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On Quasilinear Non-Uniformly Parabolic Equations in General Form

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The present paper is concerned with the first boundary value problem for a certain class of quasilinear non-uniformly parabolic equations. New a priori estimates of the solution and of its gradient are obtained. These are independent of the smoothness

INTRODUCTION

The primary question for quasilinear uniformly and non-uniformly parabolic equations consists in obtaining an a priori estimate for the gradient of the solution $|\nabla u|$. In the case of one space variable this estimate have been obtained without any restriction on the smoothness of the coefficients (see [1]). In the higher dimensional case the main idea, that goes back to S. N. Bernstein, involves the preliminary boundary estimate of $|\nabla u|$, differentiation of the equation and application of the maximum principle (see [2]).

In the present paper (Section 3) we consider the first boundary value problem for a certain class of quasilinear non-uniformly parabolic equations for cylinders with a strictly convex base. We obtain an estimate for $|\nabla u|$ in the whole domain without differentiation of the equation. The estimate is independent of the ellipticity coefficient, of the smoothness of the coefficients of the equation and is found explicitly. Based on this a priori estimate and known results [1, 3, 4] the existence theorem is proved. In deriving an a priori estimate for $|\nabla u|$ the idea of introducing an additional spatial variable [1] is used (see also [5–8]).

In Section 1 and 2 we consider the first boundary value problem for the general quasilinear non-uniformly parabolic equations in arbitrary domains. A new sufficient condition for the boundedness of a classical solution is obtained. The generalization of the uniqueness theorem (Theorem 2.8 from [2]) is given.

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1. ESTIMATE OF THE SOLUTION

Consider the following problem

$$a_{ij}(t, \mathbf{x}, u, \nabla u)\, u_{x_i x_j} - u_t = f(t, \mathbf{x}, u, \nabla u) \qquad \text{in} \quad Q_T = \Omega \times (0, T), \quad (1.1)$$

(we assume the summation convention)

$$u(0, \mathbf{x}) = \phi(\mathbf{x}) \qquad \text{on } \Omega \qquad \text{and} \qquad u = \chi(s) \qquad \text{on } S = \partial\Omega \times [0, T], \quad (1.2)$$

where $\Omega \subset R^n$ is bounded domain, $\mathbf{x} = (x_1, x_2, ..., x_n)$, $\nabla u = (u_{x_1}, ..., u_{x_n})$, $a_{ij} = a_{ji}$, $i, j = 1, 2, ..., n$. Assume that the functions a_{ij}, f are defined on the set $\bar{Q}_T \times R \times R^2$, are bounded for $(t, \mathbf{x}) \in Q_T$ and for finite u, ∇u and

$$a_{ij} \xi_i \xi_j \geqslant 0, \qquad \forall \xi \in R^n. \quad (1.3)$$

Suppose that there exist an index i_0, without loss of generality set $i_0 = 1$, such that f satisfies, for $(t, \mathbf{x}) \in Q_T$ and any u, p_1 the structure condition

$$|f(t, \mathbf{x}, u, p_1, 0, ..., 0)| \leqslant a_{11}(t, \mathbf{x}, u, p_1, 0, ..., 0)\, \psi(|p_1|) \quad (1.4)$$

where $\psi(\rho)$ is a continuously differentiable function such that $\psi(\rho) \geqslant 1$ for $\rho \geqslant 0$ and

$$\int^{+\infty} \frac{d\rho}{\psi(\rho)} = +\infty. \quad (1.5)$$

LEMMA 1.1. *Let* $u(t, \mathbf{x})$ *be a classical solution* ($u(t, \mathbf{x}) \in C^0(\bar{Q}_T) \cap C^{1,2}_{t,\mathbf{x}}(\bar{Q}_T \backslash (S \cup \Omega))$) *of the problem* (1.1), (1.2) *and assume that conditions* (1.3), (1.4), (1.5) *hold. Then*

$$\sup_{Q_T} |u| \leqslant M,$$

where the constant M *depends only on* ψ, $m = \max\{\sup |\phi|, \sup |\chi|\}$ *and* d, d *is the size of the domain* Ω *in the direction of the variable* x_1.

Proof. Without loss of generality assume that Ω is lying in the slab $0 < x_1 < d$. Let

$$L(u) \equiv a_{11}(u_{x_1 x_1} + \psi(|u_{x_1}|)) + \sum_{i+j=3}^{2n} a_{ij} u_{x_i x_j} - u_t.$$

Introduce the function $h(x_1)$ such that

$$h'' + \psi(|h'|) = 0, \qquad h(0) = m, \qquad h(d) = H,$$

where H is a constant to be specified below. Represent the solution in parametrical form (which is an easy matter when using the substitution $h' = q$)

$$h(q) = \int_q^{q_1} \frac{\rho\, d\rho}{\psi(\rho)} + m, \qquad x_1(q) = \int_q^{q_1} \frac{d\rho}{\psi(\rho)},$$

where $q \in [q_0, q_1]$ and q_0, q_1 are chosen so as to have $q_1 > q_0 > 0$ and

$$x_1(q) = \int_{q_0}^{q_1} \frac{d\rho}{\psi(\rho)} = d,$$

which is possible due to (1.5). We put

$$H = \int_{q_0}^{q_1} \frac{\rho\, d\rho}{\psi(\rho)} + m.$$

Obviously, $h'(x_1) = q > 0$ and hence $u - h \leqslant 0$ on $\Omega \cup S$. Besides $L(h) = 0$ and thus for $w = u - h$ we have

$$\begin{aligned} L(u) - L(h) \equiv L_0(w) &\equiv a_{11}(w_{x_1x_1} + \beta w_{x_1}) + \sum_{i+j=3}^{2n} a_{ij} w_{x_ix_j} - w_t \\ &= f(t, \mathbf{x}, u, \nabla u) + a_{11}(t, \mathbf{x}, u, \nabla u)\, \psi(|u_{x_1}|). \end{aligned}$$

In view of the fact that $u(t, \mathbf{x})$ is a classical solution and that the function $\psi(\rho)$ is a C^1 function we obtain that $|\beta| < +\infty$ in $\bar{Q}_T \backslash (S \cup \Omega)$.

