On Commutative Nonassociative Algebras*

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In this paper we study a class of commutative nonassociative algebras which includes those special Jordan algebras which arise as the set of all elements fixed by an involution in a primitive ring with nonzero socle and with centralizer which is a field of characteristic not 2. Perhaps the most unusual feature of our approach is that we do not assume that our algebras satisfy any identity, but only that enough primitive idempotents exist satisfying certain properties that follow from the Jordan identity. (For an introduction to the standard theory, see [6].) We also need an axiom which insures that the discrete topology (Jacobson’s finite topology) will suffice.

More specifically, we assume that the following four axioms on idempotents hold in our algebra $A$ over the field $\Phi$ ($A$ will be assumed to be commutative and of characteristic not 2 hereafter):

(i) If $e$ is any idempotent of $A$ and if $A_e(\lambda) = \{x \mid x \in A, ex = \lambda x\}$ for each $\lambda \in \Phi$, then $A = A_e(1) + A_e(\frac{1}{2}) + A_e(0)$. Furthermore, $A_e(\lambda)A_e(\lambda) \subseteq A_e(\lambda)$ and $A_e(\lambda)A_e(\frac{1}{2}) \subseteq A_e(\frac{1}{2}) + A_e(1 - \lambda)$ for $\lambda = 0$ and 1, and $A_e(1)A_e(0) = 0$ and $A_e(\frac{1}{2})A_e(\frac{1}{2}) \subseteq A_e(1) - A_e(0)$.

(ii) $A$ contains a set $I$ of mutually orthogonal primitive idempotents with the property that no nonzero element of $A$ is orthogonal to all the elements of $I$.

(iii) If $e$ is a primitive idempotent of $A$ which is orthogonal to all but a finite number of the elements of $I$, then $A_e(1) = \Phi e$.

(iv) Any scalar extension of $A$ of degree 2, 4, or 8 also satisfies (i), and any scalar extension of degree 2 or 4 satisfies (iii).

Letting $F$ denote the set of elements of $A$ which are orthogonal to all but a finite number of the elements of $I$, and letting the idempotents of $I$ be indexed by the set $S$, we prove from these axioms that $F = \sum A_i + \sum A_{ij} + \sum A_{i0}$, where

$$A_i = A_{e_i(1)}, \quad A_{ij} = A_{e_i(\frac{1}{2})} \cap A_{e_j(\frac{1}{2})}.$$
for $i, j \in S$ and $i \neq j$, and where these subspaces multiply as in a Jordan algebra.

Given any element $x \in A$ it is natural to define its component in $A_i$ or $A_{ij}$ by $x_{ii} = (xe_i)(2e_i - 1)$ and $x_{ij} = 4(xe_i)e_i$ respectively. If $x$ is in $F$, we may also define

$$x_{i0} = 2xe_i - \sum_{j \in S} x_{ij} \in A_{i0}$$

and verify that the components of the product of two elements $x, y \in F$ are given by

$$(xy)_{ii} = \sum_{k} x_{ik}y_{ki}, \quad (xy)_{ij} = \sum_{k} x_{ik}y_{kj} + \sum_{k} x_{jk}y_{ki} \quad (1)$$

for each $i \in S$ and each $j \in S \cup \{0\}$ not equal to $i$, where $k$ ranges over all elements of $S \cup \{0\}$. However, in order to be able to define a notion of the component of $x$ in $A_{i0}$ for $x \notin F$ and to insure that the behavior of $x \notin F$ is determined by its components, we need to add another axiom.

(v) For each $i, j \in S$ and for each $x, y \in A$, only a finite number of the products $x_{ik}y_{ki}$ and $x_{ik}y_{ki}$ are nonzero. Furthermore, there exists a vector space homomorphism $\psi : A \to A$ satisfying $\psi(xe_k) = \psi(x)e_k \in A_{k0}$ for each $k$, such that for each $i \in S$ only a finite number of the products $x_{ik}y_{ki}$ are nonzero where $y_{k0} = 4\psi(ye_k)$, and such that the equations (1) are satisfied for each $x, y \in A$.

Letting $\varphi$ be a homomorphism of an algebra $A$ satisfying Axioms (i)–(v), we shall call $\varphi$ admissible if the set $\varphi(I)$ of distinct images of elements of $I$ is a set of orthogonal idempotents of $\varphi(A)$ satisfying Axiom (ii). By the radical of an algebra $A$ satisfying Axioms (i)–(v) we shall mean any ideal $R$ of $A$ which is maximal with respect to the property of not containing any primitive idempotents of the subalgebra $F$ (we shall see later on that the radical is unique and may also be characterized in several other ways.) We define an algebra $A^*$ to be primitive if it is the homomorphic image of an algebra $A$ satisfying Axioms (i)–(v) under an admissible homomorphism $\varphi$ whose kernel contains the radical of $A$, and if, given any two subscripts $i$ and $j$ in an indexing set $S^*$ of $\varphi(I)$, there exists a finite set of elements $k_0 = i, k_1, \cdots, k_n = j$ of $S^*$ such that

$$A_{\varphi^r}^* \neq 0 \quad \text{for} \quad r = 0, 1, \cdots, n - 1.$$
and $A$ is called regular if each subspace $A_{ij}$ in $A$ is regular. We are now finally able to state our main results.

**Theorem 1.** If $A$ is a commutative algebra satisfying Axioms (i)-(v) and if $R$ is the radical of $A$, then $A/R$ is a subdirect sum of primitive algebras $A_i$. The subalgebra $F + R/R$ is a direct sum of the corresponding algebras $F_i$ in the $A_i$'s.

**Theorem 2.** A regular primitive algebra is either the split exceptional Jordan algebra, or can be realized as a set of matrices under the Jordan product. In the latter case, $F$ is faithfully represented as a simple reduced Jordan algebra of type $A_1, A_2, B, C_1, C_2$, or $D$.

The matrices in this theorem are understood to be not only possibly infinite but also possibly uncountable. When we say that a set of matrices is closed under the Jordan product we imply that for each pair of matrices $a_{ij}$ and $b_{ij}$ of the set and for each pair of subscripts $i, j$ there are only a finite number of subscripts $k$ such that the product $a_{ik} b_{kj}$ is nonzero. The reduced Jordan algebras of type $A_1, A_2, B, C_1$, and $C_2$ mentioned in the theorem are the obvious generalizations of the usual finite-dimensional reduced Jordan algebras of type $A_1, A_2, B$, and $C$ obtained by replacing the $n \times n$ matrices by the set of all matrices of a given cardinality which have only a finite number of nonzero entries. The generalization of the notion of a Jordan algebra of type $D$ to the infinite dimensional case is equally straightforward.

We do not know whether a set of matrices closed under the Jordan product is necessarily a subset of a set of matrices closed under the associative product. Whenever it is, Theorem 2 may be sharpened.

**Corollary.** If the set of matrices in Theorem 2 corresponding to a given regular primitive algebra $A$ are a subset of an associative algebra of matrices, then $A$ is isomorphic to one of the following:

(a) A Jordan subalgebra $K$ of a primitive ring $G$ with nonzero socle whose centralizer is a field, where $K$ contains the set of all elements of $G$ which are represented by matrices with finitely many nonzero entries in some representation of $G$.

(b) A Jordan subalgebra $K$ of a primitive ring $G$ with nonzero socle whose centralizer is a field, where the elements of $K$ are fixed under an involution $J$ of $G$, and where $K$ contains the set of all elements of $G$ fixed by $J$ which are represented by matrices with finitely many nonzero entries in some representation of $G$.

(c) A Jordan subalgebra $K$ of a primitive ring $G$ with nonzero socle whose centralizer is a generalized quaternion division algebra $\Gamma$, where the elements of $K$ are fixed under an involution $J$ of $G$, where $K$ contains the set of all elements.
of $G$ fixed by $J$ which are represented by matrices with finitely many nonzero entries in some representation of $G$, and where the elements of $\Gamma$ fixed under the involution of $\Gamma$ induced by $J$ form the center of $\Gamma$.

Conversely, each of these algebras $K$ satisfies Axioms (i)-(v).

Our final result deals with the nonregular case.

**Theorem 3.** Let $A$ be a primitive algebra which is not regular. Then each of the subspaces $A_{ij}$ is either zero or is one-dimensional and not regular.

Some examples of nonregular primitive algebras are also given to show that they actually exist and that they do not have uniform local structure as the regular primitive algebras have.

1. In this section we shall develop most of the basic machinery and prove Theorem 1. We begin by investigating the implications of our first axiom.

**Lemma 1.** Let $e_1, \ldots, e_n$ be a set of mutually orthogonal idempotents in an algebra $A$ satisfying Axiom (i), and let $A_i = A e_i$, (1),

$$A_{ij} = A e_i \left(\frac{1}{2}\right) \cap A e_j \left(\frac{1}{2}\right), \quad A_{i0} = A e_i \left(\frac{1}{2}\right) \cap \bigcap_{k \neq i} A e_k(0),$$

and

$$A_0 = \bigcap_i A e_i(0) \quad \text{for} \quad i, j, k = 1, 2, \ldots, n,$$

and $i \neq j$. Then

$$A = \sum_{i=1}^{n} A_i + \sum_{i \neq j=0}^{n} A_{ij} + \sum_{i=1}^{n} A_{i0} + A_0$$

is a vector space direct sum, and $A_{ij} A_{jk} \subset A_{ik}$, $A_{ij} A_{i0} \subset A_{i0}$, $A_{i0} A_{j0} \subset A_{ij}$, $A_{ij} A_{ij} \subset A_i + A_j$, and $A_{ij} A_{kl} = A_{ij} A_{k0} = A_{ij} A_{k} = A_i A_k = 0$ for $i, j, k, l$ distinct.

It will be convenient to prove this lemma under the assumption that $A_{ij}$ is given by the slightly modified definition

$$A_{ij} = A e_i \left(\frac{1}{2}\right) \cap A e_j \left(\frac{1}{2}\right) \cap \bigcap_{k \neq i, j} A e_k(0),$$

and then to show that the two definitions are equivalent. Suppose first that $0 = \sum a_i + \sum a_{ij} + \sum a_{i0} + a_0$, where each symbol in this equation is
assumed to be in the subspace with the same subscripts. Multiplying this equation by $e_i$ gives

$$0 = a_i = \frac{1}{2} \sum_j a_{ij} + \frac{1}{2} a_{i0}$$

since $e_i a_k \in A_{e_i}(0)A_{e_k}(1) = 0$ for $k \neq i$, and again applying $e_i$ gives

$$0 = a_i = \frac{1}{4} \sum_j a_{ij} + \frac{1}{4} a_{i0}.$$  

Thus,

$$0 = a_i = \sum_j a_{ij} + a_{i0},$$

and multiplying this by $e_j$ gives $0 = a_{ij}$ for each $j \neq i$. It now follows easily that $a_{i0} = a_0 = 0$, and hence that the right side of (2) is a vector-space direct sum.

Next let $e = e_1 + \cdots + e_n$ and let $a = a_1 + a_2 + a_0$ be the decomposition postulated in Axiom (i) of an arbitrary element $a$ of $A$ with respect to $e$. In order to establish (2) it is sufficient to prove by induction on $n$ that $$a_i \in \sum A_i + \sum A_i,$$

and $a_0 \in A_0$. The last relation is immediate, since $e_i a_0 \in A_{e_i}(1)A_{e_i}(0) = 0$, giving

$$a_0 \in \bigcap_i A_{e_i}(0) = A_0.$$  

To show that $a_i \in \sum A_0$, let $e' = e_1 + \cdots + e_{n-1}$ and let $a_i = b_1 + b_1 + b_0$ be the decomposition of $a_i$ with respect to $e'$. Then $e_i b_1 \in A_{e_i}(0)A_{e_i}(1) = 0$ and $e_i b_1 = (e' + e_n) b_1 = b_1$, so that $b_1 \in A_{e_i}(1)$. On the other hand,

$$b_1 = (a_i e')(2 e' - 1) \in [A_{e_i}(1)A_{e_i}(1)](2 e' - 1) \subset [A_{e_i}(1) + A_{e_i}(0)](2 e' - 1) \subset A_{e_i}(1) + A_{e_i}(0).$$

Hence, $b_1 = 0$ and $a_i = b_2 + b_0$, which leads to

$$\frac{1}{2} b_0 = \frac{1}{2} a_i - \frac{1}{2} b_1 = (e - e') a_i = e_i a_i + e_i b_0.$$

Since $e_i b_1 \in A_{e_i}(0)A_{e_i}(\frac{1}{2}) \subset A_{e_i}(\frac{1}{2}) + A_{e_i}(1)$ and $e_i b_0 \in A_{e_i}(0)A_{e_i}(0) \subset A_{e_i}(0)$, the last equation breaks into $e_i b_2 = 0$ and $e_i b_0 = \frac{1}{2} b_0$, or $b_i \in A_{e_i}(0)$ and $b_0 \in A_{e_i}(\frac{1}{2})$. But

$$b_i \in \sum_{i \neq n} [A_{e_i}(\frac{1}{2}) \cap \bigcap_{k \neq i, n} A_{e_k}(0)].$$
and

\[ b_0 \subset \bigcap_{i \neq n} A_{e_i}(0) \]

by induction, giving

\[ a_i = b_i + b_0 \in \sum_{i \neq n} \left[ A_{e_i} \left( \frac{1}{2} \right) \cap \bigcap_{k \neq i} A_{e_k}(0) \right] \]

\[ + A_{e_n} \left( \frac{1}{2} \right) \cap \bigcap_{i \neq n} A_{e_i}(0) = \sum A_{i0} . \]

To show that \( a_1 \in \sum A_i + \sum A_{ij} \), we let \( a_1 = c_1 + c_2 + c_0 \) be the decomposition of \( a_1 \) with respect to \( e' \). Since \( a_1, e' \in A_e(1) \) and since each component of \( a_1 \) can be expressed as \( a \) operated on by a polynomial in \( R_e \) (right multiplication by \( e' \)), we see that \( c_1, c_2, c_0 \in A_e(1) \). Then

\[ e_n c_0 = (e - e') c_0 = c_0 \]

or \( c_0 \in A_n \), and \( e_n c_\frac{1}{2} = (e - e') c_\frac{1}{2} = c_\frac{1}{2} - \frac{1}{2} c_\frac{1}{2} = \frac{1}{2} c_\frac{1}{2} \) or \( c_\frac{1}{2} \in A_{e_\frac{1}{2}}(\frac{1}{2}) \). But \( c_1 \in \sum A_i + \sum A_{ij} \) and \( c_\frac{1}{2} \in \sum \left[ A_{e_\frac{1}{2}}(\frac{1}{2}) \cap \bigcap A_{e_\frac{1}{2}}(0) \right] \) for \( i, j \neq n \) by induction, giving \( a_1 = c_1 + c_\frac{1}{2} + c_0 \in \sum A_i + \sum A_{ij} \) as desired.

