Fixed point subalgebras of extended affine Lie algebras

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Abstract

It is a well-known result that the fixed point subalgebra of a finite dimensional complex simple Lie algebra under a finite order automorphism is a reductive Lie algebra so it is a direct sum of finite dimensional simple Lie subalgebras and an abelian subalgebra. We consider this for the class of extended affine Lie algebras and are able to show that the fixed point subalgebra of an extended affine Lie algebra under a finite order automorphism (which satisfies certain natural properties) is a sum of extended affine Lie algebras (up to existence of some isolated root spaces), a subspace of the center and a subspace which is contained in the centralizer of the core. Moreover, we show that the core of the fixed point subalgebra modulo its center is isomorphic to the direct sum of the cores modulo centers of the involved summands.

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0. Introduction

In 1955, A. Borel and G.D. Mostow [7] proved that the fixed point subalgebra of a finite dimensional complex simple Lie algebra under a finite order automorphism is a reductive Lie algebra. A natural question which arises here is what can we say about the fixed points of a finite order automorphism of an extended affine Lie algebra (EALA for short). EALAs are natural generalizations of finite dimensional complex simple Lie algebras and affine Kac–Moody Lie algebras. They are axiomatically defined (see Definition 1.5) and the axioms guarantee the existence of analogues of Cartan subalgebras, root systems, invariant forms, etc. A root of an EALA is called isotropic if it is orthogonal to itself, with respect to the form. The dimension of the real span of the isotropic roots is called the nullity of the Lie algebra. A finite dimensional simple Lie algebra is an EALA of nullity zero, and a tame EALA is an affine Lie algebra if and only if its nullity is 1 (see [1] for details). Thus EALAs form a natural class of algebras in which to consider extensions of the result of Borel and Mostow.

Here we would like to explain a procedure which has been the most general theme of constructing affine Lie algebras and their generalizations, since the birth of Kac–Moody Lie algebras in 1968.

Let $(\mathcal{G}, (\cdot, \cdot), \mathcal{H})$ be an EALA with root system $R$ (in particular, $\mathcal{G}$ can be a finite dimensional simple Lie algebra or an affine Lie algebra). Let $\sigma$ be a finite order automorphism of $\mathcal{G}$ which stabilizes $\mathcal{H}$ and leaves the form invariant. Assume also that the fixed point subalgebra of $\mathcal{H}$ (with respect to $\sigma$) is self-centralizing in the fixed point subalgebra of $\mathcal{G}$.

Consider the Lie algebra $\text{Aff}(\mathcal{G}) := (\mathcal{G} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d$, where $c$ is central, $d = t \frac{d}{dt}$ is the degree derivation so that $[d, x \otimes t^n] = nx \otimes t^n$, and multiplication is given by

$$\left[ x \otimes t^n, y \otimes t^m \right] = [x, y] \otimes t^{n+m} + n(x, y)\delta_{m+n,0}c.$$ 

Extend the form $(\cdot, \cdot)$ to $\text{Aff}(\mathcal{G})$ so that $c$ and $d$ are naturally paired. Set $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathbb{C}c \oplus \mathbb{C}d$. Then the triple

$$(\text{Aff}(\mathcal{G}), (\cdot, \cdot), \tilde{\mathcal{H}})$$

is again an EALA with root system $\tilde{R} = R + \mathbb{Z}\delta$ where $\delta$ is the linear functional on $\tilde{\mathcal{H}}$ defined by $\delta(d) = 1$ and $\delta(\mathcal{H} \oplus \mathbb{C}c) = 0$. Extend $\sigma$ to an automorphism of $\text{Aff}(\mathcal{G})$ by

$$\sigma(x \otimes t^i + rc + sd) = \xi^{-i}\sigma(x) \otimes t^i + rc + sd,$$

where $\xi = e^{2\pi \sqrt{-1}/m}$ and $\sigma^m = \text{id}$. Let $\text{Aff}(\mathcal{G})^\sigma$ be the fixed points of $\sigma$ and $\tilde{\mathcal{H}}^\sigma$ be the fixed points of $\tilde{\mathcal{H}}$ under $\sigma$. It follows that $\text{Aff}(\mathcal{G})^\sigma$ has a root space decomposition with respect to $\tilde{\mathcal{H}}^\sigma$ such that if the corresponding root system has some nonisotropic roots, then

$$(\text{Aff}(\mathcal{G})^\sigma, (\cdot, \cdot), \tilde{\mathcal{H}}^\sigma)$$
is an EALA (see [2] and Example 3.81). As one can see in [10] and [8], when $G$ is a finite dimensional simple Lie algebra, all affine Kac–Moody Lie algebras can be constructed this way. In fact all the examples of EALA’s which are presented in [9,11,13] can be obtained from the above procedure. We will also see that most of the examples of EALA’s constructed in [1] can be put in the above context (see Section 3 and [4]).

In this paper, we consider the theorem of [7] for the class of EALA’s. Namely, we investigate the fixed point subalgebra of an EALA under a finite order automorphism which satisfies certain conditions. In a very real sense this paper found it is inspiration from the paper [2] which also studies Lie algebras constructed from finite order automorphisms of an EALA. The difference is this; in [2] the algebras studied are affinizations of the EALA one starts with while here we study fixed point subalgebras. But our basic techniques are similar to [2] and, as the reader will see, we will rely on some of the results from [2] in certain crucial places.

In Section 1, we give the definition and record some basic properties of an extended affine Lie algebra. We also state a modified version of the definition of an extended affine root system (EARS for short). The usual definition has indecomposability built into it. We need to consider root systems which are unions of such root systems and so the usual definition needs to be extended. This should not cause the reader any difficulties. The definition given here is basically the same as [1] except for a rearrangement of the axioms. Part of the reason this is done is to make more compatible the two different definitions of EARSs found in the literature. It is shown that an EARS, modulo some isotropic roots, is a union of a finite number of irreducible EARS’s which are orthogonal with respect to the form. (In [5] the relations between the two terms of EARSs found in the literature are clarified.)

In Section 2, which forms the core of the paper, we show that the fixed point subalgebra $G^\sigma$ of an EALA $G$ under a finite order automorphism $\sigma$ which satisfies certain conditions is of the form $G^\sigma = (\sum_{i=1}^{k} G^\sigma_i) \oplus W \oplus I$, where for each $i$, $G^\sigma_i$ satisfies axioms EA1–EA5(a) of an EALA, $I$ is a subspace of $G^\sigma$, and $W$ is a subspace of the center of $G^\sigma$ (see Theorem 2.65). This agrees with the result of [7] when $G$ is a finite dimensional simple Lie algebra over $\mathbb{C}$, as in this case $I = \{0\}$ and for each $i$, $G^\sigma_i$ is a finite dimensional simple Lie algebra (see Corollary 2.66). Even though we have not been able to figure out the Lie bracket multiplication behavior of some isotropic root spaces of $G^\sigma$, however we have shown that the core of the fixed point subalgebra modulo its center is isomorphic to the direct sum of the cores modulo centers of the involved summands. In Lemma 2.56 certain relations between the core of $G$ and the core of $G^\sigma$ and some results regarding the tameness of $G^\sigma$ in terms of the tameness of $G$ are obtained (see Definition 1.4 for terminologies).

In Section 3, the last section, we present some examples related to the results obtained in Section 2. In Examples 3.70, 3.74, and 3.75 we discuss the terms $G^\sigma_1$, $G^\sigma_2$, $W$ and $I$ (see Theorem 2.65) for some particular automorphisms of certain EALA. In 3.78 and 3.79 two examples from [4] are restated in such a way that they fit in the setting presented in Section 2. In 3.81, an example regarding the results in [2] is provided. This example shows how the affinizations in [2] can be viewed as fixed point subalgebras of the type considered here.
1. Terminology and prerequisites

In this section we give the definitions of an extended affine Lie algebra and an extended affine root system. We state some basic properties of these objects which will be of our use in the sequel. The reader can consult [1] for any results stated without proof.

Consider a triple $(G, (\cdot, \cdot), H)$ where $G$ is a Lie algebra over $\mathbb{C}$, $(\cdot, \cdot) : G \times G \to \mathbb{C}$ is a bilinear form, and $H$ is a subalgebra of $G$. Consider the following five axioms for this triple:

EA1. The form $(\cdot, \cdot)$ is symmetric, nondegenerate, and invariant on $G$.

EA2. $H$ is a nontrivial finite dimensional abelian subalgebra of $G$ which is self-centralizing and $\text{ad}(h)$ is diagonalizable for all $h \in H$.

Let $H^*$ be the dual space of $H$ and set

$$ G_\alpha = \{ x \in G \mid [h, x] = \alpha(h)x \text{ for all } h \in H \}. $$

Then we have from EA2 that $G = \bigoplus_{\alpha \in R} G_\alpha$ where

$$ R = \{ \alpha \in H^* \mid G_\alpha \neq \{0\} \} $$

is the so called root system of $G$. It follows from EA1 and EA2 that $G_0 = H$, $0 \in R$, and $(G_\alpha, G_\beta) = \{0\}$ unless $\alpha + \beta = 0$. In particular, $R = -R$ and the form restricted to $H$ is nondegenerate. This allows us to transfer the form to $H^*$ by letting

$$ (\alpha, \beta) = (t_\alpha, t_\beta) \quad \text{for } \alpha, \beta \in H^*, $$

where $t_\alpha$ is the unique element in $H$ with $\alpha(h) = (t_\alpha, h)$ for all $h \in H$. Now having the form invariant on $G$ and nondegenerate on $H$, one obtains

$$ [x_\alpha, x_{-\alpha}] = (x_\alpha, x_{-\alpha})t_\alpha \quad \text{for any } \alpha \in R, \ x_{\pm \alpha} \in G_{\pm \alpha}. \quad (1.1) $$

Then for $\alpha \in R^*$, one can find $e_{\pm \alpha} \in G_{\pm \alpha}$ such that

$$ (e_\alpha, e_{-\alpha}, h_\alpha := 2t_\alpha/(\alpha, \alpha)) \quad \text{is an } \mathfrak{sl}_2\text{-triple.} \quad (1.2) $$

Let $R^0$ be the set of isotropic roots, roots $\alpha \in R$ with $(\alpha, \alpha) = 0$, and $R^* = R \setminus R^0$.

The next axiom states that

EA3. For any $\alpha \in H^*$ and $x \in G_\alpha$, $\text{ad}_G(x)$ is locally nilpotent on $G$.

Assuming EA1–EA3, we have from [1, Theorem I.1.29] and [3, Lemma 1.3] that

$$ \dim(G_\alpha) = 1 \quad (\alpha \in R^*) \quad \text{and} \quad [G_\alpha, G_\beta] = G_{\alpha + \beta} \quad (\alpha, \beta, \alpha + \beta \in R^*). \quad (1.3) $$

Our last two axioms are related to the root system $R$:
EA4. $R$ is a discrete subset of $\mathcal{H}^*$.  

EA5. $R$ is irreducible, in the sense that it satisfies the following two conditions:

(a) $R^\times$ is indecomposable, that is $R^\times = R_1 \cup R_2$ with $(R_1, R_2) = \{0\}$ implies $R_1 = \emptyset$ or $R_2 = \emptyset$.  

(b) For any $\delta \in R^0$ there is some $\alpha \in R^\times$ with $\alpha + \delta \in R$.  

When there is no confusion we write $\mathcal{G}$ instead of a triple $(\mathcal{G}, (\cdot, \cdot), \mathcal{H})$ satisfying the above axioms (or a part of them).

Definition 1.4. Let $(\mathcal{G}, (\cdot, \cdot), \mathcal{H})$ satisfy axioms EA1–EA4. The core of $\mathcal{G}$ is by definition the subalgebra $\mathcal{G}_c$ of $\mathcal{G}$ generated by root spaces $\mathcal{G}_\alpha$, $\alpha \in R^\times$. $\mathcal{G}$ is called tame if $\mathcal{G}_c$ contains its centralizer in $\mathcal{G}$. Equivalently $\mathcal{G}$ is tame if the centralizer $C_\mathcal{G}(\mathcal{G}_c)$ of $\mathcal{G}_c$ equals the center $Z(\mathcal{G}_c)$ of $\mathcal{G}_c$. $\mathcal{G}$ is called nondegenerate if the real span of and the complex span of $R_0$ in $\mathcal{H}^*$ have the same dimension. It follows from the proof of [2, Lemma 3.26] that axioms EA1–EA4 plus tameness implies EA5(b).

An argument similar to [6, Lemma 1.26] shows that $\mathcal{G}_c$ is a perfect ideal of $\mathcal{G}$.

Definition 1.5. A triple $(\mathcal{G}, (\cdot, \cdot), \mathcal{H})$ satisfying axioms EA1–EA5 is called an extended affine Lie algebra (EALA for short).

Fix an EALA $(\mathcal{G}, (\cdot, \cdot), \mathcal{H})$ and let $V$ be the real span of $R$. Multiplying with some nonzero scaler, if necessary, we may assume that the form on $\mathcal{H}^*$ restricted to $V$ is real valued and positive semidefinite. Let $V^0$ be the real span of $R^0$ and let $\bar{\cdot} : V \rightarrow \bar{V} = V/V^0$ be the natural map. Then the induced form on $\bar{V}$ is positive definite and $\bar{R}$ is an irreducible finite root system in $\bar{V}$ ($\bar{R}$ contains 0 and is possibly nonreduced). The type of $\mathcal{G}$ is by definition the type of $\bar{R}$ and the nullity of $\mathcal{G}$ is defined to be the dimension of $V^0$. Using the finite root system $\bar{R}$, it is shown in [1] that $\mathcal{G}$ is an irreducible reduced extended affine root system in the sense of the following definition.