Consider the function $\tilde{w} = we^{-t}$. We have that

$$\begin{aligned} L_0(w) &= a_{11}(\tilde{w}_{x_1x_1} + \beta \tilde{w}_{x_1}) + \sum_{i+j=3}^{2n} a_{ij} \tilde{w}_{x_ix_j} - \tilde{w} - \tilde{w}_t \\ &= e^{-t}(f(t, \mathbf{x}, u, \nabla u) + a_{11}(t, \mathbf{x}, u, \nabla u)\, \psi(|u_{x_1}|)). \end{aligned}$$

If the function $\tilde{w}$ attains its positive maximum at the point $N \in \bar{Q}_T \backslash (S \cup \Omega)$ then at this point $\nabla \tilde{w} = 0$ i.e. $u_{x_1} = h'$, $u_{x_i} = 0$ for $i = 2, 3, ...n$. By virtue of (1.4) we obtain

$$\begin{aligned} & a_{11}(\tilde{w}_{x_1x_1} + \beta \tilde{w}_{x_1}) + \sum_{i+j=3}^{2n} a_{ij} \tilde{w}_{x_ix_j} - \tilde{w} - \tilde{w}_t |_N \\ &\quad = e^{-t}(f(t, \mathbf{x}, u, u_{x_1}, 0, ..., 0) + a_{11}(t, \mathbf{x}, u, u_{x_1}, 0, ..., 0)\, \psi(|u_{x_1}|))|_N \geqslant 0. \end{aligned}$$

This contradicts the fact that $\tilde{w}$ attains its positive maximum in $\bar{Q}_T \backslash (S \cup \Omega)$. In view of the non-positivity of $\tilde{w}$ on the $S \cup \Omega$ we conclude that $\tilde{w} \leqslant 0$ on $\bar{Q}_T$ and consequently $w = u - h \leqslant 0$ on $\bar{Q}_T$.

Since $h' > 0$, the function $v \equiv u + h$ is not less than zero on $S \cup \Omega$. Obviously

$$L_1(u) \equiv a_{ij} u_{x_i x_j} - u_t = f(t, \mathbf{x}, u, \nabla u) \quad \text{and} \quad L_1(h) = -a_{11}(t, \mathbf{x}, u, \nabla u)\, \psi(|h'|).$$

It is clear that for $\tilde{v} = v e^{-t}$ we have

$$a_{ij} \tilde{v}_{x_i x_j} - \tilde{v} - \tilde{v}_t = e^{-t}(f(t, \mathbf{x}, u, \nabla u) - a_{11}(t, \mathbf{x}, u, \nabla u)\, \psi(|h'|)).$$

If the function $\tilde{v}$ attains its negative minimum at the point $N \in \bar{Q}_T \backslash (S \cup \Omega)$ then at this point $\nabla \tilde{v} = 0$ i.e. $u_{x_1} = -h'$, $u_{x_i} = 0$ for $i = 2, 3, \ldots n$. By virtue of (1.4)

$$a_{ij} \tilde{v}_{x_i x_j} - \tilde{v} - \tilde{v}_t|_N = e^{-t}(f(t, \mathbf{x}, u, -h', 0, \ldots, 0)$$
$$- a_{11}(t, \mathbf{x}, u, -h', 0, \ldots, 0)\, \psi(|h'|))|_N \leqslant 0.$$

This contradicts the fact that $\tilde{v}$ attains a negative minimum at N. Taking into account that $\tilde{v} = (u + h)\, e^{-t} \geqslant 0$ on $S \cup \Omega$ we conclude that $\tilde{v} \geqslant 0$ on $\bar{Q}_T$ and hence $v \geqslant 0$ on $\bar{Q}_T$. This completes the proof of the lemma.

Remark 1.1. Instead of (1.5) it is sufficient to require the existence of α_1 and α_2 ($\alpha_2 > \alpha_1 > 0$) such that

$$\int_{\alpha_1}^{\alpha_2} \frac{d\rho}{\psi(\rho)} = d.$$

Remark 1.2. If the condition (1.5) is fulfilled for $\psi(\rho)$ such that $\psi(0) = 0$ (instead of $\psi(\rho) \geqslant 1$), then (1.5) can be replaced by the following condition

$$\int_0^{+\infty} \frac{d\rho}{\psi(\rho)} = +\infty.$$

For example we can take $\psi(|u_{x_1}|) = |u_{x_1}|^k$ for any non-negative constant $k \in R$.

We will now prove the estimates of the solution near the boundary in a special case. These will be used in Section 3 for the estimation of the gradient of the solution.

