Introduction by means of the following symbols and concepts:

1. Names of matrices are written in capital letters, names of vectors in small letters.
2. An algebraic vector is identified by a single bar and a matrix by two bars.
3. Row and column vectors are distinguished by the over or under position of the bar. This eliminates the use of the transpose to indicate a row vector.
4. A physical vector is identified by an arrow and a tensor by two arrows, thus distinguishing between physical vectors and tensors and their arrays of scalar quantities.
5. The base is formally introduced as the collection of direction vectors. With this quantity a formal relation between a tensor and its representation is established.
6. Vectors and tensors are identified by the same name in any base. Their representations as arrays in different bases are distinguished by adding the base name as part of the symbol.
7. A transposed form of the physical vector notation is introduced as a necessary concept for a complete matrix algebra.
8. The concept of representation in mixed bases is introduced.
9. The base itself is identified as a unit tensor quantity.
10. The tensor transformation matrix is identified as a representation of a tensor, contrary to customary tensor algebra.
11. Transformations are done with a corresponding algebra of base symbols which is similar to, but with a different meaning, tensor algebra.
12. The transpose is the only extra burden of matrix algebra compared to tensor algebra for up to second order tensors, but its use always indicates an operation in unsuitable or unknown bases.
13. The meaning of the transpose symbol on vector arrays is changed from the conventional to the same meaning it has for matrices.
14. Transformations to rotating bases are facilitated by this notation.
15. The notation is equally valid in unit and in non-unit bases but takes advantage of the simplicity of orthonormal base. Contravariant and covariant vectors are expressed by the simple device of using column and row vectors.
16. The notation can be applied to functions as vectors in function space. The introduction of a space vector for a function completes the function space analogy.
REFERENCES