Definition 1.6. Let $V$ be a finite dimensional real vector space with a nontrivial positive semidefinite symmetric bilinear form $(\cdot, \cdot)$ and let $R$ be a subset of $V$. Let

$$R^\times = \{ \alpha \in R \mid (\alpha, \alpha) \neq 0 \} \quad \text{and} \quad R^0 = \{ \alpha \in R \mid (\alpha, \alpha) = 0 \}.$$  

Then $R = R^\times \uplus R^0$ where $\uplus$ means disjoint union. Then we will say $R$ is an extended affine root system (EARS) in $V$ if $R$ satisfies the following four axioms:

(R1) $R = -R$,

(R2) $R$ spans $V$,

(R3) $R$ is discrete in $V$,

(R4) if $\alpha \in R^\times$ and $\beta \in R$, then there exist $d, u \in \mathbb{Z}_{\geq 0}$ such that

$$\{ \beta + n\alpha \mid n \in \mathbb{Z} \} \cap R = \{ \beta - d\alpha, \ldots, \beta + u\alpha \} \quad \text{and} \quad d - u = \frac{2(\alpha, \beta)}{(\alpha, \alpha)}.$$
The EARS $R$ is called **tame** if it satisfies:

(R5) For any $\delta \in R^0$, there exists $\alpha \in R^\times$ such that $\alpha + \delta \in R$. We say a root $\delta \in R^0$ satisfying this condition is **nonisolated** and call isotropic roots which do not satisfy this condition **isolated**.

The EARS $R$ is called **indecomposable** if it satisfies:

(R6) $R^\times$ cannot be decomposed into a disjoint union of two nonempty subsets which are orthogonal with respect to the form.

A tame indecomposable EARS $R$ is called **irreducible**. Finally, the EARS $R$ is called **reduced** if it satisfies:

(R7) $\alpha \in R^\times \Rightarrow 2\alpha \notin R$.

Since the form is nontrivial, it follows from (R2) that $R^\times \neq \emptyset$. This, together with (R1) and (R4), implies that $0 \in R$. Note that an EARS as used here could have both isolated and nonisolated roots. Thus axiom (R5) indicates that in a tame EARS isotropic roots are non-isolated. By [5], there is a one to one correspondence between irreducible (reduced) EARS and indecomposable (reduced) extended affine root systems defined by K. Saito [12]. As we will see in the sequel, such root systems will naturally arise as the root systems of the fixed point subalgebras (under certain finite order automorphisms) of some extended affine Lie algebras. Also, such root systems arise in [14].

Let $R$ be an irreducible EARS in $\mathcal{V}$. As before let $\mathcal{V}^0$ be the radical of the form $(\cdot, \cdot)$ on $\mathcal{V}$ and $\bar{\mathcal{V}} := \mathcal{V}/\mathcal{V}^0$, and let $\bar{v}$ be the image of an element $v$ of $\mathcal{V}$ under the projection map $\mathcal{V} \to \bar{\mathcal{V}}$. For $\alpha, \beta \in \mathcal{V}$ define $(\bar{\alpha}, \bar{\beta}) := (\alpha, \beta)$. Then $(\cdot, \cdot)$ is positive definite on $\bar{\mathcal{V}}$ and $\bar{R}$ is an irreducible finite root system in $\bar{\mathcal{V}}$ (which might not be necessarily reduced). One may find a preimage $\bar{R}$ of $\bar{R}$ in $\bar{\mathcal{V}}$ such that if $\bar{\mathcal{V}} = \bar{\mathcal{V}}^0 \oplus \bar{\mathcal{V}}^1$ and the map $\bar{\mathcal{V}}$ maps $\bar{R}$ isometrically onto $\bar{R}$, so $\bar{R}$ is a finite root system in $\bar{\mathcal{V}}$ isomorphic to $\bar{R}$. It is shown in [1, Chapter II] that the structure of $R$ can be described in terms of $\bar{R}$ and some subsets of $\mathcal{V}^0$ attached to roots in $\bar{R}$. In particular, if $\bar{\alpha} \in \bar{R} \setminus \{0\}$ is of minimal length and

$$S = \{ \sigma \in \mathcal{V}^0 \mid \bar{\alpha} + \sigma \in R \},$$

then $S$ is a semilattice in $\mathcal{V}^0$ of rank $v = \dim(\mathcal{V}^0)$. That is, $S$ is a discrete subset of $\mathcal{V}^0$ which contains 0, spans $\mathcal{V}^0$ and $S \pm 2S \subset S$ (see [1, II.§1] for details). Moreover, by [1, Corollary II.2.31],

$$R^0 = S + S. \quad (1.7)$$

Let us denote by $T^\times$ the set of nonisotropic roots of any subset $T$ of an EARS $R$.

**Lemma 1.8.** Let $R$ be an EARS and $R_1$ be a subset of $R$ with $R_1^\times \neq \emptyset$. Suppose that
(a) \( R_1 = -R_1 \),
(b) \( \{ \delta \in R^0 \mid \alpha' + \delta \in R_1 \text{ for some } \alpha' \in R_1^\times \} \subseteq R_1 \),
(c) \( \alpha' \in R_1, \beta \in R, (\alpha', \beta) \neq 0 \Rightarrow \beta \in R_1 \).

Then \( R_1 \) is an EARS in its real span. Moreover, if we set

\[
R_1' = R_1^\times \cup (\langle R_1 \rangle \cap R^0),
\]

then \( R_1' \) is also an EARS in the real span of \( R_1 \) ((\( R_1 \)) denotes the \( \mathbb{Z} \)-span of \( R_1 \)).

Proof. Clearly (R1)–(R3) hold for \( R_1 \). We now check (R4). Let \( \alpha' \in R_1^\times \) and \( \beta' \in R_1 \).

Since (R4) holds for \( R \), it is enough to show that for \( n \in \mathbb{Z} \),

\[
\beta' + n\alpha' \in R \implies \beta' + n\alpha' \in R_1.
\]

Since \( \beta' \in R_1 \), we may assume that \( n \neq 0 \), and by (a), we may also assume that \( n > 0 \). So let \( \beta' + na' \in R, n > 0 \). If \( \beta' + na' \in R^\times \), then \( (\beta' + na', \beta') \neq 0 \) or \( (\beta' + na', \alpha') \neq 0 \). In either case, we get from (c) that \( \beta' + na' \in R_1 \). Next, let \( \beta' + na' \in R^0 \). Since (R4) holds for \( R \) and \( n > 0 \), we have \( \beta' + (n - 1)\alpha' \in R^\times \). So repeating our previous argument, we get \( \beta' + (n - 1)\alpha' \in R_1 \). Since

\[
\beta' + na' + (-\alpha') = \beta' + (n - 1)\alpha' \in R_1^\times,
\]

it follows from (a) and (b) that \( \beta' + na' \in R_1 \). This completes the proof of the first assertion.

Next let \( R_1' \) be as in the statement. Clearly \( R_1 \) and \( R_1' \) have the same real span. Since \( R_1^\times = (R_1')^\times \), it is easy to check that \( R_1' \) satisfies conditions (a)–(c), and so is an EARS.

For an EARS \( R \), we set

\[
R_{\text{iso}} = \{ \delta \in R^0 \mid \alpha + \delta \notin R \text{ for any } \alpha \in R^\times \},
\]

\[
R_{\text{niso}} = \{ \delta \in R^0 \mid \alpha + \delta \in R \text{ for some } \alpha \in R^\times \} = R^0 \setminus R_{\text{iso}}.
\]

That is \( R_{\text{iso}} (R_{\text{niso}}) \) is the set of isolated (nonisolated) isotropic roots of \( R \), so \( R \) is tame if and only if \( R_{\text{iso}} = \emptyset \). We also set

\[
R_1 = R^\times \cup R_{\text{iso}}.
\]

Then \( R_1^\times = R^\times \) and

\[
R = R^\times \cup R_{\text{niso}} \cup R_{\text{iso}} = R_1 \cup R_{\text{iso}}.
\]

Lemma 1.10. Let \( R \) be an EARS. Then

(i) \( R_1 \) is a tame EARS in its real span.
(ii) $R = \bigcup_{i=1}^{k} R_i \cup R_{iso}$, where each $R_i$ is an irreducible EARS. Moreover, $R^\times = \bigcup_{i=1}^{k} R_i^\times$ where for $i \neq j$, $R_i$ and $R_j$ are orthogonal with respect to the form. Furthermore, if we set

$$R_i' = R_i^\times \cup (R_i \cap R^0),$$

(1.11) then $R_i'$ is an indecomposable EARS.

**Proof.** (i) It follows from definition of $R_t$ that conditions (a)–(c) of Lemma 1.8 hold for $R_t$. So $R_t$ is an EARS. It is clear from definition that $R_t$ is tame.

(ii) We have $R = R_t \cup R_{iso}$, and by part (i), $R_t$ is a tame EARS. Since $\bar{R}_t = \bar{R}$ is a finite root system, we have $\bar{R}_t \setminus \{0\} = \bigcup_{i=1}^{k} \bar{R}_i$, where for each $i$, $R_i \cup \{0\}$ is an irreducible finite root system, and $\bar{R}_i$’s are orthogonal with respect to the form. Let $R_i^\times$ be the preimage in $R^\times$ of $\bar{R}_i$ under the projection map $\bar{\cdot}$. Then

$$R_i^\times = \bigcup_{i=1}^{k} R_i^\times \quad \text{and} \quad (R_i^\times, R_j^\times) = \{0\} \quad \text{if} \quad i \neq j.$$  

(1.12)

Set

$$R_i = R_i^\times \cup \{\delta \in R^0_i \mid \delta + \alpha \in R_i \text{ for some } \alpha \in R_i^\times\}.$$ (1.13)

Since $R_i$ is tame, the isotropic roots of $R_i$ are nonisolated so

$$R_i = \bigcup_{i=1}^{k} R_i.$$  

It is easy to see that conditions (a)–(c) of Lemma 1.8 hold for $R_i$. Thus $R_i$ is an EARS. □

2. Fixed point subalgebras

In this section we study the structure of fixed point subalgebra of an extended affine Lie algebra under a finite order automorphism which satisfies certain conditions.

Consider a fixed EALA $(\mathcal{G}, (\cdot, \cdot), \mathcal{H})$ with root system $R$ and the corresponding root space decomposition $\mathcal{G} = \bigoplus_{\alpha \in \mathcal{H}^*} \mathcal{G}_{\alpha}$. Consider a fixed automorphism $\sigma$ of $\mathcal{G}$ and set

$$\mathcal{G}^{\sigma} = \{x \in \mathcal{G} \mid \sigma(x) = x\} \quad \text{and} \quad \mathcal{H}^{\sigma} = \{h \in \mathcal{H} \mid \sigma(h) = h\}.$$ (2.14)

That is $\mathcal{G}^{\sigma}$ (respectively $\mathcal{H}^{\sigma}$) is the fixed point subalgebra of $\mathcal{G}$ (respectively $\mathcal{H}$) with respect to $\sigma$.

Let $m \geq 1$ and suppose that

(A1) $\sigma^m = 1$.

(A2) $\sigma(\mathcal{H}) = \mathcal{H}$.
(A3) \((\sigma(x), \sigma(y)) = (x, y)\) for all \(x, y \in G\).

(A4) \(C_{G^\sigma}(\mathcal{H}^\sigma) = \mathcal{H}^\sigma\).

We start by recording some facts related to (A1)–(A3), so we only assume that (A1)–(A3) hold for now. Let \(i\) denote the image of \(i \in \mathbb{Z}\) in \(\mathbb{Z}/m\mathbb{Z}\). From (A1) and (A2), we have

\[
G = \bigoplus_{i=0}^{m-1} G_i \quad \text{and} \quad H = \bigoplus_{i=0}^{m-1} H_i,
\]

(2.15)

where \(G_i\) (respectively \(H_i\)) is the eigenspace corresponding to the \(i\)th power of the \(m\)th root of unity \(\zeta = e^{2\sqrt{-1}\pi/m}\). Then

\[
G_i = \{ x \in G \mid \sigma(x) = \zeta^i x \}.
\]

Note that \(G^\sigma = G_0\) and \(H^\sigma = H_0\). It follows from (A3) that

\[
(G_i, G_j) = [0] = (H_i, H_j) \quad \text{if} \ i + j \neq 0.
\]

(2.16)

Set \(G^c = \bigoplus_{i=1}^{m-1} G_i\) and \(H^c = \bigoplus_{i=1}^{m-1} H_i\). Then \(G = G^\sigma \oplus G^c\) and \(H = H^\sigma \oplus H^c\). Moreover,

\[
(G^\sigma, G^c) = [0] = (H^\sigma, H^c).
\]

(2.17)

Note also that \(\sigma\) induces an automorphism \(\sigma \in \text{Aut}(\mathcal{H}^*)\) by

\[
\sigma(\alpha)(h) = \alpha(\sigma^{-1}(h)) \quad \text{for} \ \alpha \in \mathcal{H}^* \text{ and } h \in \mathcal{H}.
\]

(2.18)

Let \(\pi\) denote the orthogonal projection map from \(G\) (respectively \(H\)) onto \(G^\sigma\) (respectively \(H^\sigma\)). Then for \(x \in G\) we have \(x - \pi(x) \in G^c\). Since \(G^\sigma\) is stable under \(\sigma\), we obtain

\[
\sum_{i=0}^{m-1} \sigma^i(x - \pi(x)) \in G^c \cap G^\sigma = [0].
\]

(2.19)

So since \(\sigma \pi = \pi\),

\[
\pi(x) = \frac{1}{m} \sum_{i=0}^{m-1} \sigma^i(x) \quad (x \in G).
\]

Similarly,

\[
\pi(h) = \frac{1}{m} \sum_{i=0}^{m-1} \sigma^i(h) \quad (h \in \mathcal{H}).
\]
The map $\pi$ induces a map, denoted again by $\pi$, from the dual space $\mathcal{H}^*$ of $\mathcal{H}$ onto the dual space $(\mathcal{H}^\sigma)^*$ of $\mathcal{H}^\sigma$. Namely, for $\alpha \in \mathcal{H}^*$ and $h \in \mathcal{H}^\sigma$, define

$$\pi(\alpha)(h) = \alpha(h),$$

(2.21)

that is $\pi(\alpha)$ for $\alpha \in \mathcal{H}^*$ is the restriction of $\alpha$ to $\mathcal{H}^\sigma$. Then from (2.18) we have, for $\alpha \in \mathcal{H}^*$ and $h \in \mathcal{H}^\sigma$,

$$\pi(\alpha)(h) = \alpha(\pi(h)) = \alpha\left(\frac{1}{m} \sum_{i=0}^{m-1} \sigma^i(h)\right) = \frac{1}{m} \sum_{i=0}^{m-1} \sigma^i(\alpha)(h).$$