Suppose now that \( a \in A_{e_\frac{1}{2}}(\frac{1}{2}) \cap A_{e_\frac{1}{2}}(\frac{1}{2}) \) and that

\[ a = \sum a_i + \sum a_{ij} + \sum a_{i0} + a_0 \]

is the decomposition of a given by (2) (still using our modified definition of \( A_{ij} \)). Multiplying this equation first by \( e_i \) and then by \( e_j \) gives \( \frac{1}{2} a = \frac{1}{2} a_{ij} \), showing that our two definitions of \( A_{ij} \) are indeed equivalent.

Finally, we have

\[ A_{ij} A_{jk} \subset A_{e_i} \left( \frac{1}{2} \right) A_{e_j}(0) \subset A_{e_i} \left( \frac{1}{2} \right) + A_{e_j}(1) = A_i + \sum A_{ii} + A_{i0} , \]

and, by symmetry,

\[ A_{ij} A_{jk} \subset A_k + \sum A_{ki} + A_{k0} \]

for \( i, j, k \) distinct. Thus \( A_{ij} A_{jk} \subset A_{ik} \), and by an identical calculation, \( A_{i0} A_{j0} \subset A_{ij} \). Next,

\[ A_{ij} A_{j0} \subset A_{e_i} \left( \frac{1}{2} \right) A_{e_j}(0) \subset A_i + \sum A_{ii} + A_{i0} \]

as above, \( A_{ij} A_{j0} \subset A_{e_j}(\frac{1}{2}) A_{e_j}(\frac{1}{2}) \subset A_{e_j}(1) + A_{e_j}(0) \), and

\[ A_{ij} A_{j0} \subset A_{k}(0) A_{k}(0) \subset A_{k}(0) \] for \( k \neq i, j \), giving \( A_{ij} A_{j0} \subset A_i + A_{i0} \).
But we also have

\[ A_{ij}A_{j0} \subseteq A_{e_i + e_j} (1) A_{e_i + e_j} (\frac{1}{2}) \subseteq A_{e_i + e_j} (\frac{1}{2}) + A_{e_i + e_j} (0), \]

which leads to \( A_{ij}A_{j0} \subseteq A_{i0} \). Now,

\[ A_{ij}A_{ij} \subseteq A_{e_i} (\frac{1}{2}) A_{e_i} (\frac{1}{2}) \subseteq A_{e_i} (1) + A_{e_i} (0) \]

and, by symmetry, \( A_{ij}A_{ij} \subseteq A_{e_j} (1) + A_{e_j} (0) \) yielding \( A_{ij}A_{ij} \subseteq A_i + A_j \).

Since \( A_{ij} \subseteq A_{e_i + e_j} (1) \) and \( A_{kl} \subseteq A_{e_k + e_l} (0) \), we also have \( A_{ij}A_{kl} = A_{ij}A_{k0} = A_{ij}A_k = A_iA_k = 0 \) for \( i, j, k, l \) distinct, to finish the proof of Lemma 1.

Suppose now that \( A \) satisfies Axioms (ii) and (iii) in addition to (i) and let \( e_i, e_j, e_k, \ldots \) denote the idempotents of \( I \) where \( i, j, k \) lie in some indexing set \( S \). Defining the subspaces \( A_i, A_{ij}, \) and \( A_{i0} \) as in Lemma 1 for distinct \( i, j \in S \) (note that \( A_{i0} = 0 \) by (ii) this time), we see that these subspaces are again independent and that products of these subspaces satisfy the relations given in Lemma 1 (since all calculations can be done with a finite number of indices). In addition, (iii) implies that \( A_iA_{ij} \subseteq A_{ij} \). Hence, letting \( F \) denote the set of all elements of \( A \) which are finite sums of elements in the subspaces \( A_i, A_{ij}, A_{i0} \) for \( i, j \in S \), we see that \( F \) is a subalgebra of \( A \).

In order to avoid having to treat the subspaces of type \( A_{ij} \) and of type \( A_{i0} \) separately in our proofs hereafter, let us observe that \( A \) may be embedded in the algebra \( A_0 \) which is additively the direct sum of \( A \) and a one-dimensional algebra \( e_0 \), where multiplication in \( A_0 \) is given by

\[ (\alpha e_0 + x)(\beta e_0 + y) = \alpha \beta e_0 + [\alpha \varphi (y) + \beta \varphi (x) + xy]. \]

It is easy to check that \( A_0 \) satisfies Axioms (i)-(v) with \( I_0 = I \cup \{ e_0 \} \) playing the role of \( I \), and that \( A_0 \) contains no elements which are in the halfspace for one idempotent of \( I_0 \) and in the zero-space for all the others. Since the process of embedding \( A \) in \( A_0 \) is quite similar to the process of adjoining an identity element to an algebra (and is identical if the cardinality of \( I \) is finite), we shall refer to it as the process of adjoining an idempotent to \( A \). We have proved

**Proposition 1.** After possibly adjoining an idempotent to \( A \), the subalgebra \( F = \{ x \mid x \in A, xe_i \neq 0 \text{ for finitely many } e_i \text{'s in } I \} \) is additively the direct sum of the subspaces \( A_i = \Phi e_i \) and the subspaces \( A_{ij} = A_i (\frac{1}{2}) \cap A_j (\frac{1}{2}) \) for \( i, j \in S \) and \( j \neq i \). Also, \( A_{ij}A_{jk} \subseteq A_{ik} \), \( A_{ij}A_{kl} = 0 \), and

\[ A_{ij} \subseteq A_i + A_j \]

for \( i, j, k, l \) distinct.
We are now ready to begin a study of $F$ which will lead to Theorem 1. We shall first give a complete proof under the assumption that $F = \sum A_i + \sum A_{ij}$, and then it will be easy to show that the theorem also holds if the spaces $A_{i0}$ are not all zero.

**Lemma 2.** An element $u \neq e_i + e_j$ of $A_i + A_{ij} + A_j$ is idempotent if and only if $u$ has the form $u = \alpha e_i + (1 - \alpha) e_j + a$ for some $\alpha, \gamma \in \Phi$ satisfying $a^2 = \gamma (e_i + e_j) + \alpha - \alpha + \gamma = 0$.

Suppose first that $u \in A_i + A_{ij} + A_j$ is idempotent. Then $u = \alpha e_i + \beta e_j + a$ for some $\alpha, \beta \in \Phi$ and $a \in A_{ij}$, and

$$u^2 = \alpha^2 e_i + \beta^2 e_j + a^2 + (\alpha + \beta)a = \alpha e_i + \beta e_j + a.$$  

If, for each $x \in A_{ij}$, $[x]_i$ denotes that element of $\Phi$ which satisfies the relation $x_i = [x]_i e_i$, we see that the last equation may be separated into its components in $A_i$, $A_j$, and $A_{ij}$ to give respectively $\alpha^2 + [a^2]_i = \alpha, \beta^2 + [a^2]_j = \beta$, and $(\alpha + \beta)a = a$. If $a = 0$ the desired conclusion follows from these equations trivially, so we may take $a \neq 0$. Thus $\alpha + \beta = 1$, and

$$[a^2]_i = \alpha, \quad \alpha^2 = (1 - \beta), \quad (1 - \beta)^2 - \beta, \quad \beta = [a^2]_j.$$  

Letting $\gamma = [a^2]_i$, we have $a^2 = \gamma (e_i + e_j)$ and $\alpha^2 - \alpha + \gamma = 0$ as desired. Conversely, if $a^2 = \gamma (e_i + e_j)$ and $\alpha^2 - \alpha + \gamma = 0$, then

$$u = \alpha e_i + (1 - \alpha) e_j + a$$  

is clearly idempotent.

**Lemma 3.** Let $u \neq e_i + e_j$ be an idempotent in $A_i + A_{ij} + A_j$. Then $A_u(1) \cap [A_i + A_{ij} + A_j] = \Phi u$.

By Lemma 2 we have $u = \alpha e_i + (1 - \alpha) e_j + a$ where $a^2 = \gamma (e_i + e_j)$ and $\alpha^2 - \alpha + \gamma = 0$. If $w = \beta_1 e_i + \beta_2 e_j + b$ is an element of

$$A_u(1) \cap [A_i + A_{ij} + A_j],$$

then

$$w = uw = \alpha_1 e_i + (1 - \alpha) \beta_2 e_j + ab + \frac{1}{2}(\beta_1 + \beta_2)a + \frac{1}{2}b = \beta_1 e_i + \beta_2 e_j + b.$$  

Separating this equation into its components in $A_i$, $A_j$, and $A_{ij}$ gives $\alpha \beta_1 + [ab]_i = \beta_1, (1 - \alpha) \beta_2 + [ab]_j = \beta_2, \frac{1}{2}(\beta_1 + \beta_2)a + \frac{1}{2}b = b$. The last equation reduces to $b = (\beta_1 + \beta_2)a$, and substituting this into the other two equations gives $\alpha \beta_1 + (\alpha - \alpha^2)(\beta_1 + \beta_2) = \beta_1$ and

$$(1 - \alpha) \beta_2 + (\alpha - \alpha^2)(\beta_1 + \beta_2) = \beta_2.$$
Thus, \( \beta_1 = \alpha \beta_1 + \beta_2 - (1 - \alpha) \beta_2 = \alpha (\beta_1 + \beta_2) \) and 
\[
\beta_2 = (1 - \alpha) \beta_2 + \beta_1 - \alpha \beta_1 = (1 - \alpha) (\beta_1 + \beta_2),
\]
showing that \( v = (\beta_1 + \beta_2) u \in \Phi u \) as desired.

**Proposition 2.** Either \( A_{ij} \) has dimension one or \( A_i + A_{ij} + A_j \) is a Jordan algebra.

Using the fact that any algebra \( B \) occurs in a natural way as a subring of any scalar extension of \( B \), it is easy to see that Proposition 2 will follow for \( B - A_i + A_{ij} + A_j \) if we can prove it for some scalar extension of \( B \). Since Axioms (i) and (iii) are true in any scalar extension of \( A \) of degree 2 or 4 by hypothesis and since (ii) and (v) automatically remain true in any scalar extension of \( A \), we may make two quadratic extensions during our proof and still continue to make use of anything derived just from Axioms (i)-(iii) and (v), or of anything derived from Axioms (i)-(v) which remains true under scalar extension.

Now if the dimension of \( A_{ij} \) is 0 or 1 Proposition 2 is clearly true, so we may assume that there exist linearly independent elements \( b \) and \( c \) in \( A_{ij} \). Then \( b^2 = \alpha_1 e_i + \beta_1 e_j \), \( bc = \alpha_2 e_i + \beta_2 e_j \), and \( c^2 = \alpha_3 e_i + \beta_3 e_j \) for some \( \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \Phi \). Letting \( a = \lambda b + \mu c \) for some scalars \( \lambda, \mu \) yet to be chosen, we have 
\[
a^2 = \lambda^2 b^2 + 2 \lambda \mu b c + \mu^2 c^2 = (\alpha_1 \lambda^2 + 2 \alpha_2 \lambda \mu + \alpha_3 \mu^2) e_i + (\beta_1 \lambda^2 + 2 \beta_2 \lambda \mu + \beta_3 \mu^2) e_j.
\]
The coefficients of \( e_i \) and \( e_j \) in this equation will be equal if and only if 
\[
(\alpha_1 - \alpha_2) \lambda^2 + 2(\alpha_2 - \beta_2) \lambda \mu + (\alpha_3 - \beta_2) \mu^2 = 0,
\]
which can always be achieved by some choice of \( \lambda \) and \( \mu \) in \( \Phi \) or in an appropriate quadratic scalar extension of \( \Phi \). Thus, after possibly making a quadratic scalar extension of \( \Phi \), \( A_{ij} \) contains a nonzero element \( a \) such that \( a^2 = \gamma (e_i + e_j) \) for some scalar \( \gamma \). Letting \( \alpha \) be a scalar satisfying the relation \( \alpha^2 - \alpha + \gamma = 0 \) (a second quadratic scalar extension may be necessary for such an \( \alpha \) to exist), we see from Lemma 2 that \( B = A_i + A_{ij} + A_j \) contains the idempotent 
\[
u = \alpha e_i + (1 - \alpha) e_j + a \text{ where } a \neq 0.
\]
But if \( u = \alpha e_i + (1 - \alpha) e_j + a \in B \), then \( v = (1 - \alpha) e_i + \alpha e_j - a \) is an idempotent of \( B \) orthogonal to \( u \), and \( u + v = e_i + e_j \) is the identity element of \( B \). By Lemma 3, \( B_{\alpha}(1) \) and \( B_{\alpha}(1) \) are one-dimensional, so that 
\[
B = \Phi u + B_{uv} + \Phi v
\]
where \( B_{uv} = B_{u}(\frac{1}{2}) \cap B_{v}(\frac{1}{2}) \), and where \( B_{uv}^2 \subset \Phi u + \Phi v \) by Lemma 1.
Suppose now that $c$ is any element of $A_{ij}$ and that $c = ou + rv + r$ where $r = \delta e_i + ee_j + d \in B_{uv}$ and $d \in A_{ij}$. Then $ur = \frac{1}{2}r$, or
\[
(\alpha e_i + (1 - \alpha)e_j + d)(\delta e_i + ee_j + d)
= \alpha \delta e_i + (1 - \alpha)e_j + ad + \frac{1}{2}d + \frac{1}{2}(\delta + \epsilon)a
= \frac{1}{2}(\delta e_i + ee_j + d).
\]
Separating components, we get $\alpha \delta + [ad]_a = \frac{1}{2} \delta, (1 - \alpha)\epsilon + [ad]_a = \frac{1}{2} \epsilon$, and $\frac{1}{2}(\delta + \epsilon)a = 0$. Since $a \neq 0$, we have $\epsilon = -\delta$, leading to
\[
ad = [ad]_a e_i + [ad]_a e_j = (\frac{1}{2} - \alpha)\delta e_i + (\alpha - \frac{1}{2})\epsilon e_j = (\frac{1}{2} - \alpha)\delta e_i + \epsilon e_j.
\]
We also have $r^2 = (\delta e_i - \delta e_j + d)^2 = \delta^2(e_i + e_j) + d^2 \in A_i + A_j$, while $r \in B_{uv}$ implies that
\[
r^2 = vu + rv = [v\alpha + \rho(1 - \alpha)]e_i + [v(1 - \alpha) + \rho\epsilon]e_j + [v - \rho]a.
\]
But then $[v - \rho]a = 0$ or $\nu = \rho$, yielding
\[
d^2 = r^2 - \delta^2(e_i + e_j) = (\rho - \delta^2)(e_i + e_j).
\]
Observing that the component of the equation $c = ou + rv + r$ in $A_{ij}$ is $c = o\alpha - rv + d$, we finally have
\[
c^2 = [(\sigma - \tau)a + d]^2 = (\delta - \tau)^2a^2 + 2(\sigma - \tau)ad + d^2
= [(\sigma - \tau)^2 + 2(\sigma - \tau)(\alpha - \frac{1}{2})\delta + (\rho - \delta^2)](e_i + e_j).
\]
Thus, the square of any element of $A_{ij}$ has equal coefficients for $e_i$ and $e_j$. As we mentioned in the introduction, we shall call $A_{ij}$ regular in this case.

