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Cosmological reconstruction of realistic modified $F(R)$ gravities

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ABSTRACT

The cosmological reconstruction scheme for modified $F(R)$ gravity is developed in terms of e-folding (or, redshift). It is demonstrated how any FRW cosmology may emerge from specific $F(R)$ theory. The specific examples of well-known cosmological evolution are reconstructed, including ΛCDM cosmology, deceleration with transition to phantom superacceleration era which may develop singularity or be transient. The application of this scheme to viable $F(R)$ gravities unifying inflation with dark energy era is proposed. The additional reconstruction of such models leads to non-leading gravitational correction mainly relevant at the early/late universe and helping to pass the cosmological bounds (if necessary). It is also shown how cosmological reconstruction scheme may be generalized in the presence of scalar field.

1. Introduction

Modified gravity approach suggests the gravitational alternative for unified description of inflation, dark energy and dark matter without the need to introduce by hands the inflaton and extra dark components. Moreover, the easy explanation of inflationary or dark energy phase in such scenario follows: the corresponding era is emerging due to dominance of the specific gravitational sector in the course of the universe expansion. In other words, the early-time and late-time acceleration is governed by the universe expansion within the specific modified gravity theory. Special interest in this gravitational paradigm for the description of the universe evolution is related with $F(R)$ gravity (for a general review, see [1]) due to its quite simple structure if compare with more general modified gravity which includes all curvature invariants as well as non-local terms. Nevertheless, even in frames of $F(R)$ gravity the background evolution (due to high non-linearity of the problem) is often non-explicit and/or non-analytic process. From another side, any realistic modified gravity should pass not only the local tests but also the observational cosmological bounds. To comply with cosmological bounds, the reconstruction program in any modified gravity has been developed [2].

The cosmological reconstruction of $F(R)$ gravity has been considered in Refs. [2–5]. It turns out that in most cases this reconstruction is done in the presence of the auxiliary scalar which may be excluded at the final step so that any FRW cosmology may be realized within specific reconstructed $F(R)$ gravity. However, the weak point of so developed reconstruction scheme is that the final function $F(R)$ represents usually some polynomial in the positive/negative powers of scalar curvature. On the same time, the viable models have strongly non-linear structure.

In the present Letter we develop the new scheme for cosmological reconstruction of $F(R)$ gravity in terms of e-folding (or, redshift z) so that there is no need to use more complicated formulation with auxiliary scalar [2,3,5]. Using such technique the number of examples are presented where $F(R)$ gravity is reconstructed so that it gives the well-known cosmological evolution: ΛCDM epoch, deceleration/acceleration epoch which is equivalent to presence of phantom and non-phantom matter, late-time acceleration with the crossing of phantom-divide line, transient phantom epoch and oscillating universe. It is shown that some generalization of such technique for viable $F(R)$ gravity is possible, so that local tests are usually satisfied. In this way, modified gravity unifying inflation, radiation/matter dominance and dark energy eras may be further reconstructed in the early or in the late universe so that the future evolution may be different. This opens the way to non-linear reconstruction of realistic $F(R)$ gravity. Moreover, it is demonstrated that cosmological reconstruction of viable modified gravity may help in the formulation of non-singular models in

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finite-time future. The reconstruction suggests the way to change some cosmological predictions of the theory in the past or in the future so that it becomes easier to pass the available observational data. Finally, we show that our method works also for modified gravity with scalar theory and any requested cosmology may be realized within such theory too.

2. Cosmological reconstruction of modified $F(R)$ gravity

Let us demonstrate that any FRW cosmology may be realized in specific $F(R)$ gravity. The starting action of the $F(R)$ gravity (for general review, see [1]) is given by

$$S = \int d^4x\sqrt{-g}\left(\frac{F(R)}{2\kappa^2} + \mathcal{L}_{\mathrm{matter}}\right). \tag{1}$$