Thus for $\alpha \in \mathcal{H}^*$, we have

$$\pi(\alpha) = \frac{1}{m} \sum_{i=0}^{m-1} \sigma^i(\alpha).$$

(2.22)

Recall that we have used the notation $\mathcal{G}_\alpha$, $\alpha \in \mathcal{H}^*$, for the $\alpha$-root spaces relative to the adjoint action of $\mathcal{H}$ on $\mathcal{G}$. Let us also denote the same notation for the root spaces of $\mathcal{G}^\sigma$ corresponding to the adjoint action of $\mathcal{H}^\sigma$ on $\mathcal{G}^\sigma$. This might be confusing since we have identified $(\mathcal{H}^\sigma)^*$ as a subspace of $\mathcal{H}^*$. However, to prevent this we always use the notation $\tilde{\alpha}$ or $\pi(\alpha)$ for roots in $(\mathcal{H}^\sigma)^*$ and so we write

$$\mathcal{G}_{\tilde{\alpha}} = \{x \in \mathcal{G} \mid [h, x] = \tilde{\alpha}(h)x \text{ for all } h \in \mathcal{H}^\sigma\}.$$

Since both $\mathcal{G}$ and $\mathcal{G}^\sigma$ are $\mathcal{H}^\sigma$-modules, we have

$$\mathcal{G} = \sum_{\tilde{\alpha} \in (\mathcal{H}^\sigma)^*} \mathcal{G}_{\tilde{\alpha}} = \sum_{\alpha \in \mathcal{H}^*} \mathcal{G}_{\pi(\alpha)} = \sum_{\alpha \in R} \mathcal{G}_{\pi(\alpha)} \quad \text{and} \quad \mathcal{G}^\sigma = \sum_{\alpha \in R} \mathcal{G}_{\pi(\alpha)},$$

(2.23)

where

$$\mathcal{G}_{\pi(\alpha)} = \mathcal{G}_\alpha \cap \mathcal{G}_{\pi(\alpha)}.$$

Note that $\mathcal{G}^\sigma$ as an $\mathcal{H}^\sigma$-submodule of $\mathcal{G}$ has a weight space decomposition

$$\mathcal{G}^\sigma = \bigoplus_{\tilde{\alpha} \in (\mathcal{H}^\sigma)^*} \mathcal{G}_{\tilde{\alpha}}^\sigma,$$

(2.24)

where

$$\mathcal{G}_{\tilde{\alpha}}^\sigma = \mathcal{G}^\sigma \cap \mathcal{G}_{\tilde{\alpha}}.$$

Set

$$R^\sigma = \{\tilde{\alpha} \in (\mathcal{H}^\sigma)^* \mid \mathcal{G}_{\tilde{\alpha}}^\sigma \neq [0]\} \subseteq \pi(R).$$
We will see in the next section that in general $R^\sigma$ may be a proper subset of the fixed point set $\pi(R)$ of $\sigma$ in $R$, so here the upper index $\sigma$ does not stand for the fixed points of $\sigma$ on $R$.

Denote the set of nonisotropic (isotropic) roots of $R^\sigma$ by $(R^\sigma)^\times$ ($(R^\sigma)^0$), respectively. That is,

$$(R^\sigma)^\times = \{ \tilde{\alpha} \in R^\sigma \mid (\tilde{\alpha}, \tilde{\alpha}) \neq 0 \} \quad \text{and} \quad (R^\sigma)^0 = \{ \tilde{\alpha} \in R^\sigma \mid (\tilde{\alpha}, \tilde{\alpha}) = 0 \}.$$  

We should mention that the form $(\cdot, \cdot)$ on $(H^\sigma)^\star$ here has two possible interpretations. We may consider it either as the form transferred from $H^\sigma$ or the form obtained from $H^\star$ by restriction. However, both are the same.

It is easy to see that

$$\sigma(G_\alpha) = G_{\sigma(\alpha)} \quad (\alpha \in R), \quad (2.25)$$

and so

$$\sigma(R) = R.$$  

Also we have

$$\pi(G_\alpha) \subseteq G_{\sigma(\pi(\alpha))}.$$  

Thus

$$G_{\sigma(\pi(\alpha))} = \sum_{\beta \in R \atop \pi(\beta) = \pi(\alpha)} \pi(G_\beta). \quad (2.26)$$

Since the form $(\cdot, \cdot)$ is nondegenerate (by (2.17)) and invariant on $G^\sigma$, it follows that

$$(G^\sigma_\alpha, G^\sigma_\beta) = \{ 0 \} \quad \text{unless} \quad \bar{\alpha} + \bar{\beta} = 0. \quad (2.27)$$

In particular, we have

$$R^\sigma = -R^\sigma. \quad (2.28)$$

Also for any $\alpha, \beta \in R$,

$$[G^\sigma_{\pi(\alpha)}, G^\sigma_{\pi(\beta)}] = 0 \quad \text{or} \quad [G^\sigma_{\pi(\alpha)}, G^\sigma_{\pi(\beta)}] \subseteq G^\sigma_{\pi(\gamma)}, \quad (2.29)$$

for some $\gamma \in R$ with $\pi(\alpha) + \pi(\beta) = \pi(\gamma)$. Then from (2.26) and (2.24), we have

$$G^\sigma = \sum_{\alpha \in R} G^\sigma_{\pi(\alpha)} = \bigoplus_{\tilde{\alpha} \in (H^\star)^\times} G^\sigma_{\tilde{\alpha}} = \bigoplus_{\tilde{\alpha} \in R^\sigma} G^\sigma_{\tilde{\alpha}}. \quad (2.30)$$
For \( \alpha \in H^* \) let \( t_\alpha \) be the unique element in \( H \) which represents \( \alpha \) via the nondegenerate form \( (\cdot, \cdot) \). That is,

\[ \alpha(h) = (t_\alpha, h) \quad (h \in H). \]

Then for \( \alpha \in H^* \),

\[ \sigma(t_\alpha) = t_{\sigma(\alpha)} \quad \text{and} \quad \pi(t_\alpha) = t_{\pi(\alpha)}. \quad (2.31) \]

So

\[ H^\sigma = \{ t_\alpha \mid \alpha \in (H^\sigma)^* \} \quad \text{and} \quad H^c = \{ t_\alpha \mid \alpha \in (H^c)^* \}. \quad (2.32) \]

Transfer the form \( (\cdot, \cdot) \) to \( H^* \) by

\[ (\alpha, \beta) = (t_\alpha, t_\beta) \quad (\alpha, \beta \in H^*). \]

Now, similar to (2.17), we have

\[ H^* = (H^*)^\sigma \oplus (H^*)^c \quad \text{and} \quad ((H^*)^\sigma, (H^*)^c) = \{0\}. \quad (2.33) \]

Identifying \( (H^*)^\sigma \) with \( (H^*^\sigma)^\sigma \) and \( (H^*)^c \) with \( (H^*^c)^c \), we see that the map \( \pi \) on \( H^* \) is in fact the restriction to \( (H^*^\sigma)^\sigma \).

**Lemma 2.34.** Let \( \alpha \in R \) and \( (\pi(\alpha), \pi(\alpha)) \neq 0 \). Then for \( x \in G^\sigma_{\pi(\alpha)} \), \( \text{ad}_{G^\sigma} (x) \) acts locally nilpotently on \( G^\sigma \).

**Proof.** Fix \( 0 \neq x \in G^\sigma_{\pi(\alpha)} \). By (2.30), it is enough to show that if \( \beta \in R \) and \( y \in G^\sigma_{\pi(\beta)} \), then \( (\text{ad} x)^N(y) = 0 \) for sufficiently large \( N \). By (2.31) and (2.32), \( h := t_{\pi(\alpha)} \in H^\sigma \) and so

\[ [h, (\text{ad} x)^k(y)] = (k(\pi(\alpha), \pi(\alpha)) + \pi(\beta)(h))(\text{ad} x)^k(y). \]

According to (2.30) the eigenvalues of \( \text{ad}_{G^\sigma} h \) are of the form \( \pi(\gamma)(h), \gamma \in R \). Now

\[ \pi(\gamma)(h) = (\pi(\alpha), \pi(\gamma)) = \frac{1}{m} \sum_{i=0}^{m-1} (\alpha, \sigma^i(\gamma)). \]

But \( \tilde{R} \) is a finite root system, so \( \{(\alpha, \gamma) \mid \gamma \in R\} = \{\tilde{\alpha}, \tilde{\gamma} \mid \gamma \in R\} \) is a finite set. This shows that \( \text{ad}_{G^\sigma} h \) has only a finite number of eigenvalues, and so \( (\text{ad} x)^N(y) = 0 \) for sufficiently large \( N \). \( \square \)

The parts (ii) and (iii) of the following lemma are proved in [2].

**Lemma 2.35.** Let \( \pi(R)^\sigma \neq \emptyset \). Then
(i) \( \pi(R) \) is a discrete subset of \((H^\sigma)^*\). In particular \( R^\sigma \) as a subset of \( \pi(R) \) is discrete.

(ii) For \( \beta \in R^\sigma \) there exists \( \alpha \in R \) such that \( \pi(\alpha), \pi(\alpha) \neq 0 \) and \( (\alpha, \beta) \neq 0 \).

(iii) \( \pi(R)^\times \) is indecomposable. That is, \( \pi(R)^\times \) cannot be written as a disjoint union of two nonempty subsets which are orthogonal with respect to the form.

**Proof.** (i) This is clear as

\[
\pi(\alpha) = \frac{1}{m} \sum_{i=0}^{m-1} \sigma^i(\alpha) \in \frac{1}{m} \langle R \rangle,
\]

and \( \langle R \rangle \) is a discrete subset of \( H^\sigma \). For parts (ii) and (iii) see [2, Lemma 3.49].

Up to now we have only used properties (A1)–(A3) of the automorphism \( \sigma \). From now on we assume that \( \sigma \) satisfies (A1)–(A4).

As we will see in the next proposition, the conditions (A1)–(A4) imply that \((G^\sigma, (\cdot, \cdot), H^\sigma)\) satisfies axioms EA1–EA4 of an EALA. It is shown in [1, Chapter I] that most of the structural properties of an EALA rely on these axioms.

Since \( \sigma(R) = R \), \( \sigma \) restricted to \( V \) is an automorphism of \( V \). Then \( \pi(V) \) is the fixed point subspace of \( V \) under \( \sigma \). Set

\[
\nu^\sigma := \text{span}_R R^\sigma \quad \text{and} \quad \nu^\pi := \text{span}_R \pi(R) = \pi(V).
\]

We will see in the next section that in general \( \nu^\sigma \) may be a proper subspace of \( \nu^\pi \), so here the upper index \( \sigma \) does not stand for the fixed points of \( \sigma \) on \( V \).

**Proposition 2.37.** Let \((G, (\cdot, \cdot), H)\) be an EALA and \( \sigma \) be an automorphism of \( G \) such that (A1)–(A4) hold.

(i) If \( (R^\sigma)^\times \neq \emptyset \), then \((G^\sigma, (\cdot, \cdot), H^\sigma)\) satisfies axioms EA1–EA4 of an EALA.

(ii) If \( (R^\sigma)^\times \neq \emptyset \), then \( R^\sigma \) is a reduced EARS in \( \nu^\sigma \) (in the sense of Definition 1.6).

**Proof.** (i) EA1 holds for \( G^\sigma \) by (2.17) and the fact that the form \((\cdot, \cdot)\) is nondegenerate and invariant on \( G \). EA2 holds by (A4) and (2.30). EA3 holds by Lemma 2.34. Finally, EA4 holds by Lemma 2.35.

(ii) Since \( (R^\sigma)^\times \neq \emptyset \), the form is nontrivial and positive semidefinite on \( \nu^\sigma \). By (2.28), (R1) holds for \( R^\sigma \). (R2) holds by the way \( \nu^\sigma \) is defined. Since \( R^\sigma \subseteq \pi(R) \), (R3) holds by Lemma 2.35. By part (i), \( G^\sigma \) satisfies the axioms EA1–EA4 of an EALA. Therefore by [1, Theorem 1.29], the root system \( R^\sigma \) of \( G^\sigma \) satisfies (R4) and (R7).

As in (1.9), \( R^\sigma_{\text{iso}} (R^\sigma_{\text{nio}}) \) denotes the set of isolated (nonisolated) isotropic roots of \( R^\sigma \).

**Corollary 2.38.** Let \( (R^\sigma)^\times \neq \emptyset \). Then

\[
R^\sigma = \left( \bigcup_{i=1}^{k} R^\sigma_i \right) \cup R^\sigma_{\text{iso}},
\]

(2.39)
where for each $i$, $R_σ^i$ is an irreducible reduced EARS in the real span $V_σ^i$ of $R_σ^i$, with $(R_σ^i, R_σ^j) = \{0\}$ if $i \neq j$. Moreover, if

$$ R_i := R_σ^i \cup (\langle R_σ^i \rangle \cap (R_σ^i)^0), $$

(2.40)

then $R_i$ is a reduced indecomposable EARS in $V_σ^i$.

**Proof.** See Proposition 2.37(ii) and Lemma 1.10. □

Note that $R_σ^i \subseteq R_i$ and $\text{Span}_R R_i = \text{Span}_R R_σ^i = V_σ^i$.