Definition. We will say that *the domain Ω satisfies the condition* (a) if Ω is strictly convex, $\Omega \subset \{\mathbf{x} : A_i \leqslant x_i \leqslant B_i,\ i = 1, \ldots n\}$, where $A_i < 0$, $B_i > 0$, $\partial\Omega \in C^2$ and $(A_1, 0, 0, \ldots, , 0, 0)$, $(0, A_2, 0, \ldots, 0, 0)$, ..., $(0, 0, 0, \ldots, 0, A_n)$, $(B_1, 0, 0, \ldots, , 0, 0)$, $(0, B_2, 0, \ldots, 0, 0)$, ..., $(0, 0, 0, \ldots, 0, B_n) \in \partial\Omega$.

Obviously if Ω satisfies the condition (a) then it is possible to represent the parts of $\partial\Omega$ which are lying in the half-spaces $x_i \leqslant 0$ and $x_i \geqslant 0$,

$i=1, ..., n$ in the form of the functions $x_i=F_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$, $x_i=G_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ respectively, where F_i, G_i $i=1, ..., n$ are C^2 functions. Introduce the functions $h_{1i}(x_i)$, $h_{2i}(x_i)$, $i=1, 2, ..., n$ by the following

$$\begin{aligned} &h''_{1i}+\psi(|h'_{1i}|)=0, \qquad h_{1i}(A_i)=0, \qquad h_{1i}(A_i+\tau_0)=2M, \\ &h''_{2i}+\psi(|h'_{2i}|)=0, \qquad h_{2i}(B_i-\tau_0)=2M, \qquad h_{2i}(B_i)=0, \end{aligned} \tag{1.6}$$

where ψ now satisfies the following condition

$$\int^{+\infty} \frac{\rho\, d\rho}{\psi(\rho)}=+\infty \tag{1.7}$$

and $M=\sup_{Q_T}|u|$. To define τ_0 represent h_{1i} in parametrical form

$$h_{1i}(q)=\int_q^{q_1} \frac{\rho\, d\rho}{\psi(\rho)}, \qquad x_i(q)=\int_q^{q_1} \frac{d\rho}{\psi(\rho)},$$

where $q\in[q_0, q_1]$ and q_0, q_1 are chosen so as to have $q_1>q_0\geqslant m_i\geqslant 0$, $i=1, ..., n$ and

$$h_{1i}(q)=\int_{q_0}^{q_1} \frac{\rho\, d\rho}{\psi(\rho)}=2M$$

which is possible due to (1.7). The constants m_i will be specified in Section 3. We put

$$\tau_0=\int_{q_0}^{q_1} \frac{d\rho}{\psi(\rho)}.$$

Suppose that instead of (1.4) the following condition is fulfilled

$$|f(t, \mathbf{x}, u, \mathbf{p})|\leqslant a_{11}(t, \mathbf{x}, u, \mathbf{p})\,\psi(|p_1|) \tag{1.8}$$

for $(t, \mathbf{x})\in Q_T$, $|u|<M$ and any $\mathbf{p}$, where $\mathbf{p}=(p_1, ..., p_n)$.

LEMMA 1.2. *Suppose that Ω satisfies the condition* (a). *In addition suppose that $\chi\equiv 0$, $a_{ij}\equiv 0$ for $i\neq j$ and*

$$|\phi(\mathbf{x})|\leqslant h_{k1}(\xi_{k1}) \qquad \text{in } D_{k1}, \qquad k=1, 2.$$

where $\xi_{11}=x_1-F_1(\mathbf{x}')+A_1$, $\xi_{21}=x_1-G_1(\mathbf{x}')+B_1$, $\mathbf{x}'=(x_2, ..., x_n)$,

$$D_{11}=\Omega\cap\{\mathbf{x}: F_1(\mathbf{x}')<x_1<F_1(\mathbf{x}')+\tau_0;\ A_i<x_i<B_i,\ i=2, ..., n\},$$

$$D_{21}=\Omega\cap\{\mathbf{x}: G_1(\mathbf{x}')-\tau_0<x_1<G_1(\mathbf{x}');\ A_i<x_i<B_i,\ i=2, ..., n\}.$$

If the conditions (1.3), (1.7), (1.8) *are fulfilled then for any classical solution* $u(t, \mathbf{x})$ *of the problem* (1.1), (1.2) *we have the estimates*

$$|u(t, \mathbf{x})| \leqslant h_{k1}(\xi_{k1}) \qquad \text{in} \quad \bar{D}_T^{k1} = \bar{D}_{k1} \times [0, T], \qquad k = 1, 2.$$

Proof. Obviously, $|u(t, \mathbf{x})| \leqslant h_{k1}(\xi_{k1})$ on $D_{k1} \cup (\partial D_{k1} \times [0, T])$, $k = 1, 2$. Let

$$L_0(u) \equiv a_{11}(u_{x_1 x_1} + \psi |u_{x_1}|) + \sum_{j=2}^{n} a_{jj} u_{x_j x_j} - u_t.$$

We have $L_0(u) \geqslant 0$ and

$$L_0(h_{11}(\xi_{11})) = \sum_{j=2}^{n} a_{jj}(h''_{11} F^2_{1x_j} - h'_{11} F_{1x_j x_j}) \leqslant 0,$$

$$L_0(h_{21}(\xi_{21})) = \sum_{j=2}^{n} a_{jj}(h''_{21} G^2_{1x_j} - h'_{21} G_{1x_j x_j}) \leqslant 0,$$

since Ω is convex domain, $h'_{11} > 0$, $h'_{21} < 0$ and $h''_{11} < 0$, $h''_{21} < 0$. Arguing in the same manner as in Lemma 1.1 we obtain that $u(t, \mathbf{x}) \leqslant h_{k1}(\xi_{k1})$, $k = 1, 2$ on $\bar{D}_{k1T}$.