If an element $c \in A_{ij}$ satisfies $c^2 = \beta(e_i + e_j)$ for some $\beta \in \Phi$, we shall say that $c$ has a norm and write $N(c) = \beta$. Thus, $A_{ij}$ is regular if and only if every element of $A_{ij}$ has a norm. Whenever $A_{ij}$ is regular, we can linearize $N$ to get the symmetric inner product $(b, c) = \frac{1}{2}[N(b + c) - N(b) - N(c)]$ defined on $A_{ij}$. This inner product can be equally well defined by the relation $bc = (b, c)(e_i + e_j)$. Letting $s_2, s_3, \cdots$ be any orthogonal basis of $A_{ij}$ under this inner product and letting $s_0 = e_i + e_j$ and $s_1 = e_i - e_j$, we have the usual basis for a simple Jordan algebra of class $D$ with the two differences that our algebra is not necessarily finite-dimensional and that our algebra is not necessarily simple (since the inner product doesn’t have to be non-singular).

Using the notion of regularity, Proposition 2 may be restated as

**Proposition 2’.** For each distinct $i$ and $j$, $A_{ij}$ is either regular or is spanned by an element $a_{ij}$ such that $a_{ij}^2 = \beta_1 e_i + \beta_2 e_j$ where $\beta_1 \neq \beta_2$. 
Lemma 4. Let \( a \in A_{ij}, b \in A_{ik}, \) and \( d \in A_{jk}, \) with \( a \neq 0 \) and \( a^2 = \gamma(e_i + e_j), \) and let \( u = \alpha e_i + (1 - \alpha) e_j + a \) and \( v = (1 - \alpha) e_i + \alpha e_j - a \) where \( \alpha \) satisfies \( x^2 - \alpha - \gamma = 0. \) Then \( b \) and \( d \) may each be expressed (uniquely) as the sum of an element in \( A_{u\frac{1}{2}} \cap A_{v\frac{1}{2}} \) and an element in \( A_{e_i\frac{1}{2}} \cap A_{e_j\frac{1}{2}}. \) This decomposition is given by \( b = [\alpha b \ 2ab] + [(1 - \alpha) b - 2ab] \) and \( d = [2ad + (1 - \alpha) d] \) respectively. Furthermore, \( 4a(ab) = \gamma b \) and \( 4a(ad) = \gamma d. \)

Since \( A_{ik} + A_{jk} = A_{\frac{1}{2}} \cap A_{e_i\frac{1}{2}} \cap A_{e_j\frac{1}{2}} \), we may write \( b = (x + y) + (w + z) \) where \( x + y \in A_{\frac{1}{2}} \cap A_{e_i\frac{1}{2}} \) and

\[
w + z = A_{e_i\frac{1}{2}} \cap A_{e_j\frac{1}{2}}
\]

and where \( x, y \in A_{ik} \) and \( y, z \in A_{jk} \). Then \( u(x + y) = \frac{1}{2} \alpha x + \alpha x + \frac{1}{2}(1 - \alpha)y + \alpha y = \frac{1}{2}x + \frac{1}{2}y, \) and separating components gives \( ay = \frac{1}{2}(1 - \alpha)x \) and \( ax = \frac{1}{2}ay. \)

Similarly, the relation \( v(w + z) = \frac{1}{2}w - \frac{1}{2}z \) leads to \( aw = \frac{1}{2}(1 - \alpha)z \) and \( az = -\frac{1}{2}\alpha w. \) But separating our original equation into the components \( b = x - w \) and \( y + z = 0, \) we obtain

\[
0 = a(x + z) = ay + az = \frac{1}{2}(1 - \alpha)x - \frac{1}{2}\alpha w.
\]

which implies that \( x = \alpha x + \alpha w + \alpha b \) and that

\[
w = (1 - \alpha)x + (1 - \alpha)w = (1 - \alpha)b.
\]

If \( \alpha \neq 0, \) we also have \( y = (2/\alpha)ax = 2ab \) and \( z = -y = -2ab. \) The same result follows for \( \alpha = 0 \) by switching the roles of \( y \) and \( z, \) to finish the derivation of the formula for \( b. \)

When \( \alpha \neq 0 \) we may compute that \( 4a(ab) = (4/\alpha)a(ax) = (4/\alpha)(a(1/2\alpha y)) \)

\[
= 2ay = (1 - \alpha)x = (\alpha - a\alpha)b = \gamma b. \]

This equation may be derived for \( \alpha = 0 \) by switching the roles of \( u \) and \( v \) (which switches \( \alpha \) and \( 1 - \alpha \)). Then \( 4a(ad) = \gamma d \) also holds by symmetry. The validity of the decomposition for \( d \) now follows from the two calculations

\[
u[2ad + (1 - \alpha)d] = \alpha ad + 2a(ad) + \frac{1}{2}(1 - \alpha)^2 d + (1 - \alpha)ad = ad + \frac{1}{2}\gamma d + \frac{1}{2}(1 - 2\alpha + \alpha^2)d = \frac{1}{2}[2ad + (1 - \alpha)d]
\]

\[
v[-2ad + \alpha d] = -(1 - \alpha)ad + 2a(ad) + \frac{1}{2}\alpha^2 d - \alpha ad = \frac{1}{2}[-2ad + \alpha d].
\]

Proposition 3. If \( a \in A_{ij}, \ b \in A_{ik} \) and if \( a^2 = \gamma(e_i + e_j) \) and \( b^2 = \beta_1 e_i + \beta_2 e_k, \) then \( (ab)^2 = \frac{1}{4}N(b)e_j + \beta_2 e_k, \) if \( \beta_1 = \beta_2, \) then \( ab \) has a norm and \( N(ab) = \frac{1}{4}N(a)N(b). \)

As in the proof of Proposition 2, it is sufficient to assume that \( \Phi \) contains an element \( \alpha \) such that \( \alpha^2 - \alpha + \gamma = 0. \) Then defining \( u \) and \( v \) as above, we
may write \( b = p + q \) where \( p \in A_u(\frac{1}{2}) \cap A_v(\frac{1}{2}) \) and \( q \in A_v(\frac{1}{2}) \cap A_w(\frac{1}{2}) \). Letting \( p^2 = \sigma_1 u + \sigma_2 e_k \), \( q^2 = \tau_1 v + \tau_2 e_k \), and \( 2pq = \delta e_i + \varepsilon e_0 + d \) for \( d \in A_{ij} \), we have
\[
\beta_1 e_i + \beta_2 e_k = b^2 = p^2 + q^2 + 2pq
= \sigma_1 u + \sigma_2 e_k + \tau_1 v + \tau_2 e_k + \delta e_i + \varepsilon e_0 + d.
\]
But the component of \( e_k \) in this equation is \( \beta_2 = \sigma_2 + \tau_2 \), while the sum of the components of \( e_i \) and \( e_j \) is \( \beta_1 = \sigma_1 + \tau_1 + \delta + \varepsilon \). As in the proof of proposition 2, the relation \( u(\delta e_i + \varepsilon e_j + \delta d) = \frac{1}{2}(\delta e_i + \varepsilon e_j + d) \) leads easily to \( \varepsilon = -\delta \), allowing us to write \( \beta_1 = \sigma_1 + \tau_1 \).

Setting \((ab)^2 = \eta_1 e_j + \eta_2 e_k\) and noting that Lemma 3 gives \( p = ab + 2ab \) and \( q = (1 - \alpha)b - 2ab \), we may apply the argument just above with the roles of \( e_i \), \( e_j \), and \( u \), \( v \) switched to get \( \sigma_1 = \alpha^2 \beta_1 + 4\eta_1 \), \( \sigma_2 = \alpha^2 \beta_2 + 4\eta_2 \), \( \tau_1 = (1 - \alpha)^2 \beta_1 + 4\eta_1 \), and \( \tau_2 = (1 - \alpha)^2 \beta_2 + 4\eta_2 \). Thus, \( \beta_1 = \sigma_1 + \tau_1 = \alpha^2 \beta_1 + 4\eta_1 + (1 - \alpha)^2 \beta_2 + 4\eta_1 \), \( \eta_1 = \frac{1}{2} \beta_1 \), and similarly \( \eta_2 = \frac{1}{2} \beta_2 \). If \( \beta_1 = \beta_2 \), then \( \eta_1 = \eta_2 \) and \( N(ab) = N_1 = \frac{1}{2} \beta_1 \).

If \( A_{ij} \) is regular, we have seen that the norm function induces a symmetric inner product on \( A_{ij} \) which we denoted by \((a, b)\) for \( a, b \in A_{ij} \). We now need to derive a few formulæ involving this inner product.

**Lemma 5.** Let \( a \in A_{ij} \), \( b \) and \( c \in A_{ik} \), \( d \in A_{jk} \), \( f \in A_{il} \), \( g \in A_{il} \) for \( i, j, k, l \) distinct, and suppose that all of these subspaces are regular. Then \( \frac{1}{2}(b, c)a = b \cdot ca + c \cdot ba \), \((ab, ac) = \frac{1}{2}N(a)(b, c)\), \((b, ad) = (ba, d)\), and \( ab \cdot af = \frac{1}{2}N(a)bf \). Furthermore, if \( A_{ij} \) contains an element of nonzero norm, then \( b \cdot ag = ba \cdot g \).

For the first formula of Lemma 5, we have
\[
\begin{align*}
\frac{1}{2}(b, c)a &= \frac{1}{4}[N(b + c) - N(b) - N(c)]a = \frac{1}{4}N(b + c)a - \frac{1}{4}N(b)a - \frac{1}{4}N(c)a \\
&= (b + c)[(b + c)a] - b(ba) - c(ca) = b(ca) + c(ba),
\end{align*}
\]
using Lemma 4 in the next to last step. The second formula follows from the calculation
\[
(ab, ac) = \frac{1}{2}[N(ab + ac) - N(ab) - N(ac)]
= \frac{1}{2}\left[\frac{1}{4}N(a)N(b + c) - \frac{1}{4}N(a)N(b) - \frac{1}{4}N(a)N(c)\right] = \frac{1}{4}N(a)(b, c).
\]

In proving the third formula it suffices to prove \((b, ad)a = (ba, d)a\). But using the first formula and Lemma 4, \( (b, ad)a = 2b(ad \cdot a) + 2ad \cdot ba \)
\[
= \frac{1}{2}N(a)bd + 2ab \cdot ad, \text{ and by symmetry } (ba, d)a = \frac{1}{2}N(a)bd + 2ab \cdot ad. \text{ also.}
\]

Observe now from Lemma 4 that \( b = [ab + 2ab] + [(1 - \alpha)b - 2ab] \)
and \( f = [(x + 2a)f] + [(1 - \alpha)f + 2af], \) where the four expressions in brackets are respectively in 

\[
A_{e_1}(\frac{1}{2}) \cap A_{e_2}(\frac{1}{2}), \quad A_{e_3}(\frac{1}{2}) \cap A_{e_4}(\frac{1}{2}), \quad A_{e_5}(\frac{1}{2}) \cap A_{e_6}(\frac{1}{2}),
\]

and \( A_{e_7}(\frac{1}{2}) \cap A_{e_8}(\frac{1}{2}). \) Thus

\[
bf = [(ab + 2af)][xf + 2af] + [(1 - \alpha)b + 2ab][(1 - \alpha)f + 2ab],
\]

since the cross terms vanish. Noting that \( b \cdot af = ab \cdot f = 0, \) this equation simplifies to \( bf = x^2bf + 4ab \cdot af + (1 - \alpha)^2bf + 4ab \cdot af, \) or \( 8ab \cdot af = (2x - 2x^2)bf = 2\gamma bf = 2N(a)bf. \) If \( N(a) \neq 0, \) we can derive the last formula of Lemma 5 by letting \( f = ag \) in the previous formula:

\[
N(a)b \cdot ag = N(a)bf = 4ab \cdot af = 4(ab)(a \cdot ag) = N(a)bu \cdot g.
\]

If \( N(a) = 0 \) but \( A_{ij} \) contains an element with nonzero norm, we may express \( a \) as the sum of two elements with nonzero norm and the formula follows from the fact that it is linear in \( a. \)

For any distinct \( i \) and \( j, \) let \( A'_{ij} = \{b : b \in A_{ij}, \ ab = 0 \ \text{for all} \ a \in A_{ij}\}. \) Then \( A'_{ij} \) is trivially a subspace of \( A_{ij} \) which can only be nonzero if \( A_{ij} \) is regular. In this case, \( A'_{ij} \) is the elements of \( A_{ij} \) which are orthogonal to all of \( A_{ij} \) under the inner product.

**Proposition 4.** If \( A_{ij} \) contains an element of nonzero norm, then \( \dim A_{ik} = \dim A_{jk}, \dim A'_{ik} = \dim A'_{jk}, \) and \( A_{ik} \) is regular if and only if \( A_{jk} \) is regular.