(2.41)

Since $R_σ^i$ is an EARS in $V_σ^i$, we may use the same notation as in Section 1. So $\hat{R}_σ^i$ is an irreducible finite root system in $\hat{V}_σ^i$. Fix a choice of a fundamental system $\hat{\Pi}_i$ for $\hat{R}_σ^i$. Also choose a fixed preimage $\hat{\Pi}_i$ in $R_σ^i$ of $\hat{\Pi}_i$, under $\hat{\cdot}$. Let $\hat{V}_σ^i$ be the real span of $\hat{\Pi}_i$. Then

$$ V_σ^i = \hat{V}_σ^i \oplus (V_σ^i)^0 \quad \text{and} \quad (V_σ^i, V_σ^j) = \{0\} \quad \text{for} \quad i \neq j. $$

(2.42)

Then $\hat{\cdot}$ restricts to an isometry of $\hat{V}_σ^i$ onto $\hat{V}_σ^i$. Let

$$ \hat{R}_σ^i = \{ \hat{\alpha} \in \hat{V}_σ^i \mid \hat{\alpha} + \delta \in R_σ^i \text{ for some } \delta \in (V_σ^i)^0 \}. $$

Then $\hat{R}_σ^i$ is a finite root system in $\hat{V}_σ^i$ which is isometrically isomorphic under $\hat{\cdot}$ to $\hat{R}_σ^i$. Set

$$ (R_σ^i)^0 = R_σ^i \cap (V_σ^i)^0. $$

Let $v_i = \dim(V_σ^i)^0$ and $l_i = \dim \hat{V}_σ^i$. Then by [1, Corollary II.2.31], $(R_σ^i)^0 = S_i + S_1$, where $S_i$ is a semilattice in $(V_σ^i)^0$ of rank $v_i$ (see also 1.7). Also the $\mathbb{Z}$-span of $S_i$ is a free abelian group of finite rank with a basis $B_i \subseteq S_i$. Now

$$ B_i \cup \hat{\Pi}_i \subseteq R_i \subseteq (\hat{\mathcal{H}}_σ)^* \quad \text{and} \quad R_σ^i \subseteq (B_i \cup \hat{\Pi}_i). $$

(2.43)

Corresponding to the subspaces $\hat{V}_σ^i$ and $(V_σ^i)^0$ of $(\hat{\mathcal{H}}_σ)^*$ we consider two subspaces $(\hat{V}_σ^i)_C$ and $(V_σ^i)^0_C$ of $\mathcal{H}_σ$ as follows. Let

$$ (\hat{V}_σ^i)_C = \sum_{a \in \hat{\Pi}_i} C\alpha_t \subseteq \mathcal{H}_σ \quad \text{and} \quad (V_σ^i)^0_C = \sum_{\delta \in B_i} C\delta \subseteq \mathcal{H}_σ. $$

(2.44)

Set

$$ \hat{V}_σ^i_C = \sum_{i=1}^k (\hat{V}_σ^i)_C \quad \text{and} \quad (V_σ^i)^0_C = \sum_{i=1}^k (V_σ^i)^0_C. $$
Lemma 2.45. Suppose $(R^n)^\times \neq \emptyset$. Then

$$\hat{V}_C^0 \oplus (V_C^0)^0 = \sum_{\tilde{\alpha} \in (R^n)^\times} [G_{\tilde{\alpha}}, G_{-\tilde{\alpha}}].$$

Proof. Since $G^n$ satisfies axioms EA1–EA4 of an EALA (see Proposition 2.37), we have from [1, Chapter I] that if $\tilde{\alpha} \in (R^n)^\times$ then $[G_{\tilde{\alpha}}, G_{-\tilde{\alpha}}] = C_{t\tilde{\alpha}}$. So if $\tilde{\alpha} \in \tilde{H}_i \subset (R^n)^\times$, then $t\tilde{\alpha} \in [G_{\tilde{\alpha}}, G_{-\tilde{\alpha}}]$. If $\delta \in S_i$ then from (1.7), we have $\tilde{\alpha} + \delta \in (R^n)^\times \subseteq (R^n)^\times$ for some $\tilde{\alpha} \in \tilde{H}_i$. Then $t\tilde{\alpha} + \delta \in [G_{\tilde{\alpha} + \delta}, G_{-\tilde{\alpha} - \delta}]$. Thus $t\tilde{\alpha}$ is contained in the $C$-span of $[G_{\tilde{\alpha}}, G_{-\tilde{\alpha}}]$, $\tilde{\alpha} \in (R^n)^\times$ and so $\hat{V}_C^0 \oplus (V_C^0)^0$ is contained in the sum appearing in the statement. Conversely, let $\tilde{\alpha} \in (R^n)^\times$. Since $R^n_{\text{iso}} = \emptyset$, $\tilde{\alpha} \in R^n_i$ for some $1 \leq i \leq k$. So $\tilde{\alpha}$ is in the real span of $\tilde{H}_i$. So we choose a basis $B$ of $V_{\tilde{H}_i}$.

Set

$$(R^n)_0^0 = R^n_{\text{iso}} \setminus \left( \bigcup_{i=1}^k (R_i)_{\text{iso}} \right)$$

Assume that $(R^n)^\times \neq \emptyset$. Since the form $(\cdot, \cdot)$ is real valued and positive definite on $\hat{V}_C^0$, it follows that the form on $\hat{H}_C$ restricted to $(\hat{V}_C^0)_C$ is nondegenerate, and so by (2.42) the form on $\hat{V}_C^0$ is nondegenerate. Since the form $(\cdot, \cdot)$ is nondegenerate on $\hat{H}_C^0$, and $(\hat{V}_C^0 \oplus (V_C^0)^0, (V_C^0)^0) = [0]$, it follows that there exists a subspace $D$ of $\hat{H}_C$ such that

$$\dim D = \dim (V_C^0)^0, \quad (D, D + \hat{V}_C^0) = [0],$$

$(\cdot, \cdot)$ is nondegenerate on $\hat{V}_C^0 \oplus (V_C^0)^0 \oplus D$. \hfill (2.46)

So we may choose a subspace $W$ of $\hat{H}_C$ such that

$$\hat{H}_C = \hat{V}_C^0 \oplus (V_C^0)^0 \oplus D \oplus W,$$

$$(\hat{V}_C^0 \oplus (V_C^0)^0 \oplus D, W) = [0],$$

$(\cdot, \cdot)$ is nondegenerate on $W$. \hfill (2.47)

Next consider a basis $B = \{h_1, \ldots, h_m\}$ of $(V_C^0)^0$ such that $B$ contains a basis of $(V_C^0)^0$ for each $i$. Using (2.46), we can pick a basis $B' = \{d_1, \ldots, d_m\}$ of $D$ such that $(h_i, d_j) = \delta_{ij}$. Set

$$D_i = \text{span}_C \{d_j \in B' \mid h_j \in (V_C^0)^0\}.$$  

Then $D = \sum_{i=1}^k D_i$ and
\[ \dim D_i = \dim (V_i^\sigma)_C^0, \quad (D_i, D \oplus \hat{V}_i^\sigma) = \{0\}. \]

(\cdot, \cdot) is nondegenerate on \((V_i^\sigma)_C^0 \oplus (V_i^\sigma)_C \oplus D_i. \tag{2.48} \]

Set
\[ \mathcal{H}_i^\sigma = (\hat{V}_i^\sigma)_C \oplus (V_i^\sigma)_C \oplus D_i. \tag{2.49} \]

Then the form on \(\mathcal{H}_i^\sigma\) is nondegenerate and
\[ \mathcal{H}^\sigma = \left( \sum_{i=1}^k \mathcal{H}_i^\sigma \right) \oplus \mathcal{W}. \]

Put
\[ \mathcal{G}_i^\sigma = \mathcal{H}_i^\sigma \oplus \left( \sum_{\tilde{\alpha} \in R_i \setminus \{0\}} \mathcal{G}_{\tilde{\alpha}}^\sigma \right).

**Proposition 2.50.**

(i) \(\mathcal{G}_i^\sigma\) is a subalgebra of \(\mathcal{G}^\sigma\).

(ii) \(\mathcal{H}_i^\sigma\) is an abelian subalgebra of \(\mathcal{G}_i^\sigma\). Moreover, the form restricted to \(\mathcal{H}_i^\sigma\) is nondegenerate.

(iii) \(C_{\mathcal{G}^\sigma} (\mathcal{H}_i^\sigma) = \mathcal{H}_i^\sigma\).

**Proof.** (i) Set
\[ K_i = \sum_{\tilde{\alpha} \in R_i \setminus \{0\}} \mathcal{G}_{\tilde{\alpha}}^\sigma \quad \text{and} \quad T_i = \sum_{\tilde{\alpha} \in R_i} C_{\tilde{\alpha}} = \sum_{\tilde{\alpha} \in R_i} C_{\tilde{\alpha}} = \sum_{\tilde{\alpha} \in (R_i^\sigma)^{\times}} C_{\tilde{\alpha}}, \]

and let \(\mathcal{M}_i\) be the subalgebra of \(\mathcal{G}^\sigma\) generated by \(K_i\). By (2.43) and (2.44),
\[ T_i = (\hat{V}_i^\sigma)_C \oplus (V_i^\sigma)_C^0. \]

We first claim that \(\mathcal{M}_i = K_i \oplus T_i\). To see this note that by Proposition 2.37, EA1–EA4 hold for \(\mathcal{G}^\sigma\), therefore for any \(\tilde{\alpha} \in R^\sigma\),
\[ [\mathcal{G}_{\tilde{\alpha}}^\sigma, \mathcal{G}_{\tilde{\beta}}^\sigma] = C_{\tilde{\alpha}} \]
(see (1.1)). So \(T_i \subseteq \mathcal{M}_i\). Thus \(T_i \oplus K_i \subseteq \mathcal{M}_i\). To see the equality it is enough to show that \(T_i \oplus K_i\) is closed under [ , ]. But \(T_i \subseteq \mathcal{H}^\sigma\), so \([T_i, T_i \oplus K_i] \subseteq K_i\). To see \([K_i, K_i] \subseteq T_i \oplus K_i\), let \(\tilde{\alpha}, \tilde{\beta} \in R_i \setminus \{0\}\), and \([\mathcal{G}_{\tilde{\alpha}}^\sigma, \mathcal{G}_{\tilde{\beta}}^\sigma] \neq \{0\}\). In particular, \(\tilde{\alpha} + \tilde{\beta} \in R^\sigma\). If \(\tilde{\beta} = -\tilde{\alpha}\), then \([\mathcal{G}_{\tilde{\alpha}}^\sigma, \mathcal{G}_{\tilde{\beta}}^\sigma] = C_{\tilde{\alpha}} \subseteq T_i\). So we may assume that \(\tilde{\beta} + \tilde{\alpha} \neq 0\). If \(\tilde{\alpha} + \tilde{\beta}\) is nonisotropic, then
either \((\tilde{\alpha} + \tilde{\beta}, \tilde{\alpha}) \neq 0\) or \((\tilde{\alpha} + \tilde{\beta}, \tilde{\beta}) \neq 0\). In either case \(\tilde{\alpha} + \tilde{\beta}\) is a nonisotropic root in \(R^\sigma\) which is not orthogonal to \(\tilde{\alpha} \in R^\sigma_i\) or \(\tilde{\beta} \in R^\sigma_i\), so \(\tilde{\alpha} + \tilde{\beta} \in (R^\sigma_i)^\times\). Thus

\[
\left[ G^\sigma_a, G^\sigma_\beta \right] \subseteq G^\sigma_{\tilde{\alpha} + \tilde{\beta}} \subseteq K_i.
\]

If \(\tilde{\alpha} + \tilde{\beta}\) is isotropic, then \(\tilde{\alpha} + \tilde{\beta} \in (R^\sigma_i) \cap (R^\sigma)^0 \subseteq R_i\). Thus \([G^\sigma_a, G^\sigma_\beta] \subseteq G^\sigma_{\tilde{\alpha} + \tilde{\beta}} \subseteq K_i\), as \(\tilde{\alpha} + \tilde{\beta} \in R_i \setminus \{0\}\). This completes the proof of our claim.

Next note that

\[
G^\sigma_i = \mathcal{H}^\sigma_i \oplus \sum_{\tilde{\alpha} \in R_i \setminus \{0\}} G^\sigma_{\tilde{\alpha}} = T_i \oplus K_i \oplus D_i = M_i \oplus D_i.
\]

Since \(D_i \subseteq \mathcal{H}^\sigma_i \subseteq \mathcal{H}^\sigma\), \([D_i : D_i \oplus K_i] \subseteq K_i\). But \(M_i\) is the subalgebra generated by \(K_i\), so \([D_i : M_i : D_i \oplus M_i] \subseteq M_i\).

(ii) The first assertion is clear. It follows from (2.48) that the form is nondegenerate.

(iii) Suppose to the contrary that \(C G^\sigma_{\tilde{\alpha}}(\mathcal{H}^\sigma_i) \not\subseteq \mathcal{H}^\sigma_i\). Then there exists \(x = x_0 + x_{\tilde{\alpha}_1} + \cdots + x_{\tilde{\alpha}_t} \in C G^\sigma_{\tilde{\alpha}}(\mathcal{H}^\sigma_i)\) such that \(x_0 \in \mathcal{H}^\sigma_i, x_{\tilde{\alpha}_j}\)'s are nonzero and \(\tilde{\alpha}_j\)'s are distinct roots of \(R_i \setminus \{0\}\). Then for any \(h \in \mathcal{H}^\sigma_i\),

\[
0 = [h_x] = \tilde{\alpha}_1(h)x_{\tilde{\alpha}_1} + \cdots + \tilde{\alpha}_t(h)x_{\tilde{\alpha}_t}.
\]

Thus \(\tilde{\alpha}_j(h) = 0\) for \(1 \leq j \leq t, h \in \mathcal{H}^\sigma_i\). That is \((t_{\tilde{\alpha}_j}, \mathcal{H}^\sigma_i) = \{0\}\). Since \(t_{\tilde{\alpha}_j} \in T_i \subseteq \mathcal{H}^\sigma_i\), it follows from part (ii) that \(\tilde{\alpha}_j = 0\) for all \(j\), which is a contradiction. \(\Box\)

**Proposition 2.51.** Suppose that \((R^\sigma)^\times \neq \emptyset\). Then \((G^\sigma_i, \langle \cdot, \cdot \rangle, \mathcal{H}^\sigma_i)\) satisfies axioms EA1–EA5(a) of an EAAL and its root system is \(R_i\).