Now let $L(u) \equiv a_{jj} u_{x_j x_j} - u_t = f(t, \mathbf{x}, u, \nabla u)$. Obviously $L(h_{k1}(\xi_{k1})) \leqslant -a_{11}(t, \mathbf{x}, \nabla u)\ \psi(|h'_{k1}(\xi_{k1})|)$. Again arguing in the same manner as in Lemma 1.1 we obtain that $u(t, \mathbf{x}) \geqslant -h_{k1}(\xi_{k1})$, $k = 1, 2$ on $\bar{D}_T^{k1}$. The lemma is proved.

Let us state a lemma whose proof is similar to the proof of Lemma 1.2.

LEMMA 1.3. *Suppose that the conditions of Lemma* 1.2 *are fulfilled and in addition we have that*

$$|f(t, \mathbf{x}, u, \mathbf{p})| \leqslant a_{ii}(t, \mathbf{x}, u, \mathbf{p})\ \psi(|p_i|) \tag{1.9}$$

and $|\phi(\mathbf{x})| \leqslant h_{ki}(\xi_{ki})$ *on* D_{ki} *for* $i = 2, ..., n$, $k = 1, 2$ *where*

$$D_{1i} = \Omega \cap \{\mathbf{x}: F_i(\mathbf{x}') < x_i < F_i(\mathbf{x}') + \tau_0;\ A_j < x_j < B_j,$$
$$j = 1, ..., i-1, i+1, ..., n\},$$
$$D_{2i} = \Omega \cap \{\mathbf{x}: G_i(\mathbf{x}') - \tau_0 < x_i < G_i(\mathbf{x}');\ A_j < x_j < B_j,$$
$$j = 1, ..., i-1, i+1, ..., n\}.$$

Then we have

$$|u(t, \mathbf{x})| \leqslant h_{ki}(\xi_{ki}) \qquad \text{on } \bar{D}_T^{ki},$$

where $\xi_{1i}=x_i-F_i+A_i$, $\xi_{2i}=x_i-G_i+B_i$ *and* $\bar{D}_T^{ki}=\bar{D}_{ki}\times[0, T]$ *for* $i=1, 2, ..., n$, $k=1, 2$.

2. UNIQUENESS

Usually, the uniqueness is proved under the assumption of differentiability of functions $a_{ij}(t, \mathbf{x}, u, \mathbf{p})$ and $f(t, \mathbf{x}, u, \mathbf{p})$ with respect to u and $\mathbf{p}$ (see for example Theorem 2.8 from [2]). We will show that in a very simple manner the assumption on differentiability with respect to $\mathbf{p}$ can be avoided.

THEOREM 2.1. *If the functions* $a_{ij}(t, \mathbf{x}, u, \mathbf{p}), f(t, \mathbf{x}, u, \mathbf{p})$ *satisfy the condition* (1.3) *and are bounded with their partial derivatives of first order with respect to u for* $(t, \mathbf{x})\in\bar{Q}_T$ *and finite* $u, \nabla u$ *then the problem* (1.1), (1.2) *has no more than one solution in the class of functions belonging to* $C^{2,1}_{\mathbf{x},\mathbf{t}}(\bar{Q}_T)$. *If in addition the functions* a_{ij} *and f are independent of u then the problem* (1.1), (1.2) *has no more than one classical solution.*

Proof. Suppose that there exist two solutions $u, v\in C^{2,1}_{\mathbf{x},\mathbf{t}}(\bar{Q}_T)$. For the function $w\equiv u-v$ we have

$$\begin{aligned} a_{ij}(t, \mathbf{x}, u, \nabla u)\, w_{x_ix_j}-w_t=\; & f(t, \mathbf{x}, u, \nabla u)-f(t, \mathbf{x}, v, \nabla u)+f(t, \mathbf{x}, v, \nabla u)\\ & -f(t, \mathbf{x}, v, \nabla v)+(a_{ij}(t, \mathbf{x}, u, \nabla v)\\ & -a_{ij}(t, \mathbf{x}, u, \nabla u))\, v_{x_ix_j}+(a_{ij}(t, \mathbf{x}, v, \nabla v)\\ & -a_{ij}(t, \mathbf{x}, u, \nabla v))\, v_{x_ix_j}. \end{aligned}$$

Rewrite this relation in the following way

$$a_{ij}(t, \mathbf{x}, u, \nabla u)\, w_{x_ix_j}-(\tilde{f}+\tilde{a}_{ij}v_{x_ix_j})\, w-w_t=F, \tag{2.1}$$

where

$$F=f(t, \mathbf{x}, v, \nabla u)-f(t, \mathbf{x}, v, \nabla v)+(a_{ij}(t, \mathbf{x}, u, \nabla v)-a_{ij}(t, \mathbf{x}, u, \nabla u))\, v_{x_ix_j}.$$