Let \( a \in A_{ij} \) be an element of nonzero norm. If \( A_{ik} \) is not regular, it is spanned by an element \( b \) without norm, and the product \( ab \) is an element of \( A_{jk} \) without norm by Proposition 3. Thus, \( A_{ik} \) not regular implies that \( A_{jk} \) is not regular and conversely. If \( A_{ik} \) and \( A_{jk} \) are both regular, let the maps \( \phi_1 : A_{ik} \to A_{jk} \) and \( \phi_2 : A_{jk} \to A_{ik} \) be defined by \( \phi_1(x_{ik}) = ax_{ik} \) and \( \phi_2(x_{jk}) = ax_{jk} \) respectively. Then \( \phi_1 \) and \( \phi_2 \) are linear transformations and \( \phi_2\phi_1 \) and \( \phi_1\phi_2 \) are just \( \frac{1}{2} \) times the identity map on \( A_{ik} \) and \( A_{jk} \) respectively by Lemma 4. Hence, \( \phi_1 \) and \( \phi_2 \) are both one-to-one onto, and \( \dim A_{ik} = \dim A_{jk}. \) By the second formula of Lemma 5, we see that \( \varphi_i(x_{ik}) \) is in \( A'_{ik} \) if and only if \( x_{ik} \) is in \( A'_{ik}, \) showing that \( \dim A'_{ik} = \dim A'_{jk}. \)

We are now ready to introduce the radical \( R \) of \( A \) and to prove that it has the properties that the name suggests. It will be convenient to use here a different definition of radical from that given in the introduction. The two will be proven equivalent in Proposition 6. We define the radical \( R \) to be the set of all elements \( x \) of \( A \) whose components in \( A_i \) vanish for each \( i \in S, \) and whose components in \( A_{ij} \) lie in \( A'_{ij} \) for each distinct \( i, j \in S. \)
Proposition 5. R is an ideal of A.

It is clear that R is a subspace of A, so that we only need to show that the product of an element in R with any element of A lies in R. In view of Axiom (v), it is sufficient to show that $A'_{ij}A_{kl}$, $A_{ij}A_{kl}$, and $A_{ij}A_{ik}$ lie in R for all distinct $i, j, k \in S$. The first two are obvious, so that we need only show that $A_{ij}A_{ik} \subseteq A_{jk}$. We may assume that $A_{ij}$ is regular, since otherwise $A_{ij} = 0$ and we have nothing to prove. Then for any $a \in A_{ij}$ and $b \in A_{ik}$ it follows from Proposition 3 that $(ab)^2 = 0$. If $A_{jk}$ is not regular, the only element of it which squares to zero is zero, showing that $ab = 0$ and $A_{ij}A_{ik} = 0$. Hence we may suppose that $A_{jk}$ is regular. We may also assume that $A_{jk}$ contains an element of nonzero norm, since otherwise $A_{jk} = A_{jk}$ and there is nothing to prove. But then $A_{ik}$ is also regular by Proposition 4. Thus, for each $a \in A_{ij}$ and $b \in A_{ik}$, we need to prove that $(ab, d) = 0$ for all $d \in A_{jk}$. But $(ab, d) = (a, bd)$ by Lemma 5, and $(a, bd) = 0$ since $a \in A'_{ij}$.

Proposition 6. If A is an algebra satisfying Axioms (i)-(v), the following three characterizations of the radical R of A are equivalent:

1. R is the set of all elements $x$ of A whose components in $A_i$ vanish for each $i \in S$, and whose components in $A_{ij}$ lie in $A_{ij}$ for each distinct $i, j \in S$.

2. R is the unique largest ideal of A not containing any elements of I.

3. R is the unique largest ideal of A not containing any primitive idempotents of $F$.

If each primitive idempotent $e$ of A has the property that $A_e(1)$ contains no nonzero elements that square to zero, then R is independent of the set I chosen to satisfy Axiom (ii) and is characterized by the fact that

4. R is the unique largest ideal of A not containing any primitive idempotents.

We shall assume throughout the proof of Proposition 6 that R is the ideal defined by part (1). To prove this definition equivalent to the others, it suffices to prove, first of all, that every ideal not contained in R contains an element of I, and secondly, that each primitive idempotent e of R has the property that $A_e(1)$ contains a nonzero element whose square is zero.

Let C be any ideal of A not contained in R. Then C contains an element $x$ which either has a nonzero component in some $A_i$ or a component in some $A_{ij}$ which is not in $A'_{ij}$. In the first case, the component $x_{ii} = (xe_i)(2e_i - 1)$ is in C so that C contains the idempotent $e_i \in I$. In the second case, $x_{ij} = 4(xe_i)e_i$ is in C and since $x_{ij} \notin A'_{ij}$ there exists a $b \in A_{ij}$ such that $bx_{ij} = \eta x_{ij} + \eta g e_j \neq 0$. Then either $e_i (bx_{ij}) = \eta x_{ij} e_i$ or $e_i (bx_{ij}) = \eta g e_j$ is nonzero, and again C contains an element of I. Thus, every ideal of A without elements of I is contained in R.

Suppose now that $e$ is an idempotent of R with the property that $A_e(1)$ contains no nonzero elements whose square is zero. For some $e_i \in I$ let $e_i = f_1 + f_2 + f_0$ be the decomposition of $e_i$ with respect to $e$ and let
Let $e = g_1 + g_0$ be the decomposition of $e$ with respect to $e_i (g_1 = 0$ since $e \in R)$. Then $ee_i = f_1 + \frac{1}{2}f_1 = \frac{1}{2}g_1$, and the component of $e^2 = e$ in $A_g(\frac{1}{2})$ is $2g_0g_1 = g_1$. Also, $e \cdot ee_i = f_1 + \frac{1}{4}g_1 = \frac{1}{2}(g_1)^2 + \frac{1}{2}g_0g_1 = \frac{1}{2}(g_1)^2 + \frac{1}{4}g_1$, and solving for $f_1$ in terms of $g_1$ gives

$$f_1 = 2(f_1 + \frac{1}{2}f_1) - (f_1 + \frac{1}{2}f_1) = (g_1)^2 + \frac{1}{2}g_1 - \frac{1}{2}g_1 = (g_1)^2.$$ 

Substituting this into the relation $f_1 e = f_1$ yields $(g_1)^2(g_1 + g_0) = (g_1)^3$ and dropping the component in $A_g(0)$ gives $(g_1)^2g_1 = 0$. But the component of $0 = (g_1)^2g_1 = f_1(f_1 + \frac{1}{2}f_1) = f_1 + \frac{1}{2}f_1f_1$ in $A_g(1)$ is $0 = f_1^2$, showing that $f_1$ is an element of $A_g(1)$ that squares to zero. Thus, by hypothesis, $f_1 = (g_1)^2 = 0$. Denoting the components of $e$ by $g_{ij}$ for each $j, k \in S$, we observe that $g_{ij}$ is the set of all components $g_{ij}$ where $i$ is fixed and where $j$ ranges over $S$, and that $(g_1)^2$ is the set of all products $g_{ki}g_{ij}$ where $i$ is fixed and $j, k$ range over $S$. Since no two of these products lie in the same component of $A$, the relation $(g_1)^2 = 0$ implies that $g_{ki}g_{ij} = 0$ for each $j, k \in S$. But $i$ was any element of $S$, so that the product of any two components of $e$ is zero, or $e^2 = 0$. This contradiction shows that, for any idempotent $e$ of $R$, $A_g(1)$ contains a nonzero element whose square is zero, and the proof is complete.

Now that we are ready to divide out the radical of $A$, we are faced with the problem of whether $A/R$ also satisfies Axioms (i)-(v). We recall first that a homomorphism $\varphi$ of $A$ is called admissible if $\varphi$ maps the set $I$ onto a set of orthogonal idempotents $\varphi(I)$ in $\varphi(A)$ satisfying Axiom (ii). Whenever $\varphi$ is admissible, as is the case with the natural map of $A$ onto $A/R$, it is easy to see that $\varphi(A)$ satisfies (ii) and (v). Using appropriate examples of associative rings with nonzero socle under the Jordan product, it is not difficult to show that $A$ may have homomorphs which don’t satisfy (ii), as well as homomorphs that do satisfy (ii) but which don’t arise under admissible homomorphisms.

Consider next Axioms (i) and (iii) under homomorphism. If $e$ is an idempotent of $A$, then $\varphi(e)$ satisfies (i); and if $e$ is primitive, then $\varphi(e)$ satisfies (iii). Given an idempotent $e'$ of $\varphi(A)$, the only way in practice to show that it satisfies (i) seems to be to show that it is the image of an idempotent in $A$. However, we have no general method of showing that $e'$ is the image of an idempotent in $A$, even if $\varphi$ is the natural map of $A$ onto $A/R$. We suspect that such a general method may not exist because of the difficulties encountered in lifting idempotents in infinite situations even in the associative case (see [7]).

Let us call an algebra semisimple if it is isomorphic to a quotient algebra $A/R$ where $A$ is an algebra satisfying Axioms (i)-(v) and where $R$ is the radical of $A$. Then every nonzero ideal in $A$ contains a primitive idempotent, since the preimage of this ideal in $A$ contains a primitive idempotent (in fact, an element of $I$). Although we cannot show that either semisimple or primitive algebras (defined in the introduction) satisfy Axioms (i), (iii), or (iv) in
general, we still effectively have the use of these axioms in semisimple and primitive algebras, since we can lift those idempotents that arise explicitly in our study of primitive algebras, and since most of the results of this section are preserved under admissible homomorphisms.

We shall often find it convenient to denote a semisimple or primitive algebra by $A$ instead of $A/R$ or $A^* = q(A)$. Correspondingly $I$ and $S$ will be used instead of $q(I)$ and $S$, and $F$ will be used instead of $(F + R)/R$ or $q(F)$.

**Proposition 7.** Every semisimple algebra $A$ is a subdirect sum of primitive algebras. The induced representation of $F$ as a subdirect sum is in fact a direct sum.

In the semisimple algebra $A$, let us define the binary relation $T$ on the elements of $S$ by the property that $iTj$ for $i, j \in S$ if and only if either $i = j$ or there exists a finite set of elements $k_0 = i, k_1, \cdots, k_n = j$ of $S$ such that $A_{k_0k_{r+1}} \neq 0$ for $r = 0, 1, \cdots, n - 1$. Then $T$ partitions $S$ into disjoint equivalence classes. If $S_v$ is such an equivalence class, let $F_v = \{x \mid x \in F, \alpha x = 0 \text{ for all } i \notin S_v\}$ and let $\rho'_v$ be the map of $F$ into $F_v$ defined on any element $x \in F$ by dropping all components of $x$ whose subscripts do not lie in $S_v$. Since no element of $F$ has a nonzero component with one subscript in $S_v$ and the other one not in $S_v$ (by the definition of $S_v$) we see that $\rho'_v$ is a homomorphism of $F$ onto the subalgebra $F_v$. In fact it is easy to see that the set of all homomorphisms $\rho'_v$ for different equivalence classes of $S_v$ effect a direct sum decomposition of $F$.

Consider now the extension of $\rho'_v$ to a map $\rho_v$ defined on all of $A$ by dropping all components whose subscripts do not lie in $S_v$. This map preserves addition and multiplication by Axiom (v), so that $\rho_v$ is a homomorphism. Since the set $I$ of $A$ is the image of a set of idempotents satisfying Axiom (ii) in an algebra satisfying Axioms (i)-(v), and since $\rho(I)$ again satisfies Axiom (ii) in $A_v = \rho_v(A)$, we see that $A_v$ is the homomorphic image of an algebra satisfying Axioms (i)-(v) under an admissible homomorphism. It now follows easily from the construction of $\rho_v$ that $A_v$ is primitive, that $\rho_v$ restricted to $F_v$ is a homomorphism, and that $A$ is a subdirect sum of the $A_v$'s. Since the $A_v$'s are not necessarily subalgebras of $A$, we get a subdirect sum rather than a direct sum this time. Of course, any particular $A_v$ which is a subalgebra of $A$ is a direct summand.

We have now proved Theorem 1 for algebras satisfying Axioms (i)-(v) after possibly adjoining an idempotent. To show that Theorem 1 holds without this last restriction, let $e_0$ again stand for the idempotent which was adjoined to $A$ to get $A^0$. The radical $R$ of $A^0$ is an ideal of $A$ and is clearly the maximal ideal of $A$ without primitive idempotents of $F$, so that it is natural to call $R$ the radical of $A$ also. If all the subspaces $A_{i0}$ for $i \in S$ are regular, then they are all in $R$, and $A^0/R$ is just a direct sum of $A/R$ and $\Phi e_0$ where $e_0 = e_0 + R$. On the other hand, if one or more of the subspaces $A_{i0}$ are not regular, then $e_0'$
is part of a nonregular primitive subsummand of $A^0/R$. This nonregular primitive subsummand may possibly break up into a number of nonregular primitive subsummands after $e'$ has been exorcised. In either case, however, it is easy to see that the fact that $A^0/R$ is a subdirect sum of primitive algebras implies that the same is true for $A/R$.

2. We turn now to the study of primitive algebras. Except for the next proposition, we shall restrict ourselves to the study of regular primitive algebras in this section. Most of the concepts and results on algebras satisfying Axioms (i)--(v) given in the last section obviously preserve under admissible homomorphisms and will be used here without further explanation.

**PROPOSITION 8.** Let $A$ be a primitive algebra. If $A$ is regular, then the subspaces $A_{ij}$ for $i, j \in S$ all have the same dimension. If $A$ is not regular, then for every $i, j \in S$ the dimension of $A_{ij}$ is either 0 or 1.

Suppose first that for some primitive algebra $A$ there exist $i, j, k$ such that $\dim A_{ij} \geq 2$ and that $A_{ik}$ is not regular. Then $A_{ij}$ contains two orthogonal elements $a$ and $b$ of nonzero norms $\gamma$ and $\delta$ respectively, and $A_{ik}$ is spanned by an element $g$ such that $g^2 = \beta_1 e_1 + \beta_2 e_2$ for distinct $\beta_1, \beta_2 \in \Phi$. By Proposition 3, $(ag)^2 = \frac{1}{\gamma}[\beta_1 e_1 + \beta_2 e_2] \neq 0$ and $(bg)^2 = \frac{1}{\delta}[\beta_1 e_1 + \beta_2 e_2] \neq 0$. Since $A_{jk}$ is not regular, $ag$ is a nonzero multiple of $bg$ and $(ag)(bg) \neq 0$. But since $a$ and $b$ are orthogonal, we have $N(a + b) = N(a) + 2(a, b) + N(b) = \gamma + \delta$, so that $[(a + b)g]^2 = \frac{1}{\gamma\delta}[(\beta_1 e_1 + \beta_2 e_2)]$ and

$$2(\gamma)(\delta) = [(a + b)g]^2 - (ag)^2 - (bg)^2 = 0.$$ 

This contradiction proves that if $\dim A_{ij} \geq 2$ for some $i, j \in S$, then every subspace of the form $A_{ij}$ or $A_{ik}$ is regular.