**Proof.** By Proposition 2.37, EA1–EA2 hold for \(G^\sigma\). So for any \(\tilde{\alpha} \in R^\sigma\), the form restricted to \(G^\sigma_{\tilde{\alpha}} \oplus G^\sigma_{-\tilde{\alpha}}\) is nondegenerate. In particular, this holds for any \(\tilde{\alpha} \in R_i\). Thus \((\cdot, \cdot)\) is nondegenerate on \(G^\sigma_i\) and EA1 holds for \(G^\sigma_i\). From (2.43) we have \(R_i \subseteq (H^\sigma)^\times\). Now from the way \(G^\sigma_i\) is defined it follows that \(ad G^\sigma_i\) is diagonalizable for all \(h \in \mathcal{H}^\sigma_i\). This together with Proposition 2.50(iii) imply that EA2 holds for \(G^\sigma_i\). EA3 holds for \(G^\sigma_i\) as it holds for \(G^\sigma\). EA4 holds for the root system \(R_i\) of \(G^\sigma_i\) since \(R_i \subseteq R^\sigma\) and EA4 holds for \(R^\sigma\). Finally EA5(a) holds by Corollary 2.38. \(\Box\)

Set

\[
\mathcal{I} = \sum_{h \in (R^\sigma)^\times_0} G^\sigma_h.
\]

Clearly

\[
\mathcal{I} = \{0\} \quad \text{if} \quad (R^\sigma)^\times_0 = \emptyset.
\]
Note that
\[
G^\sigma = \mathcal{H}^\sigma \oplus \left( \sum_{\alpha \in R^\sigma \setminus \{0\}} G^\sigma_{\alpha} \right) = \left( \sum_{i=1}^{k} G^\sigma_i \right) \oplus \mathcal{W} \oplus \mathcal{I}.
\] (2.54)

For our next proposition we need to state some results regarding the core \(G^\sigma_c\) of \(G^\sigma\). In particular, we would like to obtain some criteria for the tameness of \(G^\sigma\).

Since \(\sigma(G_\alpha) = G^\sigma_\sigma(\alpha)\), we have \(\sigma(G_c) = G_c\). Thus
\[
G_c = \bigoplus_{i=0}^{m-1} (G_c)_i,
\] (2.55)
where \((G_c)_i = G_c \cap G_i\).

**Lemma 2.56.** Suppose \((R^\sigma)^x \neq \emptyset\). Then

(i) \(G_c \cap G^\sigma\) is the sum of spaces of the forms:
\[
G^\sigma_{\pi(\alpha)}, \quad \alpha \in R, \quad \pi(\alpha) \in \pi(R)^x,
\] (2.57)
and
\[
[G^\sigma_{\pi(\alpha)}, G^\sigma_{\pi(\beta)}], \quad \alpha, \beta \in R, \quad \pi(\alpha), \pi(\beta) \in \pi(R)^x, \quad i \in \mathbb{Z}.
\] (2.58)

(ii) \(G^\sigma_c\) is the sum of spaces of the forms (2.57) and their commutators.

(iii) \(G^\sigma_c \subseteq G^\sigma \cap G_c\).

(iv) If \(G^\sigma_c \cap \mathcal{H}^\sigma = G_c \cap \mathcal{H}^\sigma\), then \(C_{G_c} (G^\sigma_c \cap G_c) \subseteq G^\sigma \cap G_c^1\). In particular, if \(G\) is tame and \(G^\sigma_c = G^\sigma \cap G_c\), then \(G^\sigma\) is tame.

**Proof.** (i) This is an immediate consequence of (2.55) and [2, Lemma 3.53].

(ii) Let \(S\) be the sum of the spaces in (2.57) and their commutators. Then \(S \subseteq G^\sigma_c\).

To show the reverse inclusion, it is enough to show that \(S\) is closed under bracket, as \(S\) contains all the generators of \(G^\sigma_c\). So it is enough to show that for any three spaces \(G^\sigma_{\pi(\alpha)}\), \(G^\sigma_{\pi(\beta)}\), \(G^\sigma_{\pi(\gamma)}\) as in (2.57),
\[
[G^\sigma_{\pi(\alpha)}, [G^\sigma_{\pi(\beta)}, G^\sigma_{\pi(\gamma)}]] \subseteq S.
\] (2.59)

Certainly, we may assume that the two brackets involved in (2.59) are nonzero. Then by (2.29), \(\pi(\beta) + \pi(\gamma) = \pi(\alpha')\) for some \(\alpha' \in R\) and \([G^\sigma_{\pi(\beta)}, G^\sigma_{\pi(\gamma)}] \subseteq G^\sigma_{\pi(\alpha')}\). Now if \(\pi(\alpha') \in \pi(R)^x\), then
\[
[G^\sigma_{\pi(\alpha)}, [G^\sigma_{\pi(\beta)}, G^\sigma_{\pi(\gamma)}]] \subseteq [G^\sigma_{\pi(\alpha)}, G^\sigma_{\pi(\alpha')}], \quad [G^\sigma_{\pi(\alpha)}, G^\sigma_{\pi(\alpha')}] \subseteq S.
\]
If \( \pi(\alpha') \in \pi(R)^0 \), then \( \pi(\alpha) + \pi(\alpha') = \pi(\beta') \) for some \( \beta' \in R \), and \( G^\sigma_\pi(\alpha), G^\sigma_\pi(\alpha') \subseteq G^\sigma_\pi(\beta') \). But \( G^\sigma_\pi(\beta') \subseteq S \), since \( \pi(\beta') \in \pi(R)^\times \). This completes the proof of part (ii). Now (iii) is an immediate consequence of (i) and (ii).

(iv) Let \( x \in C_{G^\sigma_\pi}(G^\sigma \cap G_c) \). Then \( x = \sum_{i=0}^n x_{\tilde{\alpha}_i} \) where \( \tilde{\alpha}_i \) are distinct roots of \( R^\sigma \) with \( \tilde{\alpha}_0 = 0 \), and \( x_{\tilde{\alpha}_i} \in G^\sigma_{\tilde{\alpha}_i} \) for \( i \neq 0 \). Let \( \tilde{\alpha}, \tilde{\beta} \in (R^\sigma)^\times \). By part (i), for \( j \in \mathbb{Z} \) we have

\[
\left[ x, [G^\sigma_{j,\tilde{\alpha}}, G^\sigma_{-j,\tilde{\beta}}] \right] = \sum_{i=0}^n [x_{\tilde{\alpha}_i}, [G^\sigma_{j,\tilde{\alpha}_i}, G^\sigma_{-j,\tilde{\beta}_i}]] = [0],
\]

where \( [x_{\tilde{\alpha}_i}, G^\sigma_{\tilde{\alpha}_i}] \subseteq G^\sigma_{\tilde{\alpha}_i + \tilde{\alpha}} \) and \( [x_{\tilde{\alpha}_i}, [G^\sigma_{j,\tilde{\alpha}_i}, G^\sigma_{-j,\tilde{\beta}_i}]] \subseteq G^\sigma_{\tilde{\alpha}_i + \tilde{\alpha} + \tilde{\beta}} \). It follows that for \( 0 \leq i \leq n \),

\[
[x_{\tilde{\alpha}_i}, G^\sigma_{\tilde{\alpha}_i}] = [0] \quad \text{and} \quad [x_{\tilde{\alpha}_i}, [G^\sigma_{j,\tilde{\alpha}_i}, G^\sigma_{-j,\tilde{\beta}_i}]] = [0],
\]

and so \( x_{\tilde{\alpha}_i} \in C_{G^\sigma_\pi}(G^\sigma \cap G_c) \) for \( 0 \leq i \leq n \). Therefore, we must show that for each \( i \), \( x_{\tilde{\alpha}_i} \in G^\sigma_{\tilde{\alpha}_i} \). By (2.16), it is enough to show that for each \( i \), \( (x_{\tilde{\alpha}_i}, G^\sigma \cap G_c) = [0] \). Since the form is nondegenerate and invariant on \( G^\sigma_\pi \), and since \( G^\sigma_\pi = H^\sigma \), it follows that

\[
(x_0, G^\sigma \cap G_c) = (x_0, H^\sigma \cap G_c) = (x_0, G^\sigma_c) = (x_0, [G^\sigma_c, G^\sigma_c]) = ([x_0, G^\sigma_c], G^\sigma_c) = ([x_0, G^\sigma \cap G_c], G^\sigma_c) = [0].
\]

Thus \( x_0 \in G^\sigma_c \). Next, we show that for \( 1 \leq i \leq n \), \( x_{\tilde{\alpha}_i} \in G^\sigma_{\tilde{\alpha}_i} \) (note that \( \tilde{\alpha}_i \neq 0 \)). Fix \( 1 \leq i \leq n \). Let \( \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \in (R^\sigma)^\times \), \( j \in \mathbb{Z} \), and \( y \in G^\sigma_\tilde{\gamma} \) or \( y \in [G^\sigma_{j,\tilde{\alpha}}, G^\sigma_{-j,\tilde{\beta}}] \). By (2.57) and (2.58), it is enough to show that \( (y, x_{\tilde{\alpha}_i}) = 0 \). But by (2.27) this holds if \( y \in G^\sigma_{\tilde{\gamma}} \) and \( \tilde{\gamma} + \tilde{\alpha}_i \neq 0 \) or \( y \in [G^\sigma_{j,\tilde{\alpha}}, G^\sigma_{-j,\tilde{\beta}}] \) and \( \tilde{\alpha} + \tilde{\beta} + \tilde{\alpha}_i \neq 0 \). Suppose \( \tilde{\gamma} + \tilde{\alpha}_i = 0 \) if \( y \in G^\sigma_{\tilde{\gamma}} \) and \( \tilde{\alpha} + \tilde{\beta} + \tilde{\alpha}_i = 0 \) if \( y \in [G^\sigma_{j,\tilde{\alpha}}, G^\sigma_{-j,\tilde{\beta}}] \). Then \( y \in G^\sigma \cap G_c \), and since EA1–EA2 hold for \( G^\sigma \), we have from (1.1) that

\[
0 = [x_{\tilde{\alpha}_i}, y] = (x_{\tilde{\alpha}_i}, y)_{\tilde{\alpha}_i}.
\]

Thus \( (x_{\tilde{\alpha}_i}, y) = 0 \). This completes the proof of the first statement in (iv). Now if \( G_c \) is tame then \( G^\sigma_{\tilde{\alpha}_i} \subseteq G_c \). So the second statement of (iv) is an immediate consequence of the first statement. \( \square \)

**Remark 2.61.** As we will see in the next section, in many examples \( G^\sigma_c \) is a proper subspace of \( G^\sigma \cap G_c \).

Note that from Proposition 2.56(ii), we have

\[
G^\sigma_c \subseteq \sum_{i=1}^k G^\sigma_i \subseteq G^\sigma.
\]
Proposition 2.63.

(i) \( \mathcal{W} \subseteq C_{\mathcal{G}^\circ} \left( \sum_{i=1}^k \mathcal{G}^\circ_i \right) \).
(ii) \( \sum_{\delta \in R_{iso}^\circ} \mathcal{G}^\circ_\delta \subseteq C_{\mathcal{G}^\circ} \left( \mathcal{G}^\circ_r \right) \). In particular, \( \mathcal{W} \oplus \mathcal{T} \subseteq C_{\mathcal{G}^\circ} \left( \mathcal{G}^\circ_r \right) \).

Proof. (i) Since \( \mathcal{W} \subseteq \mathcal{H}^\circ \), we have \([\mathcal{W}, \mathcal{H}^\circ] = \{0\}\). Therefore, it remains to show that \([\mathcal{W}, \mathcal{G}^\circ_i] = \{0\}\) for any \( \tilde{\alpha} \in R_1 \setminus \{0\} \), \( 1 \leq i \leq k \). Let \( \tilde{\alpha} \in R_1 \setminus \{0\} \). Then \( \tilde{\alpha} \in (R^\circ)^\times \). By Corollary 2.38 \( R^\circ_i \) is tame, so \( (R^\circ_i)^\times = ((R^\circ_i)^\times)^\times \). By (2.43), (2.44), and (2.47), we have \( \mathcal{W} \subseteq \ker \beta \), for any \( \tilde{\beta} \in (R^\circ_i)^\times \). Therefore \([\mathcal{W}, \mathcal{G}^\circ_i] = \{0\}\).

(ii) Let \( \delta \in R_{iso}^\circ \). Since \( \delta + \tilde{\alpha} \notin R^\circ \) for any \( \tilde{\alpha} \in (R^\circ)^\times \), \( [\mathcal{G}^\circ_{\delta}, \mathcal{G}^\circ_\delta] = \{0\}\). But \( \mathcal{G}^\circ_r \) is generated by root spaces \( \mathcal{G}^\circ_{\delta} \), \( \tilde{\alpha} \in (R^\circ)^\times \), so \( [\mathcal{G}^\circ_{\delta}, \mathcal{G}^\circ_{\delta}] = \{0\}\). Now the proof is complete, using part (i) and (2.62). \( \square \)

Corollary 2.64. If \( (R^\circ)^\times \neq \emptyset \), then for each \( i \), the triple

\( (\mathcal{G}^\circ_i \oplus \mathcal{W}, (\cdot, \cdot), \mathcal{H}^\circ_i \oplus \mathcal{W}) \)

satisfies EA1–EA5(a).

Proof. By Proposition 2.63, \( \mathcal{G}^\circ_i \oplus \mathcal{W} \) is a Lie algebra. By Proposition 2.51 the form \((\cdot, \cdot)\) is nondegenerate on \( \mathcal{G}^\circ_i \), so by (2.47) and (2.48), the form \((\cdot, \cdot)\) is nondegenerate on the Lie algebra \( \mathcal{G}^\circ_i \oplus \mathcal{W} \). By Propositions 2.50(iii) and 2.63(i), \( \mathcal{H}^\circ_i \oplus \mathcal{W} \) is self-centralizing in \( \mathcal{G}^\circ_i \oplus \mathcal{W} \). By action as zero on \( \mathcal{W} \) we may identify elements of \( (\mathcal{H}^\circ)^\times \) as elements of \( (\mathcal{H}^\circ_i \oplus \mathcal{W})^\times \). Then it follows from Proposition 2.51 that EA2 holds for \( \mathcal{G}^\circ_i \oplus \mathcal{W} \) and that \( \mathcal{G}^\circ_i \) and \( \mathcal{G}^\circ_i \oplus \mathcal{W} \) have the same root system. Now it is clear that EA3–EA5(a) hold for \( \mathcal{G}^\circ_i \oplus \mathcal{W} \). \( \square \)

We summarize our results in the following theorem. We emphasize that the exact Lie bracket multiplication between some isotropic root spaces which appearing in the structure of \( \mathcal{G}^\circ \) are unknown to us, however as part (vi) of the following theorem shows, the decomposition is direct in the level of cores modulo centers. This is a remarkable fact as it is known that most of the basic structural properties of EALAs and their close counterparts can be read from the cores modulo centers.