The existence of bounded functions $\tilde{f}$ and $\tilde{a}_{ij}$ is guaranteed by the theorem of the mean and the differentiability of the functions $a_{ij}(t, \mathbf{x}, u, \mathbf{p})$ and $f(t, \mathbf{x}, u, \mathbf{p})$ with respect to the variable u. We have that $|\tilde{f}+\tilde{a}_{ij}v_{x_ix_j}|<C_0<+\infty$ because $(\nabla u, \nabla v, v_{x_ix_j})\in C(\bar{Q}_T)$. For the function $\tilde{w}=e^{-C_0t}w$ we obtain

$$L(\tilde{w})\equiv a_{ij}(t, \mathbf{x}, u, \nabla u)\, \tilde{w}_{x_ix_j}-(\tilde{f}+\tilde{a}_{ij}v_{x_ix_j}+C_0)\, \tilde{w}-\tilde{w}_t=e^{-C_0t}F. \tag{2.2}$$

Suppose that the function $\tilde{w}$ achieves its positive maximum or negative minimum at the point $N\in\bar{Q}_T\backslash(S\cup\Omega)$ where we have $\tilde{w}_{x_i}=0\ \forall i$ i.e.

$\nabla u(N) = \nabla v(N)$ and hence $F(N) = 0$. From (2.2) we have that $L(\tilde{w}) = 0$ at the point N which is impossible, so we have that in $\bar{Q}_T \backslash (S \cup \Omega)$ the function $\tilde{w}$ cannot achieve a positive maximum or a negative minimum. Taking into account that $\tilde{w} = e^{-C_0 t}(u - v) = 0$ on $S \cup \Omega$ we obtain that $u \equiv v$ on $\bar{Q}_T$.

The second statement of the theorem follows from the fact that in this case instead of (2.1) we have

$$a_{ij}(t, \mathbf{x}, u, \nabla u) w_{x_i x_j} - w_t = F.$$

Considering the function $w^1 = e^{-t} w$ we obtain the required result.

Remark 2.1. If the functions a_{ij} are independent of u and $f = f(t, \mathbf{x}, u, \nabla u)$ than the problem (1.1), (1.2) has no more then one solution in the class of functions belonging to $C^{2,1}_{\mathbf{x},\mathbf{t}}(Q_T) \cap C^{1,0}_{\mathbf{x},\mathbf{t}}(\bar{Q}_T)$, because in this case instead of (2.1) we have

$$a_{ij}(t, \mathbf{x}, \nabla u)\, w_{x_i x_j} - \tilde{f} w - w_t = f(t, \mathbf{x}, v, \nabla u) - f(t, \mathbf{x}, v, \nabla v),$$

and $|\tilde{f}| < C_0 < +\infty$.

Remark 2.2. If the function f can be written in the following form $f(t, \mathbf{x}, u, \nabla u) = g_i(t, \mathbf{x}, \nabla u)\, u_{x_i} + G(t, \mathbf{x}, u)$ where G_u is bounded for $(t, \mathbf{x}) \in \bar{Q}_T$ and finite u and the coefficients a_{ij} are independent of u then the problem (1.1), (1.2) has no more then one classical solution. In this case instead of (2.1) we have

$$a_{ij}(t, \mathbf{x}, \nabla u)\, w_{x_i x_j} - \tilde{G} w - w_t = F^1,$$

where $|\tilde{G}| < C_0 < +\infty$, and $F^1 \equiv g_i(t, \mathbf{x}, \nabla u)\, u_{x_i} - g_i(t, \mathbf{x}, \nabla v)\, v_{x_i}$. Obviously at the point N corresponding to the extremum of the function $\tilde{w} = e^{-t} w$ we have $F^1(N) = 0$.

Remark 2.3. It is not difficult to see that it is sufficient to require the boundedness of the derivative $f_u(G_u)$ from below. This restriction cannot be weakened (this follows from the linear case).

3. GRADIENT ESTIMATE AND EXISTENCE THEOREM

Consider the following problem

$$a_{11}(t, \mathbf{x}, u, \nabla u)\, u_{x_1 x_1} + \sum_{i=2}^{n} a_{ii}(t, \mathbf{x}', \nabla u)\, u_{x_i x_i} - u_t = f(t, \mathbf{x}, u, \nabla u) \qquad \text{in } Q_T, \tag{3.1}$$

$$u = 0 \quad \text{on } S, \quad \text{and} \quad u(0, \mathbf{x}) = \phi(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \Omega, \tag{3.2}$$

where $\mathbf{x}' = (x_2, ..., x_n)$, $u = u(t, \mathbf{x})$. Assume that

$$0 \leqslant a_{ii} \quad \text{for} \quad (t, \mathbf{x}) \in Q_T \quad \text{and} \quad \forall u, \nabla u. \tag{3.3}$$

LEMMA 3.1. *Suppose that Ω satisfies the condition* (a) *and that conditions* (1.7), (1.8), (3.3) *are fulfilled. Moreover suppose that $|\phi_{x_1}| \leqslant m_1$. Then, for any classical solution of the problem* (3.1), (3.2) *we have the estimate*

$$\sup_{\bar{Q}_T} |u_{x_1}| \leqslant C_1,$$

where the constant C_1 depends only on M, ψ, m_1, d_1 ($M = \sup_{Q_T} |u|$, d_1 is the size of the domain Ω in x_1 direction).