In order to prove Proposition 8 it is sufficient to prove that for any $i, j, l, m \in S$ with $i \neq j$ and $l \neq m$, the relation $\dim A_{ij} = \dim A_{im}$ holds whenever either (a) $\dim A_{ij} \geq 2$, or (b) $\dim A_{ij} = 1$ and $A$ is regular. Since $A$ is primitive, there exists a finite sequence $j = k_0, k_1, \ldots, k_n = l$ such that $A_{k_r k_{r+1}} \neq 0$ for $r = 0, 1, \ldots, n - 1$. We first show by induction that $\dim A_{i, k_r} = \dim A_{ij}$ and that $A_{i, k_r}$ is regular. Since $k_0 = j$, both statements are true for $r = 0$. But if $\dim A_{i, k_{r-1}} = \dim A_{ij}$, then $A_{k_{r-1} k_r}$ is regular either by the results of the last paragraph or because $A$ is regular. And since $A_{k_{r-1} k_r} \neq 0$ by assumption, this subspace contains an element of nonzero norm, showing that $\dim A_{i, k_r} = \dim A_{ij}$ and that $A_{i, k_r}$ is regular by Proposition 4. This completes the induction and shows that $\dim A_{ij} = \dim A_{ij}$ and that $A_{il}$ is regular. But switching the roles of $l$ and $m$ gives $\dim A_{im} = \dim A_{ij}$, and observing that $A_{il}$ contains an element of nonzero norm leads to $\dim A_{im} = \dim A_{im} = \dim A_{ij}$.

In the remainder of this section $A$ will denote a regular primitive algebra
LEMMA 6. After making a possibly infinite dimensional scalar extension of the base field, we may simultaneously select an element \( f_{ij} = f_{ii} \) for each distinct pair \( i, j \in S \) such that \( f_{ij}^2 = 4(e_i + e_j) \) and \( f_{ij}f_{jk} = f_{ik} \) for distinct \( i, j, k \in S \).

Let 1 be a fixed subscript of \( S \). Then for each \( i \in S \) different from 1 we may select an element \( f_{1i} \in A_{1i} \) of nonzero norm \( y_i \). After possibly making a scalar extension of \( \Phi \) (a possibly infinite union of quadratic extensions), we may assume that each \( y_i \) is a square in \( \Phi \), say \( y_i = \delta_i^2 \). Defining \( f_{1i} = f_{11} = 2(\delta_i)^{-1}f_{1i} \), we see that \( N(f_{1i}) = 4(\delta_i)^{-2}N(f_{1i}) = 4 \). If neither \( i \) nor \( j \) are 1, we let \( f_{ij} = f_{1i}f_{1j} \), and again \( N(f_{ij}) = \frac{1}{4}N(f_{11})N(f_{1i}) = 4 \). Also, \( f_{1i}f_{1j} = f_{1k}(f_{1i}f_{1j}) = \frac{1}{4}N(f_{1i})f_{1i}f_{1k} = f_{ik} \) for \( 1, i, j, k \) distinct.

Let us assume now that an appropriate scalar extension has been made to allow the existence of a set of \( f_{ij} \)'s satisfying Lemma 6, and let us assume that a particular set of \( f_{ij} \)'s satisfying this lemma have been selected. Then for distinct \( i, j \in S \) we define a mapping \( y \mapsto \tilde{y} \) of \( A_{ij} \) into itself by \( \tilde{y} = \frac{1}{2}(y, f_{ij})f_{ij} - y \). This mapping is clearly linear, and several more properties are given by

LEMMA 7. If \( y \in A_{ij} \), then \( N(\tilde{y}) = N(y) \), \( \tilde{y} = y \), and \( (yf_{ik} \cdot f_{ij})f_{jk} = \tilde{y} \).

For the first relation of Lemma 7, we observe that

\[
\frac{1}{4}(y, f_{ij})f_{ij}^2 = \frac{1}{4}(y, f_{ij}) \cdot 4(e_i + e_j) = f_{ij}y,
\]

giving

\[
(y)^2 = \left[\frac{1}{2}(y, f_{ij})f_{ij} - y\right]^2 = \frac{1}{4}(y, f_{ij})^2f_{ij} - (y, f_{ij})f_{ij}y + y^2 = y^2.
\]

The second relation follows from the calculation

\[
\tilde{y} = \frac{1}{2}(y, f_{ij})f_{ij} - y, f_{ij} - \left[\frac{1}{2}(y, f_{ij})f_{ij} \quad y\right] = \frac{1}{2}(y, f_{ij}) - (y, f_{ij})f_{ij} - \frac{1}{2}(y, f_{ij})f_{ij} + y = y.
\]

For the last relation, we have \( yf_{ik} \cdot f_{ij} = -y \cdot f_{ik}f_{ij} + \frac{1}{2}(y, f_{ij})f_{ik} \) by the first relation of Lemma 5, and hence

\[
(yf_{ik} \cdot f_{ij})f_{jk} = \frac{1}{2}(y, f_{ij})f_{ij} - \chi f_{jk} \cdot f_{ij} = \frac{1}{4}(y, f_{ij})f_{ij} - \frac{1}{4}N(f_{jk})y = \tilde{y}.
\]

Suppose now that the elements of \( S \) are totally ordered in some fashion such that \( S \) has a first and a second element, to be denoted by 1 and 2 respectively. For each \( i, j \in S \) \( i < j \) we shall define an algebra \( C_{ij} \) as follows: the elements of \( C_{ij} \) shall be the elements of \( A_{ij} \), addition and scalar multiplication are the same as in \( A_{ij} \), and multiplication is given by \( x * y = 2(\delta f_{jk} \cdot y)f_{ik} \).
for all \( x, y \in A_{ij} \) and for some \( k \neq i, j \). To show that multiplication is well-defined we must show that \( x \ast y \) doesn’t depend on which \( k \) was used. But, using the last relation of Lemma 5 and selecting \( l \neq i, j, k \), we have

\[
x \ast y = 2(\tilde{x} f_{jk} \cdot y)(f_{kl} f_{it}) = 2[(\tilde{x} f_{jk} \cdot y)f_{kl}] f_{it} = 2[(\tilde{x} f_{jk} \cdot f_{kl})y] f_{it}
\]

\[
= 2[(\tilde{x} \cdot f_{jk} f_{kl})y] f_{it} = 2(\tilde{x} f_{jt} \cdot y)f_{it}.
\]

**Lemma 8.** For each \( i, j \in S \) with \( i < j \), \( C_{ij} \) is a composition algebra with \( \frac{1}{2} f_{ij} \) as identity element, and all of these composition algebras are isomorphic. More specifically, the isomorphisms \( C_{12} \cong C_{1j} \) and \( C_{ij} \cong C_{ij} \) for \( 1 < i < j \) follow from the relations \( (x_{12} \ast y_{12}) f_{2j} = (x_{12} f_{2j}) \ast (y_{12} f_{2j}) \) and \( (x_{1j} \ast y_{1j}) f_{1i} = (x_{1j} f_{1i}) \ast (y_{1j} f_{1i}) \).

To show that \( \frac{1}{2} f_{ij} \) is the identity element of \( C_{ij} \) we have only to observe that

\[
x \ast \frac{1}{2} f_{ij} = (\tilde{x} f_{jk} \cdot f_{ij}) f_{ik} = x \quad \text{by Lemma 7}, \quad \text{and that}
\]

\[
\frac{1}{2} f_{ij} \ast y = (f_{ij} f_{jk} \cdot y) f_{ik} = y f_{ik} \cdot f_{ik} = y.
\]

Since \( C_{ij} \) has an identity element and since the inner product on \( C_{ij} \) is nondegenerate, \( C_{ij} \) will be a composition algebra if \( N(x \ast y) = N(x)N(y) \) for all \( x, y \in C_{ij} \) (See [5] for the exact definition of composition algebra as well as for the classification of these algebras which we will be needing shortly.)

But

\[
N(x \ast y) = N[2(\tilde{x} f_{jk} \cdot y)f_{ik}] = \frac{1}{4} N[2(\tilde{x} f_{jk} \cdot y)] N(f_{ik}) = 4N(\tilde{x} f_{jk} \cdot y)
\]

\[
= N(\tilde{x} f_{jk}) N(y) = \frac{1}{4} N(\tilde{x}) N(f_{ik}) N(y) = N(x)N(y)
\]

as desired.

The two relations mentioned in Lemma 8 may be established as follows:

\[
(x_{12} \ast y_{12}) f_{2j} = 2[(\tilde{x}_{12} f_{2j} \cdot y_{12}) f_{1k}] f_{2j} = 2[[(\frac{1}{2} (x_{12} f_{2j}) y_{12} - x_{12} f_{2j} \cdot y_{12}) f_{2j}] f_{1k}
\]

\[
= 2([(\frac{1}{2} (x_{12} f_{2j} \cdot y_{12}) f_{2j} - x_{12} f_{2j} \cdot y_{12}) f_{2j}] f_{1k}
\]

\[
= 2([\frac{1}{2} (x_{12} f_{2j} \cdot f_{jk}) f_{1k} - x_{12} f_{2j} \cdot f_{jk}] f_{1k}]
\]

\[
= 2([\frac{1}{2} (x_{12} f_{2j} \cdot f_{jk}) \cdot y_{12} f_{2j}] f_{1k} = (x_{12} f_{2j}) * (y_{12} f_{2j}).
\]

\[
(x_{1j} \ast y_{1j}) f_{1i} = 2[(\tilde{x}_{1j} f_{1i} \cdot y_{1j}) f_{ik}] f_{1i}
\]

\[
= 2[[(\frac{1}{2} (x_{1j} f_{1i}) f_{1i} - x_{1j} f_{1i} \cdot y_{1j}) f_{1i}]
\]

\[
= 2[[(\frac{1}{2} (x_{1j} f_{1i} \cdot f_{ij}) f_{1i} - x_{1j} f_{1i} \cdot y_{1j} f_{1i}] f_{1i}
\]

\[
= 2[[(\frac{1}{2} (x_{1j} f_{1i} \cdot f_{ij}) \cdot y_{1j} f_{1i}] f_{1i} = (x_{1j} f_{1i}) * (y_{1j} f_{1i}).
\]
Letting \(|S|\) denote the cardinality of \(S\), we are now in a position to prove

**Proposition 9.** Suppose that \(|S| > 2\). Then after possibly making an infinite scalar extension, \(F\) is isomorphic to the Jordan algebra \(H\) of all hermitian \(|S|\) by \(|S|\) matrices over \(C_{12}\) with a finite number of nonzero entries.

We shall first establish a vector space isomorphism between \(F\) and \(H\), and then show that it is multiplicative. We may regard the rows and columns of \(H\) as indexed by the elements of \(S\) and as being arranged in the same order as the elements of \(S\). To begin with, we let \(e_i\) correspond to the matrix of \(H\) with \(\frac{1}{2}f_{12}\) at the intersection of the \(i\)th row and the \(i\)th column and with zero elsewhere. Given an element \(x\) of \(A_{12}\), we shall correspond to it the matrix with \(x\) in the first row and second column, and with \(x\) in the second row and first column (and all other entries zero). An element \(x \in A_{ij}\) for \(j > 2\) will correspond to the matrix with \(xf_{ij}\) in the \(1, j\) position and with \(xf_{ij}\) in the \(j, 1\) position. And an element \(x \in A_{ij}\) for \(1 \leq i < j\) will correspond to the matrix with \(xf_{ij}\) in the \(i, j\) position and \(xf_{ij}\) in the \(j, i\) position. This correspondence may be uniquely extended by linearity to give a vector space isomorphism between \(F\) and \(H\).

To show that this isomorphism is multiplicative, we first observe that the products \(e_i e_j = e_i x_{jk} = x_i x_{ki} = 0\) for \(i, j, k, l\) distinct correspond to zero products in \(H\). It is also clear that the relation \(e_i x_{ij} = \frac{1}{2}x_{ij}\) is preserved under the isomorphism. To show that products of the form \(A_{ij} A_{ij}\) are preserved, it is sufficient to show that the square of an element \(x_{ij} \in A_{ij}\) corresponds to the square of the image of \(x_{ij}\). But if the matrix corresponding to \(x_{ij}\) has \(x_{12}\) in the \(i, j\) position (and hence \(x_{12}\) in the \(j, i\) position), then the square of this matrix has \(x_{12} x_{12} = 2(x_{12} f_{12} x_{12} f_{12}) = 2(\frac{1}{2}x_{12} f_{12} x_{12} f_{12}) = x_{12} f_{12}\) in the \(i, i\) position and \(x_{12} x_{12} = N(x_{12}) \cdot \frac{1}{2}f_{12}\) in the \(j, j\) position, and is thus the image of

\[
x_{ij}^2 = N(x_{ij}) e_i + N(x_{ij}) e_j = N(x_{ij}) e_i + N(x_{ij}) e_j.
\]

It remains to show that, for any \(i, j, k \in S\) such that \(i < j < k\), the products \(A_{ij} A_{jk}\), \(A_{ji} A_{ik}\), and \(A_{ik} A_{kj}\) are preserved under the isomorphism. In order to keep from having to break up each of these cases according to whether \(i\) or \(j\) take the values 1 or 2, we observe that Lemma 8 allows us to perform the necessary calculations directly using \(C_{ij}\) instead of having to move everything back into \(C_{12}\). Since \((y_{ik} f_{jk} \cdot f_{ij}) f_{21} = (y_{ik} f_{ij} \cdot f_{jk}) f_{21} = y_{ik} f_{ij} \cdot f_{ik}\) and \((z_{jk} f_{ij} \cdot f_{jk}) f_{11} \cdot f_{21} = (z_{jk} f_{ij} \cdot f_{jk}) f_{11} \cdot f_{ij} = (z_{jk} f_{ij} \cdot f_{jk}) f_{21} = z_{jk} f_{ij} \cdot f_{2k}\), we may map elements of \(A_{ik}\) and \(A_{jk}\) into \(C_{ij}\) in a manner analogous to the way we have been mapping things into \(C_{12}\), and these maps will be compatible with the isomorphism between \(C_{ij}\) and \(C_{12}\) given by Lemma 8.