Theorem 2.65. Let \( (\mathcal{G}, (\cdot, \cdot), \mathcal{H}) \) be an EALA with corresponding root system \( R \). Let \( \sigma \) be an automorphism of \( \mathcal{G} \) satisfying (A1)–(A4). Let \( \mathcal{G}^\circ (\mathcal{H})^\circ \) be the fixed point subalgebra of \( \mathcal{G} (\mathcal{H}) \), under \( \sigma \), and let \( R^\circ \) be the root system of \( \mathcal{G}^\circ \) with respect to \( \mathcal{H}^\circ \). Suppose \( (R^\circ)^\times \neq \emptyset \). Then

(i) \( (\mathcal{G}^\circ, (\cdot, \cdot), \mathcal{H}^\circ) \) satisfies axiom EA1–EA4 of an EALA and \( R^\circ \) is a reduced EARS.
(ii) \( R^\circ = \left( \bigcup_{i=1}^k R_i \right) \cup (R^\circ)^0 \) where for each \( i \), \( R_i \) is an indecomposable reduced EARS with \( (R^\circ_i, R^\circ_j) = \{0\} \) if \( i \neq j \) (the union is not necessarily disjoint), and \( (R^\circ)^0 \) is the set of isolated roots of \( R^\circ \) which are not contained in either of \( R_i \), \( 1 \leq i \leq k \).
(iii) $\mathcal{H}^\sigma = (\bigoplus_{i=1}^k \mathcal{H}^\sigma_i) \oplus W$ where $\mathcal{H}^\sigma_i$ and $W$ are some subspaces of $\mathcal{H}^\sigma$ with $(\mathcal{H}^\sigma_i, W) = (0)$.

(iv) $G^\sigma = (\bigoplus_{i=1}^k G^\sigma_i) \oplus W \oplus I$, where for each $i$, $(G^\sigma_i, \cdot, \cdot, H^\sigma_i)$ is a Lie algebra satisfying axioms EA1–EA5(a) of an EALA and has $R_i$ as its root system, and $I = \bigoplus_{i \in I} R_i R^\sigma_i$. Moreover, $\{W, G^\sigma_i\} = (0)$ for each $i$, and $I \subseteq C_{G^\sigma_i}(G^\sigma_i)$. Finally, $I = (0)$ if $R^\sigma$ has no isolated roots.

(v) If $i \neq j$, then $\{ \mathcal{I}(G^\sigma_i), \mathcal{I}(G^\sigma_j) \} = (0)$. Moreover, $G^\sigma_i = \bigoplus_{s=1}^k \mathcal{I}(G^\sigma_i) \cdot (\mathcal{I}(G^\sigma_i)_{ \cdot} \cdot)$ is the core of $G^\sigma_i$. In particular, $(\mathcal{I}(G^\sigma_i))_{ \cdot} \cdot$ is an ideal of $G^\sigma$ for each $i$.

(vi) $\mathcal{Z}(G^\sigma_i) = \bigoplus_{s=1}^k \mathcal{Z}(\mathcal{I}(G^\sigma_i)) \cdot (\mathcal{I}(G^\sigma_i)_{ \cdot} \cdot)$ and as Lie algebras

$$\frac{G^\sigma_i}{\mathcal{Z}(G^\sigma_i)} \cong \bigoplus_{i=1}^k \frac{(G^\sigma_i)_{ \cdot} \cdot}{\mathcal{Z}(\mathcal{I}(G^\sigma_i))_{ \cdot} \cdot}.$$

(vii) If $G$ is nondegenerate then for each $i$, $\dim \mathcal{H}^\sigma_i = l_i + 2v_i$.

**Proof.** For (i) see Proposition 2.37. For (ii) see Corollary 2.38. For (iii) see (2.47) and Proposition 2.63. For (iv) see (2.52), Proposition 2.51, and Proposition 2.63.

For the first statement of (v), it is enough to show that if $\pi(\alpha) \in R_i^\times$ and $\pi(\beta) \in R_j^\times$, then $\pi(\alpha) + \pi(\beta)$ is not a root in $R^\sigma$. By part (ii), $(\pi(\alpha), \pi(\beta)) = (0)$. Therefore $\pi(\alpha) + \pi(\beta)$ is not orthogonal to $\pi(\alpha)$ or to $\pi(\beta)$ and so cannot be a root (by part (ii)). This in particular shows that $\bigoplus_{i=1}^k \mathcal{I}(G^\sigma_i) \cdot$ is a Lie algebra which contains all generators of $G^\sigma_i$. So the second statement of (v) holds.

(vi) It follows immediately from (v).

(vii) It is clear from the construction of $\mathcal{H}^\sigma_i$. □

As a corollary we can state a weak version of a result which is due to [7] (see Remark 2.67).

**Corollary 2.66.** Let $(G, (\cdot, \cdot), H)$ be a finite dimensional complex simple Lie algebra, where $H$ is a Cartan subalgebra and $(\cdot, \cdot)$ is the Killing form on $G$. Let $\sigma$ be an automorphism of $G$ satisfying (A1)–(A2) and (A4). Then $(G^\sigma, (\cdot, \cdot), H^\sigma)$ is a reductive Lie algebra.

**Proof.** It is easy to see that (A3) is a consequence of (A1). $G$ has no nonzero isotropic roots and so by (2.52) and (2.53), $I = (0)$ and $G^\sigma_i = \bigoplus_{\alpha \in R_i} (G^\sigma_i)_{\alpha}$ where $R_i = R_i^\sigma$ is an irreducible finite root system in $V^\sigma_i$ (see Corollary 2.38 and Theorem 2.65(ii)), and $(G^\sigma_i)_0 = H_i$. Therefore by (1.2) and (1.3) one can see that $G^\sigma_i$ is generated by $\{h_{\alpha}, e_{\pm\alpha} | \alpha \in H_i\}$. Moreover, these $3l_i$ elements satisfy the Serre’s relations. Therefore $G^\sigma_i$ is a finite dimensional semi-simple Lie algebra. However, $R_i$ is irreducible and so $G^\sigma_i$ is simple. Finally, we have from Proposition 2.63(i) that $\mathcal{V} = \mathcal{Z}(G^\sigma_i)$. □

**Remark 2.67.** (i) According to [7], if $\sigma$ satisfies only (A1), then $G^\sigma$ is a reductive Lie algebra. A version of conjugacy for Cartan subalgebras is used in the proof and no such result is known for a general EALA.
(ii) Similar to part (ii), we may state Theorem 2.65(iv) in a different way. In fact by Corollary 2.64, for a fixed $i$ we can consider $G_\sigma^\alpha_i \oplus W$ as a Lie algebra satisfying EA1–EA5(a). So part (iv) of the theorem can be restated to say $G^\alpha$ is a direct sum of some EALAs and a subspace $I$ satisfying $I \subseteq C_{G^\alpha}(G^\alpha_i)$.

(iii) By Theorem 2.65, if $G$ is nondegenerate (see Definition 1.4) then for each $i$, $G^\alpha_i$ is nondegenerate.

**Corollary 2.68.** Let $(G, (\cdot, \cdot), \mathcal{H})$, $\sigma$ and $R^\alpha$ be as in Theorem 2.65. If $R^\alpha$ is an irreducible EARS, then $(G^\alpha, (\cdot, \cdot), \mathcal{H}^\alpha)$ is an EALA.

**Proof.** By Theorem 2.65, $G^\alpha$ satisfies EA1–EA5(a). Now EA5(b) also holds as $R^\alpha$ is irreducible.

**Corollary 2.69.** Let $(G, (\cdot, \cdot), \mathcal{H})$ be as in Theorem 2.65 and assume that $(R^\alpha)^{\times} \neq \emptyset$ and that $G$ is tame. If $G^\alpha \cap G^\sigma = G^\alpha_i$, then $G^\alpha = \sum_{i=1}^{k} G^\alpha_i$. Moreover, if $\nu \leq 2$ that each $G^\alpha_i$ is an EALA.

**Proof.** By Lemma 2.56(iv), $C_{G^\alpha}(G^\sigma) \subseteq G^\sigma_i$. Therefore, by Theorem 2.65, $(G^\alpha, (\cdot, \cdot), \mathcal{H}^\alpha)$ is a tame Lie algebra satisfying EA1–EA4. Now it follows from the proof of [2, Lemma 3.62] that $G^\alpha$ also satisfies EA5(b) (that is $R^\alpha_{\sigma_{\mathcal{O}}(\alpha)} = \emptyset$). In particular $I = \{0\}$. Since $G^\alpha$ is tame we have from Proposition 2.63 that $W \subset G^\sigma \cap G^\sigma$. By Lemma 2.45,

$$G^\sigma \cap \mathcal{H}^\alpha = \sum_{\tilde{\alpha} \in (R^\alpha)^{\times}} [G^\sigma_{\tilde{\alpha}}, G^\sigma_{-\tilde{\alpha}}] = \mathcal{V}^\sigma_{\mathcal{C}} \oplus (\mathcal{V}^\sigma)^0_{\mathcal{C}}.$$

But $\mathcal{V}^\sigma_{\mathcal{C}} \oplus (\mathcal{V}^\sigma)^0_{\mathcal{C}}$ and $W$ have zero intersection. Thus $W = \{0\}$.

If $\nu \leq 2$, then $R^\sigma_i = (R^\sigma)^0_i$, and so $R_i = R^\sigma_i$. That is, $R_i$ is an irreducible EARS. Therefore by part (iv) of Theorem 2.65 each $G^\alpha_i$ in an EALA.  

3. Examples

In this section we present several examples which elaborate on the results obtained in Section 2. In 3.70–3.75 below a large class of examples is introduced which illustrate how the terms $G^\alpha_1, \ldots, G^\alpha_k$, $W$ and $I$ (see Theorem 2.65) appear as the fixed points of automorphisms. In [4] it is shown that many examples of EALA (of types $D_\ell$, $A_1$, $B_\ell$, $C_\ell$, and $BC_\ell$) can be obtained as the fixed points of automorphisms of some other EALA (of types $A_\ell$, $D_\ell$, and $C_\ell$) which may have a simpler structure. Using Theorem 2.65, we are able to give new proofs of the results obtained in [4]. In Examples 3.78 and 3.79 we have provided the details for two cases which seems to be more delicate. The other cases can be dealt with in a similar manner. Finally, in 3.81, we present an example regarding the results in [2].
Example 3.70. Let \((\mathcal{G}, (\cdot,\cdot), \mathcal{H})\) be an EALA of type \(X_\ell\) and nullity \(v\). Let \(R\) be the corresponding irreducible extended affine root system and denote its root lattice by \(Q\). Then by [1, Proposition II.1.11] and [1, Theorem II.2.37], we have

\[ Q = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_\ell \oplus \mathbb{Z}\delta_1 \oplus \cdots \oplus \mathbb{Z}\delta_v, \]

where \(\{\alpha_1, \ldots, \alpha_\ell\}\) is a subset of \(R\) which forms a basis for a finite root system of type \(X_\ell\) and \(\{\delta_1, \ldots, \delta_v\}\) is a subset of \(R\) which forms a basis of the radical of the form on \(\mathcal{H}^*\) restricted to the real span of \(R\). Consider any group homomorphism \(\phi : Q \to \mathbb{C} \setminus \{0\}\). \(\phi\) is uniquely determined by specifying \(\phi(\alpha_i)\) for \(1 \leq i \leq \ell\) and \(\phi(\delta_j)\) for \(1 \leq j \leq v\). The homomorphism \(\phi\) induces an automorphism \(\sigma\) of \(\mathcal{G}\) by letting

\[ \sigma_{|\mathcal{G}}(\alpha) = \phi(\alpha) \text{id}_{\mathcal{G}} \quad \text{for} \quad \alpha \in R. \]

Since \(\mathcal{G}_0 = \mathcal{H}\) and \(\phi(0) = 1\), we have \(\sigma(h) = h\) for all \(h \in \mathcal{H}\). Note that \(\sigma\) is of finite order if and only if \(\phi(\alpha_i)\) for \(1 \leq i \leq \ell\) and \(\phi(\delta_j)\) for \(1 \leq j \leq v\) are roots of unity. Assume that \(\sigma^m = \text{id}\), for some positive integer \(m\). Let \(\mathcal{G}^\sigma\) be the set of fixed points of \(\sigma\). Clearly \(\sigma(\mathcal{H}) = \mathcal{H}\) and \(\mathcal{H}^\sigma = \mathcal{H}\). Since

\[ (\mathcal{G}_\alpha, \mathcal{G}_\beta) = \{0\} \quad \text{unless} \quad \alpha + \beta = 0, \]

it follows that

\[ (\sigma(x), \sigma(y)) = (x, y) \quad \text{for all} \quad x, y \in \mathcal{G}. \]

Since \(\sigma(\alpha) = \alpha\) for all \(\alpha \in (\mathcal{H}^\sigma)^*\), it follows from [2, Proposition 3.25] that \(C_{\mathcal{G}^\sigma}(\mathcal{H}) = \mathcal{H}\). Therefore \(\sigma\) satisfies conditions (A1)–(A4). Thus by Theorem 2.65, \(\mathcal{G}^\sigma\) satisfies axioms EA1–EA4. Therefore, if \((R^\sigma)^\times \neq \emptyset\), then

\[ \mathcal{G}^\sigma = \sum_{i=1}^k \mathcal{G}_\alpha^i \oplus \mathcal{W} \oplus \mathcal{I}, \quad \text{(3.71)} \]

where for each \(i\), \((\mathcal{G}_\alpha^i, (\cdot,\cdot), \mathcal{H}_\alpha^i)\) satisfies EA1–EA5(a) and \(\mathcal{W}\) and \(\mathcal{I}\) are as in Theorem 2.65. Note that \(\mathcal{G}^\sigma \cap \mathcal{G}_\alpha \neq \{0\}\) if and only if \(\phi(\alpha) = 1\). In particular,

\[ R^\sigma = \{ \alpha \in R \mid \mathcal{G}^\sigma \cap \mathcal{G}_\alpha \neq \{0\} \} = \{ \alpha \in R \mid \mathcal{G}_\alpha \subseteq \mathcal{G}^\sigma \}. \]

Thus

\[ R^\sigma = \{ \alpha \in R \mid \phi(\alpha) = 1 \} \quad \text{(3.72)} \]

and

\[ (R^\sigma)^\times \subseteq R^\times, \quad (R^\sigma)^0 \subseteq R^0. \quad \text{(3.73)} \]
Example 3.74. In Example 3.70 assume that \( X_\ell \) is reduced, that is it has one of the types \( A, D, E, B, C, F, \) or \( G \). Let \( S \) and \( L \) be semilattices which appear in the structure of \( R \) (see [1, Theorem 2.37] and (1.7)). We want to impose some restrictions on the semilattices \( S \) and \( L \), as follows. For types \( A_\ell \) \((\ell \geq 2)\), \( D \), and \( E \) we impose no restriction as we know from \([1, II.2.32]\) that for these types the semilattice \( S \) is always a lattice. For type \( A_1 \) assume that \( S \) is a lattice and for the remaining types assume that both \( S \) and \( L \) are lattices and that \( S = L \). (The root systems of toroidal Lie algebras, with some derivations added, are of this form.) We claim that for such types and under the above restrictions the axiom EA5(b) also holds for \( G \).