Proof. Consider the equation (3.1) at a point $(t, \zeta, \mathbf{x}')$, $\zeta \neq x_1$

$$a_{11}(t, \zeta, \mathbf{x}', u, \nabla u)\, u_{\zeta\zeta} + \sum_{i=2}^{n} a_{ii}(t, \mathbf{x}', \nabla u)\, u_{x_i x_i} - u_t$$
$$= f(t, \zeta, \mathbf{x}', u, \nabla u) \quad \text{in } Q_T,$$
$$u = u(t, \zeta, \mathbf{x}'), \quad \nabla u(t, \zeta, \mathbf{x}') = (u_\zeta, u_{x_2}, ..., u_{x_n}).$$

Subtracting this equation from equation (3.1) we obtain for the function $v(t, \zeta, \mathbf{x}) \equiv u(t, \mathbf{x}) - u(t, \zeta, \mathbf{x}')$

$$a_{11}^{(x)} v_{x_1 x_1} + a_{11}^{(\zeta)} v_{\zeta\zeta} - v_t + \Phi = f^{(x)} - f^{(\zeta)},$$

where $a_{11}^{(z)} \equiv a_{11}(t, z, \mathbf{x}', u(t, z, \mathbf{x}'), \nabla u(t, z, \mathbf{x}'))$, $\nabla u(t, z, \mathbf{x}') = (u_z, u_{x_2}, ..., u_{x_n})$, $f^{(z)} \equiv f(t, z, \mathbf{x}', u(t, z, \mathbf{x}'), \nabla u(t, z, \mathbf{x}'))$ and

$$\Phi \equiv \sum_{i=2}^{n} [a_{ii}(t, \mathbf{x}', \nabla u(t, \mathbf{x}))\, u_{x_i x_i}(t, \mathbf{x}) - a_{ii}(t, \mathbf{x}', \nabla u(t, \zeta, \mathbf{x}')\, u_{x_i x_i}(t, \zeta, \mathbf{x}')]. \tag{3.4}$$

From (1.8) we have

$$L(v) \equiv a_{11}^{(x)}[v_{x_1 x_1} + \psi(|v_{x_1}|)] + a_{11}^{(\zeta)}[v_{\zeta\zeta} + \psi(|v_\zeta|)] - v_t \geqslant -\Phi. \tag{3.5}$$

Let $h(\eta)$ be a solution of the problem

$$h''(\eta) + \psi(|h'(\eta)|) = 0, \quad h(0) = 0, \quad h(\tau_0) = 2M, \tag{3.6}$$

where τ_0 is the same as in (1.6) (in particular $h' \geqslant m_1$). Consider (3.5) and (3.6) in the domain $P_T = P \times (0, T)$ where

$$P = \{(\zeta, \mathbf{x}) \colon \zeta \in (F_1(\mathbf{x}'), G_1(\mathbf{x}')), x_1 \in (F_1(\mathbf{x}'), G_1(\mathbf{x}')),$$
$$\mathbf{x}' \in \Omega_0, 0 < x_1 - \zeta < \tau_0\}$$

here Ω_0 is the projection of the domain Ω on the plane $x_1=0$. Functions F_1, G_1 are defined in Section 1. For $w\equiv v-h(x_1-\zeta)$ we have

$$L(v)-L(h)=a_{11}^{(x)}[w_{x_1x_1}+\beta_1 w_{x_1}]+a_{11}^{(\zeta)}[w_{\zeta\zeta}+\beta_2 w_\zeta]-w_t\geqslant -\Phi.$$

In view of the fact that $u(t,\mathbf{x})$ is a classical solution and that the function $\psi(\rho)$ is a C^1 function we obtain that $|\beta_i|<+\infty$ in $\bar{P}_T\backslash(P\cup\Gamma)$ for $i=1,2$ where $\Gamma=\partial P\times[0,T]$. Using the substitution $w=\tilde{w}e^t$, we obtain

$$\tilde{L}(\tilde{w})\equiv a_{11}^{(x)}[\tilde{w}_{x_1x_1}+\beta_1\tilde{w}_{x_1}]+a_{11}^{(\zeta)}[\tilde{w}_{\zeta\zeta}+\beta_2\tilde{w}_\zeta]-\tilde{w}-\tilde{w}_t\geqslant -\Phi e^{-t}. \quad (3.7)$$

Suppose that $\tilde{w}$ achieves its maximum positive value at the point $N\in\bar{P}_T\backslash(P\cup\Gamma)$ then at this point we have that $\tilde{w}_{x_1}=\tilde{w}_{x_2}=\cdots=\tilde{w}_{x_n}=\tilde{w}_\zeta=0$ or $u_{x_i}(t,\mathbf{x})=u_{x_i}(t,\zeta,\mathbf{x}')$, for $i=1,\ldots,n$. Hence at this point $\nabla u(t,\mathbf{x})=\nabla u(t,\zeta,\mathbf{x}')$ and by virtue of (3.4) $e^{-t}\Phi|_N\geqslant 0$. From (3.7) we see that this contradicts the fact that $\tilde{w}$ attains its positive maximal value at $N\in\bar{P}_T\backslash(P\cup\Gamma)$.