If \(x_{ij} \in A_{ij}\) and \(y_{ik} \in A_{ik}\), then the product \(x_{ij} y_{ik}\) corresponds to the matrix
with \( x_{ik} y_{jk} \cdot f_{jk} \) in the \( i, k \) position and the conjugate of this in the \( k, i \) position. On the other hand, the Jordan product of the matrices corresponding to \( x_{ij} \) and \( y_{jk} \) has \( \frac{1}{2} x_{ij} \cdot (y_{jk} f_{ij} \cdot f_{jk}) \) in the \( i, k \) position and its conjugate in the \( k, i \) position. But

\[
\frac{1}{2} x_{ij} \cdot (y_{jk} f_{ij} \cdot f_{jk}) = \left[ (x_{ij} f_{jk}) (y_{jk} f_{ij} \cdot f_{jk}) \right] f_{ik}
\]

\[
= \frac{1}{2} (x_{ij} f_{jk}, f_{ik})(y_{jk} f_{ij} \cdot f_{jk}) - (x_{ij} f_{jk})(y_{jk} f_{ij} \cdot f_{jk}) f_{ik}
\]

\[
= \frac{1}{2} (x_{ij}, f_{ij})(y_{jk} f_{jk}) = x_{ij} f_{jk} \cdot y_{jk}
\]

\[
= \frac{1}{2} (y_{jk} f_{ij}) x_{ij} = x_{ij} f_{jk} = y_{jk} f_{ij} = x_{ij} f_{jk}.
\]

as desired. The other two cases are established by

\[
\frac{1}{2} x_{ij} \ast (y_{ik} f_{jk}) = (x_{ij} f_{jk} \cdot y_{ik} f_{jk}) f_{ik}
\]

\[
= \left[ \frac{1}{2} (x_{ij} f_{jk} y_{ik} f_{jk} - x_{ij} f_{jk} y_{jk} f_{ik}) \right] f_{ik}
\]

\[
= \frac{1}{2} (x_{ij} y_{ik}, f_{ij} f_{jk}) - (x_{ij} y_{ik})(f_{ij} f_{jk}) f_{ik}
\]

\[
= (x_{ij} y_{ik} \cdot f_{ij}) f_{jk}
\]

and

\[
\frac{1}{2} (x_{ik} f_{jk}) \ast (y_{jk} f_{ij} \cdot f_{jk}) = \left[ (x_{ik} f_{jk} \cdot f_{jk}) (y_{jk} f_{ij} \cdot f_{jk}) \right] f_{ik}
\]

\[
= \left[ \frac{1}{2} (x_{ik} f_{jk}, f_{ij})(y_{jk} f_{ij} \cdot f_{jk}) - x_{ik} f_{jk} (y_{jk} f_{ik}) f_{jk} \right] f_{ik}
\]

\[
= \frac{1}{2} (x_{ik} f_{jk}, f_{ij} y_{jk} f_{ik} - x_{ik} y_{jk} f_{ik}) f_{ik}
\]

\[
= x_{ik} \ast x_{ik} y_{jk} f_{ik} = x_{ik} y_{jk}.
\]

Now that Proposition 9 has been proved, the structure of \( F \) may be determined readily from the known structure of composition algebras. By possibly making another quadratic extension of the base field, we may assume that \( C_{12} \) is a split composition algebra, in which case \( C_{12} \) is one of four possible algebras of dimension 1, 2, 4, or 8 respectively. If \( C_{12} \) has dimension one it is easy to see that \( F \) is isomorphic to the set of all symmetric \( | S | \) by \( | S | \) matrices over \( \Phi \) with only finitely many nonzero entries under the Jordan product. Extending the standard finite dimensional terminology in the obvious way, we shall say that \( F \) is an algebra of class \( B \) in this case. If \( C_{12} \) has dimension 2, then \( C_{12} \) is the direct sum of two one-dimensional algebras over \( \Phi \), and it is easy to verify that \( F \) is isomorphic to the set of all \( | S | \) by \( | S | \) matrices over \( \Phi \) with only finitely many nonzero entries under the Jordan product. This time we call \( F \) an algebra of class \( A \).

Next suppose that the dimension of \( C_{12} \) is 4. Then \( C_{12} \) is isomorphic to the 2 \( \times \) 2 matrices over \( \Phi \) and \( F \) may easily be shown to be isomorphic to the Jordan subalgebra of the 2 \( \times \) 2 \( | S | \) by \( 2 | S | \) matrices over \( \Phi \) consisting of those
matrices with only finitely many nonzero entries in which the second
|S| \times |S| block on the diagonal is the transpose of the first diagonal
|S| \times |S| block, and in which the two off diagonal |S| \times |S| blocks are
skew-symmetric. In this case F is called an algebra of class C.

If C_{12} has dimension 8, then C_{12} is the split Cayley algebra. We shall prove
that under these circumstances |S| must be 3, showing that F can only be the
split exceptional simple Jordan algebra over \Phi. As a basis for the split Cayley
algebra we may take u, v, g_1, g_2, g_3, g_1', g_2', g_3' where u^2 - u, v^2 - v,
u g_1 = g_1 v = g_1', g_1' u = v g_1' = g_1', g_2 g_3 = -g_3 g_2 = g_2' g_3 = -g_3 g_2' = g_2
for each even permutation (i, j, k) of (1, 2, 3), and where all other products are
zero. The identity element of this algebra is 1 = u + v and conjugation is
given by \bar{u} = v, \bar{g}_i = -g_i, and \bar{g}_i' = -g_i' for i = 1, 2, 3. Suppose now that
|S| \geq 4 and consider the following three hermitian |S| \times |S| matrices
where all but four rows and the corresponding columns contain only zeros
in each case and have been omitted:
e = \begin{pmatrix}
1 & g_1' & 0 \\
-g_1' & 1 & g_2' \\
-g_2 & -g_3' & 0 \\
0 & 0 & 0
\end{pmatrix},
z = \begin{pmatrix}
0 & 0 & 0 & g_3 + g_5' \\
0 & 0 & 0 & -g_2 - g_6' \\
0 & 0 & 0 & g_1 \\
-g_3' & g_2 + g_5' & -g_1 & 0
\end{pmatrix},
and
w = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2g_1' \\
0 & 0 & 2g_1' & 0
\end{pmatrix}.

A direct computation shows that e is idempotent and that e \cdot z = w \neq 0 and
e \cdot w = 0, showing that e does not satisfy Axiom (i).

This doesn’t yet give a contradiction since the algebra F that we are dealing
at this point is only a homomorph of a possibly infinite scalar extension of an
algebra satisfying (i). However e occurs in a scalar extension of A of degree 8
over \Phi (an extension of degree 4 to allow us to find f_{12} and f_{13}, and another
extension of degree 2 to split C_{12}), and this extension of A is the homomorphic
image of an algebra A’ satisfying Axiom (i). If u = e_1 + e_2 + a + b + c
where a \in A_{12}, b \in A_{13}, and c \in A_{23} is an element of A’ mapping onto e,
then u^2 - u is in the radical R’ of A’. We may break the relation u^2 - u \in R’
to components yielding the relations N(a) + N(b) = 0, N(a) + N(c) = 0,
N(b) + N(c) = 0, a + 2bc = n_1 \in R’, 2ac = n_2 \in R’, and 2ab = n_3 \in R’.
The first three relations give N(a) = N(b) = N(c) = 0, and the fourth gives
0 = (a, n_1) = (a, a) + (a, 2bc) = 2(a, bc). But defining a’ = a - n_1, we
have N(a’) = N(a) - 2(a, n_1) + N(n_1) = 0, a’ + 2bc = 0, 2a’c =

\[ 2ac - 2n_c = 2ac - 2(a + 2bc)c = 2ac - 2ac - N(c)b = 0, \quad \text{and} \]
\[ 2a'b = 2ab - 2n,b = 2ab - 2ab - N(b)c = 0, \]
showing that the preimage \( u' = e_1 + e_2 + a' + b + c \) of \( e \) is idempotent. Then the fact that \( u' \) satisfies Axiom (i) implies that \( e \) satisfies Axiom (i), giving the desired contradiction.

Consider now the case \( |S| = 2 \) omitted from Proposition 9. We already know from Proposition 2 and its proof that \( F \) has a basis \( s_0, s_1, \ldots, s_i \ldots \)
where \( s_0^2 = s_0, s_{0}^j = s_i, s_i^2 = \alpha_i s_0 \) for some \( \alpha_i \)'s in \( \Phi \), and \( s_i s_j = 0 \) for all distinct \( i, j \neq 0 \). Since the radical of \( F \) is zero, the inner product on \( A_{12} \) is non-singular, showing that the \( \alpha_i \)'s are non-zero and that \( F \) is simple. Again, making the obvious extension of the finite dimensional terminology, we call \( F \) an algebra of type D in this case. Noting that \( F = A \) when \( |S| \) is finite, our results on primitive algebras so far are summed up by

**Proposition 10.** Let \( A \) be a regular primitive algebra and let \( F \) be its subalgebra of elements with finitely many nonzero components. Then either (1) some scalar extension of \( F \) is an algebra of class \( A, B, \) or \( C \), or (2) \( A \) is an algebra of type \( D \) or \( E \).

In order to determine the structure of \( F \) itself, it remains to investigate \( F \) under the assumption that some scalar extension of it is an algebra of class \( A, B, \) or \( C \). For finite \( |S| \) the result follows easily from the results of Jacobson and Jacobson in [3]. Luckily, the relevant arguments in [3] generalize to the infinite case without much difficulty, so that we shall content ourselves with supplying the nontrivial modifications of the argument in [3] which are needed in the infinite case.

Let \( P \) be an extension field of \( \Phi \) with the property that \( F_P \), the scalar extension of \( F \) by \( P \), is an algebra of class \( A, B, \) or \( C \). Then \( \S 1 \) of [3] shows that \( F_P \) has a universal associative algebra which is unique up to isomorphism, and the argument in \( \S 3 \) of [3] with a few obvious modifications derives what the universal algebra of \( F_P \) is in each case. In adapting the argument in \( \S 4 \) to the infinite case it is convenient to use a regular representation \( R \) of \( P \) over \( \Phi \) with a certain special property. If \( R \) maps \( P \) into \( \Phi_T \), the \( |T| \times |T| \) matrices over \( \Phi \), then we wish \( R \) to have the property that each element of \( P^R \) is a matrix of \( \Phi_T \) consisting of a finite block repeated down the diagonal for some ordering of \( T \), and that any finite number of elements of \( P^R \) may be broken into repeated finite blocks of the same size and in the same positions simultaneously.

To show that \( P \) has such a representation, we recall first that \( P \) is an extension of degree 1 or 2 of a field \( P' \) which is a union of extensions of degree 2 of \( \Phi \), where the extension from \( \Phi \) to \( P' \) is required to produce the \( f_{1,i} \)'s, and where the extension from \( P' \) to \( P \) is required to split \( C_{12} \). But if \( C_{12} \) has dimension 1 the extension from \( P' \) to \( P \) is not required, and if \( C_{12} \) has dimension 2 or 4 we could have made a quadratic scalar extension of \( \Phi \) which
introduces a nonzero element of zero norm into \( A_{12} \), after which the extension required to produce the \( f_1 \)'s automatically produces a split composition algebra. Thus, in either case, \( P \) is itself a union of fields each of which is a quadratic extension of \( \Phi \). Let \( \{ r_i \}_{i \in \nu} \) be a set of elements of \( \Phi \) whose square roots are algebraically independent and generate \( P \) over \( \Phi \). For each \( r_i \), the field \( \Phi(\sqrt{r_i}) \) may be represented over \( \Phi \) by

\[
R_i : a + b \sqrt{r_i} \rightarrow \begin{pmatrix} a & b r_i \\ b & a \end{pmatrix}
\]

for \( a, b \in \Phi \). Extending \( R_i \) to be a representation of \( \Phi(\sqrt{r_i}) \) over \( \Phi(\sqrt{r_i}) \), we may replace the elements \( \Phi(\sqrt{r_i}) \) by their images under \( R_i \) to get a representation \( R_{ij} \) of \( \Phi(\sqrt{r_j}, \sqrt{r_i}) \) in \( \Phi_3 \). Since we get the same representation \( R_{ij} \) either by substituting \( R_i \) into \( R_j \) or vice versa, we see that we may combine any set of \( R_i \)'s to get a uniquely determined representation of the appropriate subfield of \( P \). In particular, combining all of the \( R_i \)'s gives a representation \( R \) which may easily be seen to have the desired property.

Using a representation with this property, it is not difficult to verify that the central part of the argument in [3], §4 goes through with only minor modifications. More precisely, it follows that there exists an associative \( \Phi \)-algebra \( \mathfrak{A} \) with an involution \( J \) such that \( F \) is isomorphic to the Jordan algebra of \( J \)-symmetric elements of \( \mathfrak{A} \), and such that \( \mathfrak{A}_p \) is isomorphic to the universal algebra of \( F_p \). We suppose first that \( \mathfrak{A} \) is simple, which is always true if \( F_p \) is of class \( B \) or \( C \) as \( \mathfrak{A}_p \) is simple in this case. Then, since \( \mathfrak{A} \) contains the elements if \( F \), it contains an idempotent and is hence primitive. If \( e \) is an idempotent of \( I \), the right ideal \( e\mathfrak{A} \) cannot contain an infinite descending chain of right ideals since \( (e\mathfrak{A})_p = e\mathfrak{A}_p \) contains no infinite chain. Thus \( \mathfrak{A} \) has nonzero socle. But, it is known that a primitive ring with an involution and with nonzero socle may be represented as a set of continuous transformations on a vector space \( M \) over a division ring \( \Gamma \), where \( M \) is self-dual relative to an hermitian or skew-hermitian scalar product (see [4], Thms. 1 and 2, pp. 82–83). Picking a basis for \( M \) so that \( \mathfrak{A} \) is represented as matrices, we see that the simplicity of \( \mathfrak{A} \) implies that each matrix has only finitely many nonzero columns, and the self-duality of \( M \) implies that each matrix has only finitely many nonzero rows. Thus, for some set \( T \), \( \mathfrak{A} \) may be regarded as the set of all \( \begin{pmatrix} T \end{pmatrix} \times \begin{pmatrix} T \end{pmatrix} \) matrices over \( \Gamma \) with only finitely many nonzero entries.