Suppose that \((R^\sigma)^0 \neq \emptyset\). So there exists \( \alpha \in R^\times \) such that \( \phi(\alpha) = 1 \). Let \( \delta \in (R^\sigma)^0 \). That is \( \phi(\delta) = 1 \). Under the above assumptions, it follows from the structure of EARS of type \( X_\ell \) that \( \alpha + \delta \in R \). Now \( \phi(\alpha + \delta) = 1 \) and so \( \alpha + \delta \in R^\sigma \). This shows that EA5(b) holds for \( G \). Note also that since \( S \) is a lattice we have from (1.7) that \( R^0 \) is also a lattice. It now follows from (3.72) and (3.73) that \((R^\sigma)^0 \) is also a lattice.

Example 3.75. Let \( \mathcal{G} \) and \( \sigma \) be as in Example 3.70 and let \( X_\ell = A_1 \). We have

\[
R = (S + S) \cup (\pm \dot{\alpha} + S),
\]

where \( S \) is a semilattice in the real span \( V^0 \) of \( R^0 \). If \( S \) is a lattice, then by Example 3.74 axiom EA5(b) holds for \( \mathcal{G} \).

Next suppose that \( S \) is not a lattice. We show that in this case it might happen that the axiom EA5(b) does not hold for \( \mathcal{G} \). To see this let the nullity \( \nu \) of \( R \) be 3. Let

\[
S = \left\{ \sum_{i=1}^{3} m_i \delta_i \mid m_i \in \mathbb{Z} \text{ and } m_i m_j \equiv 0 \mod 2, \text{ if } i \neq j \right\}.
\]

Then \( S \) is a semilattice in \( V^0 \) and \( Q = \mathbb{Z} \dot{\alpha} \oplus \mathbb{Z} \delta_1 \oplus \mathbb{Z} \delta_2 \oplus \mathbb{Z} \delta_3 \). Define

\[
\phi(\dot{\alpha}) = -1, \quad \phi(\delta_1) = 1, \quad \phi(\delta_2) = 1, \quad \text{and} \quad \phi(\delta_3) = -1.
\]

Now \( \phi(\delta_1 + \delta_2) = 1 \) and so \( \delta := \delta_1 + \delta_2 \in (R^\sigma)^0 \). We claim that \( \delta \) is isolated, that is \( \delta \in R^\sigma_{\text{iso}} \). Suppose to the contrary that there exists \( \alpha = \pm \dot{\alpha} + m_1 \delta_1 + m_2 \delta_2 + m_3 \delta_3 \in R^\sigma \) such that \( \alpha + \delta \in R^\sigma \). We have

\[
1 = \phi(\alpha) = -\phi(\delta_3)^{m_3} = -(\alpha)^{m_3},
\]

and so \( m_3 = 2k_3 + 1 \) for some \( k_3 \in \mathbb{Z} \). Since \( \alpha \in R^\sigma \subseteq R \), we must have \( m_1 = 2k_1 \) and \( m_2 = 2k_2 \) for some \( k_1, k_2 \in \mathbb{Z} \). Then

\[
\alpha + \delta = \pm \dot{\alpha} + (2k_1 + 1)\delta_1 + (2k_2 + 1)\delta_2 + (2k_3 + 1)\delta_3 \in R^\sigma \subseteq R.
\]

But this contradicts the fact that \( R \) contains no such root. Note that for this particular example the expression (3.71) reads as \( \mathcal{G}^\sigma = \mathcal{G}^\sigma_1 \oplus W + I \). Then using (3.72), an easy computation shows that \( R_1 = R^\sigma_1 \) and so \( I \neq \emptyset \) and \( \mathcal{G}^\sigma_1 \) satisfies EA5(b).
For our next two examples we need the following setting (see [1,4,6]). Let \( v \geq 1 \). Let \( e = (e_1, \ldots, e_v) \) be a vector in \( \mathbb{C}^v \) and let \( q = (q_{ij}) \) be a \( v \times v \)-matrix such that

\[
e_i = \pm 1, \quad q_{ii} = 1 \quad \text{for} \quad 1 \leq i \leq v, \quad \text{and} \\
q_{ij} = q_{ji} = \pm 1 \quad \text{for} \quad 1 \leq i \neq j \leq v.
\]

Let \( \mathcal{A} \) be the associative algebra over \( \mathbb{C} \) with generators \( x_i, x_i^{-1} \) subject to the relations

\[
x_ix_j = q_{ij}x_jx_i.
\]

Then

\[
\mathbb{C}\langle x^\delta \rangle \quad \text{where} \quad x^\delta = x_1^{n_1} \cdots x_v^{n_v} \quad \text{for} \quad \delta = (n_1, \ldots, n_v) \in \mathbb{Z}^v.
\]

Let \( \bar{\gamma} \) be the involution on \( \mathcal{A} \) such that \( \bar{x}_i = e_i x_i, \) for all \( i. \) The pair \( (\mathcal{A}, \bar{\gamma}) \) is called the quantum torus with involution determined by the vector \( e \) and the matrix \( q. \) We have \( \mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_- \) where \( \mathcal{A}_+ = \{ h \in \mathcal{A} \mid \bar{h} = h \} \) and \( \mathcal{A}_- = \{ s \in \mathcal{A} \mid \bar{s} = -s \}. \) Set

\[
I_e = \{ i \mid 1 \leq i \leq v, \quad e_i = -1 \} \quad \text{and} \quad J_q = \{ (i, j) \mid 1 \leq i < j \leq v, \quad q_{ij} = -1 \}.
\]

Put

\[
Z_{e,q} = \left\{ \delta \in \mathbb{Z}^v \left\mid \sum_{i \in I_e} n_i + \sum_{(i,j) \in J_q} n_in_j = 0 \right\} \right. \quad \text{and} \quad Z_{e,q}^{e} = \mathbb{Z}^v \setminus Z_{e,q}.
\]

Then \( Z_{e,q} \) is a semilattice in \( \mathbb{R}^v. \)

Fix \( n \geq 1 \) and let \( M_n(\mathcal{A}) \) be the Lie algebra of \( n \times n \) matrices with entries from \( \mathcal{A}. \) Next define a bilinear form on \( M_n(\mathcal{A}) \) as follows. Define \( \varepsilon \) in the dual space of \( \mathcal{A} \) by the linear extension of \( \varepsilon(1) = 1 \) and \( \varepsilon(x^\delta) = 0 \) for any nonzero \( \delta \in \mathbb{Z}^v. \) Then for \( A, B \in M_n(\mathcal{A}) \) define \( (A, B) = \varepsilon(\text{tr}(AB)). \) This defines a symmetric nondegenerate invariant bilinear form on \( M_n(\mathcal{A}). \)

Let \( \mathcal{K} \) be the Lie subalgebra \( sl_n(\mathcal{A}) \) of \( M_n(\mathcal{A}) \) consisting of matrices \( A \) such that \( \text{tr}(A) \equiv 0 \mod [A, A]. \) Let \( \hat{\mathcal{H}} \) be the abelian subalgebra of \( sl_n(\mathcal{A}) \) with basis \( e_{ii} - e_{i+1,i+1}, \) for \( 1 \leq i \leq n - 1. \) Define \( e_i \in \hat{\mathcal{H}}^* \) by \( e_i(e_{jj} - e_{j+1,j+1}) = \delta_{ij} - \delta_{i,j+1}. \) Then, with respect to \( \hat{\mathcal{H}}, \) we have the root space decomposition

\[
\mathcal{K} = \bigoplus_{\hat{\alpha} \in \hat{\mathcal{R}}} \mathcal{K}_{\hat{\alpha}}, \quad \text{where} \quad \hat{\mathcal{R}} = \{ \pm (e_i - e_j) \mid 1 \leq i < j \leq n \} \cup \{ 0 \}.
\]

Suppose that \( \mathcal{K} \) has a \( \mathbb{Z}^v \) grading, say \( \mathcal{K} = \sum_{\delta \in \mathbb{Z}^v} \mathcal{K}^\delta, \) such that

\[
\mathcal{K}_\delta = \bigoplus_{\hat{\alpha} \in \hat{\mathcal{R}}} (\mathcal{K}_{\hat{\alpha}} \cap \mathcal{K}_\delta) \quad (\hat{\alpha} \in \hat{\mathcal{R}}),
\]

\[
\delta, \tau \in \mathbb{Z}^v, \quad \delta + \tau \neq (K^\delta, K^\tau) = \{ 0 \},
\]

\[
\mathcal{K}_0 \cap \mathcal{K}_0 = \hat{\mathcal{H}}, \quad \text{and} \quad \{ \delta \in \mathbb{Z}^v \mid K^\delta \neq \{ 0 \} \} \text{ spans } \mathbb{C}^v.
\]

(3.77)
Later we would like to consider the fixed point subalgebra of $\mathcal{K}$ under some automorphism $\sigma$. According to our previous notation, we should write $\mathcal{K}^\sigma$ for the corresponding fixed point subalgebra. But this might cause some confusion with the notation which we used for the grading on $\mathcal{K}$. To prevent this, we devote the upper index $\sigma$ only to indicate the fixed points, and we use other symbols such as $\delta$, $\tau$, $\gamma$, etc. for the grading on $\mathcal{K}$.

For $1 \leq i \leq \nu$ define $d_i \in \text{Der}(\mathcal{K})$ by $d_i(x) = n_i x$ for $x \in \mathcal{K}$. Set $\mathcal{D} = \sum_{i=1}^\nu \mathcal{C}d_i$. Consider a $\nu$-dimensional vector space $\mathcal{C} = \sum_{i=1}^\nu \mathcal{C}c_i$ and put

$$
\mathcal{G} = \mathcal{K} \oplus \mathcal{C} \oplus \mathcal{D} \quad \text{and} \quad \mathcal{H} = \mathcal{H} \oplus \mathcal{C} \oplus \mathcal{D}.
$$

Let $[\delta_1, \ldots, \delta_\nu]$ be the basis of $\mathcal{D}^\ast$ dual to $\{d_1, \ldots, d_\nu\}$ and consider them as elements of $\mathcal{H}^\ast$ by $\delta_i(\mathcal{H} \oplus \mathcal{C}) = \{0\}$. Consider $\mathbb{Z}^\nu$ as a subset of $\mathcal{H}^\ast$ through $(n_1, \ldots, n_\nu) = \sum_{i=1}^\nu n_i \delta_i$.

Extend the bracket on $\mathcal{K}$ to $\mathcal{G}$ by,

$$
[C, C]_\mathcal{G} = \{0\} = [\mathcal{D}, \mathcal{D}]_\mathcal{G}, \quad [d_i, x]_\mathcal{G} = d_i(x), \quad [x, y]_\mathcal{G} = [x, y] + \sum_{i=1}^\nu (d_i x, y)c_i,
$$

for $x, y \in \mathcal{K}$. Then $\mathcal{G}$ is a Lie algebra. Also extend the form on $\mathcal{K}$ to $\mathcal{G}$ by requiring $(\mathcal{C}, \mathcal{C}) = \{0\}$, $(\mathcal{D}, \mathcal{D}) = \{0\}$, $(\mathcal{C} \oplus \mathcal{D}, \mathcal{K}) = \{0\}$ and $(c_i, d_j) = \delta_{ij}$. Then $(\cdot, \cdot)$ is a symmetric, nondegenerate invariant bilinear form on $\mathcal{G}$. It is now easy to see that the triple $(\mathcal{G}, (\cdot, \cdot), \mathcal{H})$ satisfies the conditions (1.1)–(1.11) of [1, III.§1] and so by [1, Proposition III.1.20] $\mathcal{G}$ is a tame EALA of type $A_{n-1}$ and that $\mathcal{G} = \mathcal{K} \oplus \mathcal{C}$.

In the next two examples we consider an automorphism $\sigma$ of $\mathcal{K}$ (or of a subalgebra of $\mathcal{K}$) and we extended it to an automorphism of $\mathcal{G}$ by acting as identity on $\mathcal{C} \oplus \mathcal{D}$. Then we have $\mathcal{G}^\sigma = \mathcal{K}^\sigma \oplus \mathcal{C} \oplus \mathcal{D}$ and $\mathcal{H}^\sigma = \mathcal{H}^\sigma \oplus \mathcal{C} \oplus \mathcal{D}$, where $\mathcal{G}^\sigma$, $\mathcal{K}^\sigma$, $\mathcal{H}^\sigma$, and $\mathcal{H}^\sigma$ denote the fixed points of $\mathcal{G}$, $\mathcal{K}$, $\mathcal{H}$, and $\mathcal{H}$ under $\sigma$, respectively.