Now let us show that $\tilde{w}=(v-h(x_1-\zeta))\,e^{-t}\leqslant 0$ on $P\cup\Gamma$. For $x_1=\zeta$ we have that $v-h=0$, for $t=0$ we have that $v\leqslant h$ because $|\phi_{x_1}|\leqslant m_1\leqslant h'$. For $\zeta=F_1(\mathbf{x}')$, $x_1\in[F_1(\mathbf{x}'),F_1(\mathbf{x}')+\tau_0]$, $\mathbf{x}'\in\Omega_0$, $0<t<T$ we have that

$$v-h=u(t,\mathbf{x})-h(x_1-F_1(\mathbf{x}')).$$

We show now that $u(t,\mathbf{x})\leqslant h(x_1-F_1(\mathbf{x}'))$. For this it is sufficient to show first that $h(x_1-F_1(\mathbf{x}'))=h_{11}(x_1-F_1(\mathbf{x}')+A_1)$ and then apply Lemma 1.2. The previous equality follows directly from the fact that

$$h''(\eta)+\psi(|h'(\eta)|)=0,\qquad h(0)=0,\qquad h(\tau_0)=2M,\qquad \eta=x_1-F_1(\mathbf{x}'),$$
$$h''_{11}(\xi_{11})+\psi(|h'_{11}(\xi_{11})|)=0,\qquad h_{11}(A_1)=0,\qquad h_{11}(A_1+\tau_0)=2M,$$

where $\xi_{11}=x_1-F_1(\mathbf{x}')+A_1$.

For $x_1=G_1(\mathbf{x}')$, $\zeta\in[G_1(\mathbf{x}')-\tau_0,G_1(\mathbf{x}')]$, $\mathbf{x}'\in\Omega_0$, $0<t<T$ we have that

$$v-h=-u(t,\zeta,\mathbf{x}')-h(G_1(\mathbf{x}')-\zeta).$$

From

$$h''(\eta)+\psi(|h'(\eta)|)=0,\qquad h(0)=0,\qquad h(\tau_0)=2M,\qquad \eta=G_1(\mathbf{x}')-\zeta,$$

and

$$h''_{21}(\xi_{21})+\psi(|h'_{21}(\xi_{21})|)=0,\qquad h_{21}(B_1-\tau_0)=2M,$$
$$h_{21}(B_1)=0,\qquad \xi_{21}=\zeta-G_1(\mathbf{x}')+B_1,$$

follows that $h(G_1(\mathbf{x}')-\zeta)=h_{21}(\zeta-G_1(\mathbf{x}')+B_1)\geqslant -u(t,\zeta,\mathbf{x}')$.

Thus we have proved that $\tilde{w} \leqslant 0$ on $P \cup \Gamma$. Hence (due to the fact that $\tilde{w}$ cannot achieve a positive maximum in $\bar{P}_T \backslash (P \cup \Gamma)$) we obtain that

$$u(t, \mathbf{x}) - u(t, \zeta, \mathbf{x}') \leqslant h(x_1 - \zeta) \qquad \text{in } \bar{P}_T.$$

By analogy, taking the function $\tilde{v} \equiv u(t, \zeta, \mathbf{x}') - u(t, \mathbf{x})$ we obtain $v \geqslant -h(x_1 - \zeta)$ in $\bar{P}_T$.

In view of the symmetry of the variables x_1 and ζ in the same way we examine the case $\zeta > x_1$ in $P_{1T} = P_1 \times (0, T)$ where

$$P_1 = \{(\zeta, \mathbf{x}) : \zeta \in (F_1(\mathbf{x}'), G_1(\mathbf{x}')), x_1 \in (F_1(\mathbf{x}'), G_1(\mathbf{x}')), \mathbf{x}' \in \Omega_0, 0 < \zeta - x_1 < \tau_0\}.$$

As a result we have that in

$$\{(\zeta, \mathbf{x}) : \zeta \in [F_1(\mathbf{x}'), G_1(\mathbf{x}')], x_1 \in [F_1(\mathbf{x}'), G_1(\mathbf{x}')], \mathbf{x}' \in \bar{\Omega}_0, |x_1 - \zeta| \leqslant \tau_0\} \backslash \{x_1 = \zeta\}$$

the inequality

$$\frac{|u(t, \mathbf{x}) - u(t, \zeta, \mathbf{x}')|}{|x_1 - \zeta|} \leqslant \frac{h(|x_1 - \zeta|) - h(0)}{|x_1 - \zeta|}$$

holds, implying that $|u_{x_1}(t, \mathbf{x})| \leqslant h'(0)$ and the lemma is proved.

Now let us formulate the conditions guaranteeing a priori estimates of the gradient. Consider the following equation

$$a_{ii}(t, x_i, \nabla u)\, u_{x_i x_i} - u_t = f(t, \mathbf{x}, u, \nabla u) \qquad \text{in } Q_T. \tag{3.8}$$

LEMMA 3.2. *Suppose that* Ω *satisfies condition* (a) *and conditions* (1.7), (1.8), (1.9), (3.3) *are fulfilled. Moreover suppose that* $\sup_\Omega |\phi_{x_i}| \leqslant m_i$, $i = 1, ..., n$. *Then for any classical solution of the problem* (3.8), (3.2) *we have*

$$\sup_{Q_T} |u_{x_i}| \leqslant C_i, \qquad i = 1, ..., n,$$

where the constant C_i *depends only on* ψ, m_i *and* d_i (d_i *is the size of the domain* Ω *in the* x_i *direction*).

The proof of this lemma is similar to the proof of Lemma 3.1.