Next let us investigate the centralizer \( \Gamma' \) of \( \mathfrak{A} \). Since every element of \( \Phi \) has the same effect as some element of \( \Gamma' \), we may think of \( \Phi \) as being part of \( \Gamma' \) and of \( \Gamma \) as being an algebra over \( \Phi \). Defining the map \( f \) of \( \Gamma \) into \( \mathfrak{A} \) by \( f(\gamma) = \gamma e \) for each \( \gamma \in \Gamma \) and for some fixed \( e \in I \), we see that \( f \) is a \( \Phi \)-isomorphism of \( \Gamma \) into the subalgebra \( \mathfrak{A}_e = \{ x \mid x \in \mathfrak{A}, xe = ex = x \} \). Thus, \( [\Gamma : \Phi] \leq [\mathfrak{A}_e : \Phi] = [\mathfrak{A}_p] : P \) where \( \mathfrak{A}_p = \{ x \in \mathfrak{A}_p \mid xe = ex = x \} \). If \( F_p \) is of class \( B \), then \( [\mathfrak{A}_p] : P \) gives \( [\Gamma : \Phi] = 1 \) or \( \Gamma = \Phi \). If
$F_p$ is of class $A$, we have $[(\mathfrak{A}_p)_e : P] = 2$ or $[\Gamma : \Phi] < 2$. Since $\mathfrak{A}_e$ is an algebra over $f(\Gamma)$ and since $[\mathfrak{A}_e : f(\Gamma)]$ is a square, we must have $[\Gamma : \Phi] = 2$, showing that $\Gamma$ is a quadratic extension field of $\Phi$. Noting that $(\Gamma e) \cap F = \Phi e$, we see that $f$ is an involution of the second kind with $\Phi$ as the fixed field of $\Gamma$.

If $F_p$ is of class $C$, then $[(\mathfrak{A}_p)_e : P] = 4$ and the fact that $[\mathfrak{A}_e : f(\Gamma)]$ is a square implies that either $\Gamma = \Phi$ or $[\Gamma : \Phi] = 4$. In the latter case, $\Gamma \cong \mathfrak{A}_e$ over $\Phi$ and $F \cong (\mathfrak{A}_p)_e \cong P_2$, showing that $\Gamma$ is a generalized quaternion division algebra with center $\Phi$. In each of these cases we may find a basis for $M$ with the property that the $e_i$'s are represented by diagonal matrices. It follows from this that $T = S + S$ when $F_p$ is of class $C$ and $\Gamma = \Phi$, and that $T = S$ in the other cases.

It remains to discover the structure of $\mathfrak{A}$ when it is not simple. Here $F_p$ must be of class $A$ and $\mathfrak{A}_p = P_i(1) \oplus P_i(2)$ where $P_i(1)$ for $i = 1$ or $2$ is a copy of the set of $|S| \times |S|$ matrices over $P$ with only finitely many nonzero entries. If $\mathfrak{A}_1$ is a proper ideal of $\mathfrak{A}$, then $(\mathfrak{A}_1)_p$ must be either $P_i(1)$ or $P_i(2)$, say $(\mathfrak{A}_1)_p = P_i(1)$. Since $\mathfrak{A}$ is preserved under the fundamental involution $f$ which interchanges $P_i(1)$ and $P_i(2)$, $f$ must carry $\mathfrak{A}_1$ into another ideal $\mathfrak{A}_2$ with the property that $(\mathfrak{A}_2)_p = P_i(2)$. Then $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2$, and $F$ is the set of all elements of $\mathfrak{A}$ fixed under $f$, or the set of all elements of the form $a_1 + a_2$ for $a_1 \in \mathfrak{A}_1$. The correspondence $a_1 + a_2 \rightarrow a_1$ defines an isomorphism between $F$ and the algebra $\mathfrak{A}_1$ under the Jordan product. Here, $\mathfrak{A}_1$ contains a primitive idempotent and is a simple primitive ring with nonzero socle as in the last case. Since the subalgebra $(\mathfrak{A}_1)_e$ for $e \in I$ is one-dimensional over $\Phi$, the centralizer of $\mathfrak{A}_1$ is just $\Phi$. Representing $\mathfrak{A}_1$ as transformations on a vector space $M$ and using the elements of $I$ to find a basis for $M$, it is immediate that $\mathfrak{A}_1$ is isomorphic to $\Phi_S$.

The possibilities that we have found for the structure of $F$ are collected in

**Proposition 11.** If some scalar extension of $F$ is of class $A$, $B$, or $C$, then $F$ is isomorphic to one of the following sets of matrices under the Jordan product:

1. The set $\Phi_S$ of all $|S| \times |S|$ matrices over $\Phi$ with a finite number of nonzero entries.
2. The set of all $f$-symmetric elements of $\Phi_S$, where $f$ is an involution of the first kind preserving the diagonal elements of $\Phi_S$.
3. The set of all $f$-symmetric elements of $\Phi_{2S}$, where $f$ is an involution of the first kind which interchanges the $i$th and $(S + i)$th diagonal idempotents of $\Phi_{2S}$.
4. The set of all $f$-symmetric elements of $\Gamma_S$, where $f$ is an involution of the second kind and where $\Gamma$ is a quadratic extension of $\Phi$ with $\Phi$ as fixed field under $f$.
5. The set of all skew-hermitian elements of $\Gamma_S$, where $\Gamma$ is a generalized quaternion division algebra with center $\Phi$. 


An algebra which is isomorphic to one of the algebras (a), (b), (c), (d), or (e) under the Jordan product will be called a reduced algebra of type \( A_1, B, C_1, A_2, \) or \( C_2 \) respectively over the field \( \Phi \). A reduced algebra of type \( A_1 \) is already an algebra of class \( A \). However, a reduced algebra of type \( B \) or \( C_1 \) may require an infinite scalar extension to become an algebra of class \( B \) or \( C \) respectively. After an appropriate quadratic scalar extension, a reduced algebra of type \( A_2 \) or \( C_2 \) becomes a reduced algebra of type \( A_1 \) or \( C_1 \) respectively. Reduced algebras of type \( A_2 \) or \( C_2 \) may of course be represented as sets of matrices over \( \Phi \) using any representation of \( \Gamma \) as matrices over \( \Phi \).

Now that the structure of the subalgebra \( F \) has been determined, the result on the structure of \( A \) enunciated in Theorem 2 is immediate. For it is clear from Axiom (v) that any of the representations for \( F \) given above may be extended to a representation for \( A \) as a set of matrices under the Jordan product. If \( F \) is of type \( D \) or \( E \), then \( A = F \), and in the former case, \( A \) may be realized as matrices by an obvious generalization of the process used in the finite-dimensional case.

Suppose now that this representation for \( A \) is a subset of an associative algebra \( G \) of matrices, and let \( L \) be an ideal of \( G \). Choosing any nonzero element of \( L \) and multiplying by appropriate elements of \( I \), say \( e_i \) and \( e_j \), we see that \( L \) contains a nonzero element in the subalgebra \( G_{(ij)} = \{ x : x \in G, (e_i + e_j)x = x = x(e_i + e_j) \} \). But in each of the five cases of Proposition 11, \( G_{(ij)} \) is a simple algebra generated by the elements of \( F \) contained in it, showing that \( G_{(ij)} \subseteq L \). Then for each \( k \neq i, j \) we have \( G_{(ik)} \cap L \neq 0 \) and hence \( G_{(ik)} \subseteq L \), giving \( e_k \in L \) for every \( k \in S \). Since every ideal of \( G \) contains an idempotent, \( G \) is semisimple; and since any two ideals have nonzero intersection, \( G \) is primitive. Noting that \( e_i \) is an idempotent of finite rank in \( G \), we see that \( G \) contains primitive idempotents and hence minimal right ideals. Thus \( G \) has nonzero socle, and the first sentence of the Corollary to Theorem 2 now follows easily from Proposition 11.

Conversely, let \( G \) be any primitive ring with nonzero socle whose centralizer is the field \( \Phi \), let \( G \) be represented as row-finite matrices over \( \Phi \), and let \( A \) be any set of elements of \( G \) which includes all the elements of \( G \) represented as matrices with finitely many nonzero entries and which is closed under the Jordan product. Then \( A \) satisfies Axiom (i) since any Jordan algebra satisfies this axiom. If \( e \) is a primitive idempotent of \( G \), then it follows easily from the standard theory of primitive rings with nonzero socle that \( eGe = \Phi e \), showing that \( A \) satisfies Axiom (iii) since every primitive idempotent of \( A \) with finitely many nonzero entries is primitive in \( G \). Because \( G \) remains primitive with nonzero socle under scalar extension, we also have Axiom (iv).

Also, the elements of \( G \) with a 1 in one position on the diagonal and zero elsewhere form a set \( I \) with respect to which \( A \) satisfies Axioms (ii), (v), and primitivity.
Next, let $G$ be a primitive ring with nonzero socle whose centralizer is the division ring $\Gamma$, let $A^*$ be the set of all elements of $G$ left fixed by an involution $J$ of $G$ under the Jordan product, and let the fixed elements of $\Gamma$ under the involution of $\Gamma$ induced by $J$ be the field $\Phi$. Again it is immediate that $A^*$ satisfies Axiom (i). If $e$ is an idempotent of $A$ which is primitive in $A^*$, then $eGe$ is a primitive subalgebra of $G$ fixed under $J$ with $\Gamma$ as centralizer, but containing no idempotents of $A^*$, except $e$. Let $e_1$ be a primitive idempotent of $G$ in $eGe$, let $e_2 - e_1$, and let $M$ be a faithful irreducible right module for $G$. Then $K = \{x \mid x \in G, \ MX \subseteq Me_1 + Me_2\}$ is a subalgebra of $eGe$ fixed under $J$ and isomorphic to either $\Gamma$ or the $2 \times 2$ matrices over $\Phi$, depending on whether $e \neq e_1$ or $e_2 \neq e_1$. The identity element of $K$ is fixed under $J$ and hence must be $e$, showing that any primitive idempotent of $A^*$ has rank $1$ or $2$ in $G$. Letting $K_S$ be the set of elements of $K$ fixed under $J$, it follows from the theory of finite dimension Jordan algebras that $K_S$ has one of the forms (b), (c), (d), or (e) of Proposition 11. But if $K$ is the $2 \times 2$ matrices and if $K_S$ had the form (b) or (d), then $K_S$ would contain an idempotent besides $e$. Thus $K_S = \Phi e$, and Axiom (iii) is satisfied (in fact, we have proved that $A \Phi (1) = \Phi e$ for all primitive idempotents, not just those in $F$). As in the last case, Axiom (iv) now follows because $G$ remains primitive under scalar extension.

Suppose now that $I$ is any maximal set of orthogonal primitive idempotents of $A^*$. If there exists an idempotent $e_1$ of $G$ orthogonal to all of the elements of $I$, then $e_2 = e_1$ is also orthogonal to all the elements of $I$, and the identity element $e$ of the subalgebra $K = \{x \mid x \in G, \ MX \subseteq Me_1 + Me_2\}$ is orthogonal to the elements of $I$ as well as being in $A^*$, which contradicts the maximality of $I$. Thus $I$ is a maximal set of orthogonal (but not necessarily primitive) idempotents in $G$. But then no element of $G$ is orthogonal to every element of $I$, which implies that no element of $A^*$ is orthogonal to every element of $I$, or that Axiom (ii) holds. It is now easy to see that Axiom (v) also holds and that $A^*$ is primitive.

Let $F$ be the elements of $A^*$ with finitely many nonzero components with respect to some maximal set of orthogonal primitive idempotents of $A^*$ and let $A$ be any subalgebra of $A^*$ containing $F$. Then it is clear that the fact that $A^*$ satisfies Axioms (i)–(v) implies that $A$ does also. This completes the proof of the Corollary of Theorem 2.

3. In this final section we treat primitive algebras which are not regular. We first prove Theorem 3 and then give some examples which seem to show that Theorem 3 is as far as one can go with the structure of nonregular primitive algebras using the methods of this paper. A central tool used in the proofs of this section is

**Lemma 9.** Let $e_1, e_2, e_3$ be orthogonal idempotents of $A$, let $a \in A_{12}$,
Let $b \in A_{13}$, $c \in A_{23}$, and let $a^2 = \alpha e_1 + \beta e_2$, $b^2 = \gamma e_1 + \delta e_3$, $c^2 = \epsilon e_2 + \eta e_3$, $ab = \rho e$, $ac = \sigma b$, and $bc = \tau a$ for $\alpha, \beta, \gamma, \delta, \epsilon, \eta, \rho, \sigma, \tau \in \Phi$. Then for $\lambda_1, \cdots, \lambda_6 \in \Phi$ the element $u = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 a + \lambda_5 b + \lambda_6 c$ is idempotent if and only if the following relations hold:

$$
\begin{align*}
\lambda_1^2 - \lambda_1 + \alpha \lambda_3^2 + \gamma \lambda_5^2 &= 0, \\
\lambda_2^2 - \lambda_2 + \beta \lambda_4^2 + \epsilon \lambda_6^2 &= 0, \\
\lambda_3^2 - \lambda_3 + \delta \lambda_5^2 + \eta \lambda_6^2 &= 0, \\
(\lambda_1 + \lambda_2 - 1)\lambda_5 + 2\sigma \lambda_4 \lambda_6 &= 0, \\
(\lambda_2 + \lambda_3 - 1)\lambda_6 + 2\rho \lambda_4 \lambda_5 &= 0.
\end{align*}
$$

This lemma may be proved by checking that the equations (3) are simply the various components of the equation $u^2 - u = 0$. Since the calculation is completely straightforward, the details will be omitted.

Suppose now that $A$ is any primitive algebra which is not regular. Then every $A_{ij}$ has dimension 1 or 0 by Proposition 8, and, for some $p$ and $q$, $A_{pq}$ will not be regular. If there exists a nonzero regular $A_{ij}$ in $A$ also, then, using the finite chain of subspaces from $A_{ij}$ to $A_{pq}$ which exists by the definition of primitivity, we see that there exist $k, l, m \in I$ such that $A_{kl}$ is nonzero regular and $A_{im}$ is not regular. If we can rule out this last situation, we will have proved Theorem 3. But it is clear that the subalgebra

$$A_k + A_1 + A_m + A_{kl} + A_{km} + A_{lm}$$

is semisimple, and it is not regular since $A_{im}$ is not regular. The desired contradiction now follows from

**Proposition 12.** Let $e_1, e_2, e_3$ be orthogonal idempotents of a semisimple algebra $A$ which is not regular, and let $A = A_{12} + A_{23} + A_{13} + A_{13} + A_{23}$. Then each of the spaces $A_{12}$, $A_{13}$, or $A_{23}$ is either zero or is not regular.