**Example 3.78.** Let $(A, \cdot)$ be the quantum torus determined by $\mathbf{e}$ and $\mathbf{q}$ as in (3.76). Suppose that $n = 2\ell + m$ where $m, \ell \geq 1$. Let $\tau_1, \ldots, \tau_m$ represent distinct cosets of $2\mathbb{Z}^\nu$ in $\mathbb{Z}^\nu$ with $\tau_1 = 0$ and $\tau_i \in \mathbb{Z}_n \mathbf{q}$ for all $i$. Define a grading on $\mathcal{K}$ by $\deg(x^\mathbf{e}_p \mathbf{q}^\mathbf{r}) = 2\delta + \lambda_p - \lambda_q$ where $\lambda_1 = \cdots = \lambda_2 = 0$ and $\lambda_{2i+1} = \tau_i$ for $1 \leq i \leq m$. One can check that conditions (3.77) hold for this grading. Thus $\mathcal{G}$ is a tame EALA. Set

$$
F = \begin{pmatrix}
\tau_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \tau_m
\end{pmatrix}
$$

and

$$
G = \begin{pmatrix}
0 & I_\ell & 0 \\
I_\ell & 0 & 0 \\
0 & 0 & F
\end{pmatrix},
$$

where $I_\ell$ is the $\ell \times \ell$ identity matrix. For $Y \in \mathcal{K}$ define $\sigma(Y) = -G^{-1}\tilde{Y}^t G$. Then $\sigma$ defines a period two automorphism of $\mathcal{G}$. It is straightforward to see that $\sigma$ satisfies (A2)–(A3). Elements of $\mathcal{K}^\sigma$ are of the form

$$
X = \begin{pmatrix}
A & S & -\tilde{D}^t F \\
T & -\tilde{A}^t & -\tilde{C}^t F \\
C & D & B
\end{pmatrix},
$$

where $\tilde{S}^t = -S$, $\tilde{T}^t = -T$, $F^{-1}\tilde{B}^t F = -B$, and $\text{tr}(X) \equiv 0 \mod [A, A]$. 

We next want to check (A4). Note that \( \mathcal{H}^\sigma = \{ \sum_{i=1}^\ell a_i(e_{ii} - e_{\ell+i,\ell+i}) \mid a_i \in \mathbb{C} \} \). It is easy to see that

\[
\mathcal{G} = \bigoplus_{a \in \mathcal{R}^\sigma} \mathcal{G}_a = \sum_{\delta \in \mathcal{Z}^\sigma} \sum_{a \in \hat{R}} \mathcal{G}_{a+\delta},
\]

where \( \hat{R} = \{ \pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i \leq j \leq n \} \),

\[
\mathcal{G}_0 = \mathcal{H} \oplus \mathcal{C} \oplus \mathcal{D} \quad \text{and} \quad \mathcal{G}_{a+\delta} = \mathcal{K}_a \cap \mathcal{K}^\delta \quad \text{if} \ \alpha + \delta \neq 0.
\]

In particular, the root system \( R \) of \( \mathcal{G} \) is of the form

\[
R = \{ \alpha + \delta \mid \alpha \in \hat{R}, \ \delta \in \mathbb{Z}^\nu \ \text{and} \ \mathcal{K}_a \cap \mathcal{K}^\delta \neq [0] \}.
\]

To describe the root system of \( \mathcal{G} \) note that if \( 1 \leq i < j \leq 2\ell \), then \( \deg(e_{ij}) = \lambda_i - \lambda_j = 0 \) and so \( e_{ij} \in \mathcal{K}_{e_i-e_j} \cap \mathcal{K}^0 \). Therefore \( e_i - e_j \in \mathcal{R} \). If \( 1 \leq i \leq 2\ell \) and \( 1 \leq j \leq m \), then \( \deg(x^\delta e_{i,2\ell+j}) = 2\delta + \lambda_i - \lambda_j = 2\delta - \tau_j \). Therefore \( x^\delta e_{i,2\ell+j} \in \mathcal{K}_{e_i-2\ell+j} \cap \mathcal{K}^{2\delta-\tau_j} \). So \( e_i - e_{2\ell+j} + 2\delta - \tau_j \in \mathcal{R} \). Finally, if \( 1 \leq i \leq j \leq m \), then \( \deg(x^\delta e_{2\ell+i,2\ell+j}) = 2\delta + \tau_i - \tau_j \). So

\[
x^\delta e_{2\ell+i,2\ell+j} \in \mathcal{K}_{e_{2\ell+i}-2\ell+j} \cap \mathcal{K}^{2\delta+\tau_i-\tau_j}.
\]

Thus \( e_{2\ell+i} - e_{2\ell+j} + 2\delta + \tau_i - \tau_j \in \mathcal{R} \). This in particular shows that if \( i \neq j \) then \( e_{2\ell+i} - e_{2\ell+j} \notin \mathcal{R} \), as \( \tau_i - \tau_j \notin 2\mathbb{Z}^\nu \) if \( i \neq j \). Set

\[
e_i' = e_i \quad \text{for} \ 1 \leq i \leq 2\ell \quad \text{and} \quad \varepsilon_i' = \varepsilon_{2\ell+i} + \tau_i \quad \text{for} \ 1 \leq i \leq m.
\]

Then

\[
R = 2\mathbb{Z}^\nu \cup \{ \pm(\varepsilon_i' - \varepsilon_j') + 2\delta \mid \delta \in \mathbb{Z}^\nu, \ 1 \leq i \leq j \leq n \}.
\]

It now follows that

\[
\alpha \in R \setminus [0] \implies \pi(\alpha) = \alpha_{|2\mathbb{Z}^\nu} \neq 0.
\]

Thus by [2, Proposition 3.25], the condition (A4) also holds. It is shown in [4, Proposition 3.23], the root system \( R^\sigma \) of \( \mathcal{G}^\sigma \) with respect to \( \mathcal{H}^\sigma \) is an irreducible reduced EARS of type

\[
A_1 \quad \text{if} \ \ell = 1, \quad e = 1_v, \quad \text{and} \quad q = 1_{v \times v},
\]

\[
B_\ell \quad \text{if} \ \ell \geq 2, \quad e = 1_v, \quad \text{and} \quad q = 1_{v \times v},
\]

\[
BC_\ell \quad \text{if} \ e \neq 1_v \text{ or } q \neq 1_{v \times v}.
\]

Thus by Corollary 2.68, \( \mathcal{G}^\sigma \) is an EALA of the above type. It is not difficult to show that \( \mathcal{G}^\sigma = \mathcal{K}^\sigma \oplus \mathcal{C} \). Since \( \mathcal{G}_c = \mathcal{K} \oplus \mathcal{C} \), we have \( \mathcal{G}^\sigma_c = \mathcal{G}^\sigma \cap \mathcal{G}_c \). It now follows from Lemma 2.56 that \( \mathcal{G}^\sigma \) is a tame EALA.
Example 3.79. Let ν ≥ 1 and let \( A \) be the associative commutative Laurent polynomials in \( ν \)-variables \( x_1, \ldots, x_ν \). Let \( ℓ ≥ 1, m ≥ 1, n = ℓ + m \). Let

\[
\mathcal{K} = \left\{ \begin{pmatrix} A & B \\ C & -A' \end{pmatrix} \middle| B' = -B, \ C' = -C, \ A, B, C ∈ M_ν(\mathbb{A}) \right\}
\]

and

\[
\mathcal{H} = \left\{ \sum_{i=1}^{n} a_i (e_{ii} - e_{n+i,n+i}) \middle| a_i ∈ \mathbb{C} \right\}.
\]

Let \( τ_1, \ldots, τ_m \) represent distinct cosets of \( 2\mathbb{Z}^ν \) in \( \mathbb{Z}^ν \) with \( τ_1 = 0 \). Define a grading on \( \mathcal{K} \) by \( \deg(x_{δepq}) = 2δ + λ_p - λ_q \) where

\[
λ_i = λ_{n+i} = 0 \quad \text{for} \quad 1 ≤ i ≤ ℓ, \quad \text{and} \quad λ_ℓ+i = -τ_i, \quad λ_{n+i+i} = τ_i \quad \text{for} \quad 1 ≤ i ≤ m.
\]

Then the conditions (1.1)–(1.11) of [1, III.§1] are satisfied and so by [1, Proposition III.1.20] the triple \( (G, (·, ·), H) \) is an EALA of type \( D_n (n ≥ 4) \), where \( G = \mathcal{K} ⊕ C ⊕ D \) and \( H = \mathcal{H} ⊕ C ⊕ D \).

Put

\[
F = \begin{pmatrix} x^{τ_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x^{τ_m} \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} I_ℓ & 0 & 0 & 0 \\ 0 & 0 & 0 & F \\ 0 & 0 & I_ℓ & 0 \\ 0 & F^{-1} & 0 & 0 \end{pmatrix}.
\]

Define a period 2 automorphism of \( \mathcal{K} \) by \( σ(Y) = KYK \) and extend it to an automorphism of \( G \) as before. It is easy to check that \( σ \) satisfies (A2)–(A3). Next we check (A4). It is not difficult to see that the root system \( R \) of \( G \) is of the form

\[
R = 2\mathbb{Z}^ν \cup \{ ±(ε'_i ± ε'_j) + 2\mathbb{Z}^ν \mid 1 ≤ i < j ≤ n \},
\]

where

\[
e'_i = ε_i \quad \text{for} \quad 1 ≤ i ≤ ℓ \quad \text{and} \quad e'_i = ε_{ℓ+i} - τ_i \quad \text{for} \quad 1 ≤ i ≤ m.
\]

Also we have

\[
H^σ = \left\{ \sum_{i=1}^{ℓ} a_i (e_{ii} - e_{n+i,n+i}) \middle| a_i ∈ \mathbb{C} \right\} ⊕ C ⊕ D.
\]

It follows that

\[
α ∈ R \setminus \{0\} \implies π(α) = α|_{2ε} ≠ 0.
\]
This proves that (A4) holds for \( \sigma \). One can check that each element of \( K^\sigma \) is of the form
\[
\begin{pmatrix}
A & -D^t & S & -D^t F \\
FC & -B^t & FD & FP F \\
T & -C^t & D & -C^t F \\
C & P & D & B
\end{pmatrix}
\]
with \( S^t = -S \), \( T^t = -T \), \( P^t = -P \), \( F^{-1} B^t F = -B \), \( \ell \geq 2 \).

(3.80)

where \( A, S, T \in M_\ell(A), C, D \in M_m \times \ell(A) \) and \( B, P \in M_m(A) \). It is shown in [4, Proposition 3.24] that the root system \( R^\sigma \) of \( G^\sigma \) with respect to \( H^\sigma \) is an irreducible reduced EARS of type
\[
A_1 \quad \text{if } \ell = 1 \quad \text{and} \quad B_\ell \quad \text{if } \ell \geq 2.
\]
Thus by Corollary 2.68, \( G^\sigma \) is an EALA of the above type.

Note that \( \tilde{G} \) is a tame EALA and \( \tilde{G}_c = K \oplus \mathbb{C} \) (see [1, Chapter III]). Let \( K^\sigma_c \) be the subalgebra of \( K^\sigma \) consisting of all matrices of the form (3.80) with \( P = BF^{-1} \). By [4, Lemma 3.17], \( G^\sigma_c = K^\sigma_c \oplus \mathbb{C} \). It follows that \( G^\sigma_c = G^\sigma \cap \tilde{G}_c \) if and only if \( m = 1 \). Now Lemma 2.56 implies that \( G^\sigma \) is tame if and only if \( m = 1 \) (see [4, Corollary 3.22]). This example in particular shows that \( G^\sigma_c \) in general is a proper subalgebra of \( G^\sigma \cap \tilde{G}_c \).

Example 3.81 (See [2]). Let \( (G, (\cdot, \cdot), \mathcal{H}) \) be an EALA with root system \( R \). Consider the so called affinization
\[
\text{Aff}(G) = (G \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d,
\]
of \( G \) introduced in [2]. Then \( \text{Aff}(G) \) is a Lie algebra where \( c \) is central, \( d = t \frac{d}{dt} \) is the degree derivation so that \( [d, x \otimes t^n] = nx \otimes t^n \), and
\[
[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + n(x, y)\delta_{m+n,0}c.
\]
Extend the form \((\cdot, \cdot)\) to \( \text{Aff}(G) \) by
\[
(x \otimes t^m + rc + sd, y \otimes t^n + r'c + s'd) = \delta_{n+m,0}(x, y) + rs' + r's,
\]
where \( r, s, r', s' \in \mathbb{C} \). Then the triple
\[
(\text{Aff}(G), (\cdot, \cdot), \mathcal{H} \oplus \mathbb{C}c \oplus \mathbb{C}d)
\]
is an EALA with root system \( \tilde{R} = R + \mathbb{Z} \delta \) where \( \delta \in \tilde{\mathcal{H}}^* \) is defined by \( \delta(d) = 1 \) and \( \delta(\mathcal{H} + \mathbb{C}c) = 0 \). Moreover, \( \text{Aff}(G) \) is tame if \( G \) is tame (see [2, Lemma 3.61]). Next consider an automorphism \( \sigma \) of \( G \) satisfying (A1)–(A4). Extend \( \sigma \) to an automorphism of \( \text{Aff}(G) \) by
\[
\sigma(x \otimes t^i + rc + sd) = \zeta^{-i} \sigma(x) \otimes t^i + rc + sd.
\]
It is easy to see that $\sigma$ as an automorphism of $\text{Aff}(\mathcal{G})$ satisfies (A1)–(A4). Then by [2, Lemmas 3.49, 3.62] the root system of $\text{Aff}(\mathcal{G})^\sigma$ is irreducible (if it contains at least one nonisotropic root). Thus by Corollary 2.68, $\text{Aff}(\mathcal{G})^\sigma$ is an EALA. Therefore, all Lie algebras $\mathcal{G}$ constructed in [2] can be considered as fixed point subalgebras of the loop algebra of some EALA.

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