Remark 3.1. The fulfilment of the condition

$$|f(t, \mathbf{x}, u, \mathbf{p})| \leqslant a_{11}(t, x_1, \mathbf{p})\, \psi(|p_1|)$$

for $(t, \mathbf{x}) \in Q_T$, $|u| < M$ and any $\mathbf{p}$ implies the a priori estimate $|u_{x_1}| < C_1$. In order to obtain the a priori estimate $|u_{x_2}| < C_2$ it is sufficient to require that

$$|f(t, \mathbf{x}, u, \mathbf{p})| \leqslant a_{22}(t, x_2, \mathbf{p})\, \psi(|p_2|)$$

for $(t, \mathbf{x}) \in Q_T$, $|u| < M$, $|p_1| < C_1$ and any $p_2, ..., p_n$. To obtain the estimate $|u_{x_3}| < C_3$ we require the fulfilment of $|f(t, \mathbf{x}, u, \mathbf{p})| \leqslant a_{33}(t, x_3, \mathbf{p})\, \psi(|p_3|)$ for $(t, \mathbf{x}) \in Q_T$, $|u| < M$, $|p_1| < C_1$, $|p_2| < C_2$ and any $p_3, ..., p_n$ and so on.

Remark 3.2. The estimates in Lemmas 1.2, 1.3 give us in fact the boundary gradient estimates. In the more general case the boundary gradient estimates where obtained in [9, 10] (see also [4, 11]).

We are now in position to prove the existence theorem.

THEOREM 3.1. *Suppose that all conditions except* (3.3) *of Lemma* 3.2 *are fulfilled and assume that*

$$a_{ii} > 0 \qquad \textit{for} \quad (t, \mathbf{x}) \in \bar{Q}_T \quad \textit{and} \quad \forall \nabla u, \tag{3.9}$$

$$\phi \in C^{1+\beta}(\bar{\Omega}), \qquad \phi = 0 \qquad \textit{on } \partial\Omega.$$

Suppose that some condition which quarantees the a priori estimate of u is fulfilled (*see* [2] *or Lemma* 1.1). *Suppose in addition that for* $(t, \mathbf{x}) \in \bar{Q}_T$, $|u| \leqslant M$, $|p_i| \leqslant C_i$, $i = 1, ..., n$, *functions* $a_{ii}(t, x_i, \mathbf{p})$ *are continuously differentiable with respect to* $x_i, \mathbf{p}$ *and Hölder continuous in* t *with exponent* β, $\beta \in (0, 1)$ *and function* $f(t, \mathbf{x}, u, \mathbf{p})$ *is Hölder continuous in* $t, \mathbf{x}, u, \mathbf{p}$ *with exponent* β.

Then there exist a solution of the problem (3.8), (3.2) *belonging to* $C^{2+\beta,\, 1+\beta/2}_{\mathbf{x},\, t}(Q_T) \cap C^{1+\beta,\, (1+\beta)/2}_{\mathbf{x},\, t}(\bar{Q}_T)$.

If the derivative $f_u(t, \mathbf{x}, u, \mathbf{p})$ *is bounded then the solution is unique.*

Proof. The boundedness of $|\nabla u|$ implies the Hölder continuity of the solution with respect to t with Hölder exponent 1/2 and Hölder constant depending only on $\sup |\nabla u|$ and on the maximum of the functions $a_{ii}(t, x_i, \mathbf{p})$, $|f(t, \mathbf{x}, u, \mathbf{p})|$ on the set $D \equiv \bar{Q}_T \times [-M, M] \times [-C_1, C_1] \times \cdots \times [-C_n, C_n]$ (see [1]). The Hölder estimate for ∇u follows from [3, 4, 12] with Hölder constant and Hölder exponent depending only on $M, C_1, ..., C_n, n$, on the minimum of a_{ii} and on the maximum of a_{ii}, $|a_{iip}|$, $|a_{iix_i}|$, $|f|$ on D see [4]).

These a priori estimates and Leray–Schauder theorem imply the existence of the solution (see for example [2] or [4]).

The uniqueness follows from Theorem 2.1.

THEOREM 3.2. *Suppose that all conditions except* (3.9) *of Theorem* 3.1 *are fulfilled. Assume that*

$$a_{ii} > 0 \quad \text{for} \quad (t, \mathbf{x}) \in Q_T \quad \text{and} \quad \forall \nabla u$$

and

$$a_{ii} \geqslant 0 \quad \text{for} \quad (t, \mathbf{x}) \in S \cup \Omega \quad \text{and} \quad \forall \nabla u.$$

Then there exist a solution of the problem (3.8), (3.2) *belonging to* $C^{2+\beta, 1+\beta/2}_{\mathbf{x}, t}(Q_T) \cap C^{\alpha, 1/2}_{\mathbf{x}, t}(\bar{Q}_T)$, $\forall \alpha \in (0, 1)$.

If the derivative $f_u(t, \mathbf{x}, u, \mathbf{p})$ *is bounded then the solution is unique.*

Proof. The existence can be easily proved by adding to the left part of the equation (3.8) the term $\varepsilon \Delta u$ and then passing to the limit using the estimates of Lemma (3.2) and the above mentioned property of Hölder continuity of the solution with respect to t.

The uniqueness follows from Theorem 2.1.

Remark 3.3. The above mentioned result on Hölder continuity with respect to t was proved first for quasilinear equations in many dimensions [5], with not optimal Hölder exponent (less than 1/2), then for linear equations this result was proved with Hölder exponent 1/2 [13] and finally in [1] this result was proved for quasilinear equations with optimal Hölder exponent 1/2. In [1] the case of one space variable is considered but the proof can be applied to the higher dimensional case almost without change.

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