To prove this proposition, let us suppose that $A$ satisfies the hypotheses of Proposition 12 but not the conclusion. Let us say that $A_{12}$ is nonzero regular and $A_{23}$ is not regular. Then $A_{12}$ is also not regular by Proposition 4, and we have an algebra of the type given in Lemma 9 with the added condition that $\beta = \alpha$. By possibly making a quadratic scalar extension of $\Phi$, we may replace $a$ by an appropriate scalar multiple of $a$ so that $\alpha$ and $\beta$ may be taken to be 4. But then Proposition 4 tells us that $ab \neq 0$, so that we may replace $c$ by $ab$ to obtain $\rho = 1$. The same proposition now implies that $\gamma = \epsilon$, $\delta = \eta$, and $\sigma = 1$.

Consider now the element $u = \alpha' e_1 + (1 - \alpha') e_6 + a$ where $\alpha'$ is a root of the equation $x^2 - x + 4 = 0$ (a second quadratic scalar extension may be needed). But $u$ is idempotent by Lemma 2, and $[\alpha' b + 2c] \in A_{a\frac{1}{2}}$ and
[(1 - \alpha')b - 2c] \in A_u(0) \text{ by Lemma 4. Hence, the product}
\begin{align*}
n(x) &= [\alpha' b + 2c][(1 - \alpha')b - 2c] \\
&= \alpha'(1 - \alpha')(\epsilon_1 + \eta e_3) - 4(\epsilon_2 + \eta e_3) + (2 - 4\alpha')\tau a \\
&= 4\epsilon(\epsilon_1 - \epsilon_2) + (2 - 4\alpha')\tau a \in A_u(\frac{1}{2})
\end{align*}

by Proposition 1. But the coefficient of \epsilon_1 in the equation \(ux = \frac{1}{2}x\) is
\[4\alpha'\epsilon + 4(2 - 4\alpha')\tau = 2\epsilon,\]
giving \((2 - 4\alpha')\tau = (2 - 4\alpha')\epsilon\text{ or } 4\tau = \epsilon.\]

Suppose first that \(\tau = 0\) and hence that \(\epsilon = 0.\) After possibly adjoining
\[i = \sqrt{-1}\text{ to our base field, we may check that the element } w = e_3 + b + ic\]
is idempotent. Defining \(t = 4i\eta e_3 - ib - c\) and \(s = b + ic,\) we compute that
\[tw = 2i\eta e_3 + (2i\eta - \frac{1}{2})b + (-2\eta - \frac{1}{2})c,\]
or \(t(R_w - \frac{1}{2}I) = 2is.\) But \(\eta \neq 0\) since \(A\) is semisimple, and \(sw = \frac{1}{2}s\) by calculation, showing that \(t(R_w - \frac{1}{2}) \neq 0\) and \(t(R_w - \frac{1}{2})^2 = 0.\) Suppose now that \(A = B/R\) where \(B\) is an algebra
satisfying Axioms (i)-(v) and where \(R\) is its radical, and let \(e_1', e_2', e_3'\) be the
canonical set of primitive orthogonal idempotents in \(B\) mapping into \(e_1, e_2, e_3\)
respectively. Furthermore, let \(b'\) and \(c'\) be the unique preimages of \(b\) and \(c\) in
\(B_{13}\) and \(B_{23}\) respectively, and let \(a' = b'c' \in B_{12} \cap R.\) Then \(a'b' \in B_{23} \cap R = 0\)
and \(a'c' = 0,\) so that \(w' = e_3' + 2ia' + b' + ic'\) is an idempotent preimage
of \(w,\) and \(w\) must satisfy Axiom (i). This contradiction shows that \(\tau \neq 0.\)

Again possibly making a quadratic scalar extension, we may now replace \(b\)
and \(c\) by \((1/\sqrt{\tau})b\) and \((1/\sqrt{\tau})c\) respectively to achieve \(bc = a.\) This replace-
ment must also make \(\epsilon = 4,\) since \(\epsilon = 4\tau.\) We have shown that after a scalar
extension of degree at most eight (we do not need \(\alpha'\) in the field any longer),
the base field may be assumed to contain \(i = \sqrt{-1}\) and the multiplication
constants given in Lemma 9 may be taken to be \(\alpha = \beta = \gamma = \epsilon = 4,\)
\(\delta = \eta \neq 4,\) and \(\rho = \sigma = \tau = 1,\) with only \(\eta\) not determined. But, letting
\(e = e_1 - e_2 + e_3 - (i/2)a + \frac{1}{2}b + (i/2)c, x = (\eta - 4)e_1 + \eta e_2 - \eta e_3 - 2ic,\)
and \(y = e_1 - e_3 + (i/2)a,\) we may check that \(e^2 = e, ex = (\eta - 4)y \neq 0,\)
and \(ey = 0.\) If \(e_1', e_2', e_3', b', c'\) are the same preimages of \(e_1, e_2, e_3, b, c\)
respectively as above, and if \(a' = b'c'\) again, we may check that
\[e' = e_1' - e_2' + e_3' + i/2 a' + 1/2 b' + i/2 c',\]
is an idempotent preimage of \(e,\) so that \(e\) must satisfy Axiom (i). This contra-
diction finishes the proof of Proposition 12 and Theorem 3.

We conclude with several examples of nonregular algebras satisfying
Axioms (i)-(v). Let us begin by letting \(A\) be a 5-dimensional algebra over \(\Phi\)
spanned by the orthogonal idempotents \(e_1, e_2, e_3\) and by \(a \in A_{12}\) and \(c \in A_{23}\)
where \(ac = 0, a^2 = e_1 + \beta e_2,\) and \(c^2 = e_3 + 4\beta e_3\) for some \(\beta \in \Phi\) not
equal to 1 or \(1/2.\) By specializing Lemma 9 in the appropriate manner, we see
that the element \( u = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 a + \lambda_5 c \) of \( A \) is idempotent if and only if the \( \lambda \)'s satisfy the equations.

\[
\begin{align*}
\lambda_1^2 - \lambda_1 + \lambda_4^2 &= 0, \\
\lambda_2^2 - \lambda_2 + \beta \lambda_4^2 + \lambda_5^2 &= 0, \\
\lambda_3^2 - \lambda_3 + 4\beta \lambda_6^2 &= 0, \\
(\lambda_1 + \lambda_2 - 1)\lambda_4 &= 0, (\lambda_2 + \lambda_3 - 1)\lambda_6 &= 0.
\end{align*}
\] (4)

If \( \lambda_4 \) and \( \lambda_6 \) are both nonzero, then the last two equations of (4) give \( \lambda_1 = 1 - \lambda_2 - \lambda_3 \), showing that \( \lambda_1^2 - \lambda_1 = \lambda_2^2 - \lambda_2 = \lambda_3^2 - \lambda_3 \). The first three equations of (4) now imply \( (\beta - 1)\lambda_4^2 + \lambda_5^2 = 0 \) and \( \beta \lambda_4^2 + (1 - 4\beta)\lambda_6^2 = 0 \). Since the determinant \( (\beta - 1)(1 - 4\beta) - \beta = -(2\beta - 1) \) is not zero by hypothesis, these equations have only the trivial solution \( \lambda_4^2 = \lambda_5^2 = 0 \), showing that there is no idempotent \( u \) in \( A \) with \( \lambda_4 \) and \( \lambda_6 \) both nonzero.

Suppose now that \( \lambda_4 = 0 \). Then \( \lambda_2^2 - \lambda_1 = 0 \) or \( \lambda_2 = 0, 1 \), and \( u \) is either in \( A_2 + A_{23} + A_3 \) or is the orthogonal sum of \( e_1 \) and an idempotent in \( A_2 + A_{23} + A_3 \). But we know from Lemma 2 that the only idempotents in \( A_2 + A_{23} + A_3 \) are \( e_1, e_3 \), and \( e_1 + e_3 \) unless \( e = \eta \). Noting that the same result holds in the identical case when \( \lambda_6 = 0 \), we see that the only idempotents in \( A \) are the trivial ones made up from \( e_1, e_2 \), and \( e_3 \).

It is easy now to see that \( A \) satisfies Axioms (i) and (iii), while (iv) follows from the fact that the condition \( \beta \neq 1, \frac{1}{2} \) remains valid under field extension. \( A \) also satisfies Axioms (ii) and (v) and is a primitive algebra, and if \( \beta \neq 0 \) we may easily check that \( A \) is simple. This example shows that in a nonregular primitive algebra the subspaces \( A_{ij} \) do not all have to be the same dimension, even if the algebra is simple.

If \( \beta = 0 \) we find that the subspace spanned by \( e_1 \) and \( a \) and the subspace spanned by \( e_1, a, e_2 \), and \( c \) are both ideals. Thus, a nonregular primitive algebra need not be simple. Both of these ideals are in fact nonregular primitive algebras in their own right, giving examples of algebras satisfying Axioms (i)-(v) in which the subspaces of the form \( A_{1i} \) are nontrivial.

Our next example is formed by adding a radical to our last example. Let \( A \) have a basis consisting of the orthogonal idempotents \( e_1, e_2, e_3, \) of \( a \in A_{12} \) and \( c \in A_{23} \), and of \( b_1, b_2, \ldots \in A_{13} \), where \( ac = b_1, b_2, a = b_3, c = b_4 \), etc. for \( i = 1, 2, \ldots \), and where \( a^2 = e_1 + \beta e_2 \) and \( c^2 = e_2 + 4\beta e_3 \) for some \( \beta \neq 1, \frac{1}{2} \).

Then the element \( u = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 a + \lambda_5 b + \lambda_6 c \) for \( b \in A_{13} \) is idempotent if and only if the equations (4) and (\( \lambda_1 + \lambda_3 - 1)\lambda_5 b + 2\lambda_4 \lambda_6 c = 0 \) hold. But we have seen that the only solutions of (4) have \( \lambda_3 = \lambda_6 = 0 \), showing that any idempotent of \( A \) is in \( A_1 + A_{13} + A_3 \), or is the sum of \( e_2 \) and an idempotent in \( A_1 + A_{13} + A_4 \). Thus, by Lemma 2, every primitive idempotent of \( A \) is of the form \( e_1 + b, e_3 + b, \) or \( e_3 \) for some \( b \in A_{13} \). It is now easy to verify directly that Axioms (i)-(v) are satisfied.

We see from this example that a nonregular algebra can have a nonzero radical \( R(=A_{13}) \). Since \( ac = b, \neq 0, A \) has no subalgebra isomorphic to
For our final example we consider a 6-dimensional algebra over $A$, spanned by the orthogonal idempotents $e_1, e_2, e_3$ and by $a \in A_{12}$, $b \in A_{13}$, and $c \in A_{23}$, where $a^2 = b^2 = e_1$, $c^2 = 4e_2 + \gamma e_3$, $ab = a$, $ac = b$, and $bc = a$ for $\gamma \neq 4$. From Lemma 9 the element $u$ of $A$ will be idempotent this time only if

$$\begin{align*}
\lambda_1^2 - \lambda_1 + 4\lambda_4^2 + 4\lambda_5^2 &= 0, \lambda_2^2 - \lambda_2 + 4\lambda_6^2 = 0, \\
\lambda_3^2 - \lambda_3 + \gamma \lambda_5^2 &= 0, (\lambda_1 + \lambda_2 - 1)\lambda_4 + 2\lambda_3\lambda_6 = 0, \\
(\lambda_1 + \lambda_3 - 1)\lambda_6 + 2\lambda_4\lambda_6 &= 0, (\lambda_2 + \lambda_3 - 1)\lambda_6 + 2\lambda_4\lambda_5 = 0.
\end{align*}$$

We show first that $A$ contains no idempotents such that $\lambda_4\lambda_5\lambda_6 \neq 0$. For if $\lambda_4\lambda_5\lambda_6 \neq 0$, the last three equations of (5) may be combined to give

$$\begin{align*}
(\lambda_1 + \lambda_2 - 1)(\lambda_1 + \lambda_3 - 1) &= (-2\lambda_4\lambda_5\lambda_6)(-2\lambda_4\lambda_5\lambda_6) = 4\lambda_6^5, \\
(\lambda_1 + \lambda_2 - 1)(\lambda_2 + \lambda_3 - 1) &= 4\lambda_5^2,
\end{align*}$$

leading to

$$\begin{align*}
\lambda_1^2 - \lambda_1 - \lambda_2^2 + \lambda_2 &= (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 - 1) \\
&= [(\lambda_1 + \lambda_3 - 1) - (\lambda_2 + \lambda_3 - 1) - (\lambda_1 + \lambda_2 - 1)](\lambda_1 + \lambda_2 - 1) \\
&= 4\lambda_6^2 - 4\lambda_5^2.
\end{align*}$$

But then $\lambda_1^2 - \lambda_1 - 4\lambda_5^2 = \lambda_2^2 - \lambda_2 - 4\lambda_6^2$, which reduces to $-4\lambda_4^2 = 0$ using the first two equations of (5), contradicting the assumption that $\lambda_4\lambda_5\lambda_6 \neq 0$.

Thus, for any idempotent $u$ of $A$, at least one of the coefficients $\lambda_4, \lambda_5, \lambda_6$ is zero. However, looking at the last three equations of (5) we see that the product of the other two of these three coefficients is also zero. This shows that for some permutation $\{i, j, k\}$ of $\{1, 2, 3\}$ either $u$ is in $A_i + A_{ij} - A_j$ or $u$ is the orthogonal sum of $e_k$ and some idempotent in $A_i + A_{ij} + A_j$. Again Lemma 2 tells us that the only idempotents in $A$ are those made up of $e_1, e_2, e_3$, and again Axioms (i)-(v) and primitivity are satisfied.

This example shows that the product of independent nonregular elements in a primitive algebra does not have to be zero, and that the spaces $A_{12}, A_{13}, A_{23}$ can all be nonregular. If a nonregular primitive algebra contains a proper ideal $B$, it is clear that $B$ contains a proper subset of the idempotents of $I$, and that $e_j \in B$ and $e_k \notin B$ imply that $A_{jk} \subseteq A_j$. Our last example shows that the latter two conditions on a subspace $B$ of $A$ are not even sufficient for $B$ to be a subalgebra, let alone an ideal.
In each of the examples of nonregular primitive algebras given here as well as in all other such examples that we have been able to construct, Axioms (i)–(v) are satisfied because of the absence of any idempotent that is not a sum of elements of $I$. This very negative way of satisfying our axioms suggests that the further study of nonregular primitive algebras would not lead to much of interest.

References