The field equation corresponding to the first FRW equation is:

$$0 = -\frac{F(R)}{2} + 3\left(H^2 + \dot{H}\right)F'(R) - 18\left(4H^2\dot{H} + H\ddot{H}\right)F''(R) + \kappa^2\rho, \tag{2}$$

with $R = 6\dot{H} + 12H^2$. We now rewrite Eq. (2) by using a new variable (which is often called e-folding) instead of the cosmological time t, $N = \ln\frac{a}{a_0}$. The variable N is related with the redshift z by $e^{-N} = \frac{a_0}{a} = 1 + z$. Since $\frac{d}{dt} = H\frac{d}{dN}$ and therefore $\frac{d^2}{dt^2} = H^2\frac{d^2}{dN^2} + H\frac{dH}{dN}\frac{d}{dN}$, one can rewrite (2) by

$$0 = -\frac{F(R)}{2} + 3\left(H^2 + HH'\right)F'(R) - 18\left(4H^3H' + H^2(H')^2 + H^3H''\right)F''(R) + \kappa^2\rho. \tag{3}$$

Here $H' \equiv dH/dN$ and $H'' \equiv d^2H/dN^2$. If the matter energy density ρ is given by a sum of the fluid densities with constant EoS parameter w_i, we find

$$\rho = \sum_i \rho_{i0}a^{-3(1+w_i)} = \sum_i \rho_{i0}a_0^{-3(1+w_i)}e^{-3(1+w_i)N}. \tag{4}$$

Let the Hubble rate is given in terms of N via the function $g(N)$ as

$$H = g(N) = g\left(-\ln(1+z)\right). \tag{5}$$

Then scalar curvature takes the form: $R = 6g'(N)g(N) + 12g(N)^2$, which could be solved with respect to N as $N = N(R)$. Then by using (4) and (5), one can rewrite (3) as

$$\begin{aligned}0 = &-18\left(4g\left(N(R)\right)^3g'\left(N(R)\right) + g\left(N(R)\right)^2g'\left(N(R)\right)^2 + g\left(N(R)\right)^3g''\left(N(R)\right)\right)\frac{d^2F(R)}{dR^2} \\ &+ 3\left(g\left(N(R)\right)^2 + g'\left(N(R)\right)g\left(N(R)\right)\right)\frac{dF(R)}{dR} - \frac{F(R)}{2} \\ &+ \sum_i \rho_{i0}a_0^{-3(1+w_i)}e^{-3(1+w_i)N(R)},\end{aligned} \tag{6}$$

which constitutes a differential equation for $F(R)$, where the variable is scalar curvature R. Instead of g, if we use $G(N) \equiv g(N)^2 = H^2$, the expression (6) could be a little bit simplified:

$$\begin{aligned}0 = &-9G\left(N(R)\right)\left(4G'\left(N(R)\right) + G''\left(N(R)\right)\right)\frac{d^2F(R)}{dR^2} \\ &+ \left(3G\left(N(R)\right) + \frac{3}{2}G'\left(N(R)\right)\right)\frac{dF(R)}{dR} \\ &- \frac{F(R)}{2} + \sum_i \rho_{i0}a_0^{-3(1+w_i)}e^{-3(1+w_i)N(R)}.\end{aligned} \tag{7}$$

Note that the scalar curvature is given by $R = 3G'(N) + 12G(N)$. Hence, when we find $F(R)$ satisfying the differential equation (6) or (7), such $F(R)$ theory admits the solution (5). Hence, such $F(R)$ gravity realizes above cosmological solution.

As an example, we reconstruct the $F(R)$ gravity which reproduces the ΛCDM-era but without real matter. In the Einstein gravity, the FRW equation for the ΛCDM cosmology is given by

$$\frac{3}{\kappa^2}H^2 = \frac{3}{\kappa^2}H_0^2 + \rho_0a^{-3} = \frac{3}{\kappa^2}H_0^2 + \rho_0a_0^{-3}e^{-3N}. \tag{8}$$

Here H_0 and ρ_0 are constants. The first term in the r.h.s. corresponds to the cosmological constant and the second term to the cold dark matter (CDM). The (effective) cosmological constant Λ in the present universe is given by $\Lambda = 12H_0^2$. Then one gets

$$G(N) = H_0^2 + \frac{\kappa^2}{3}\rho_0a_0^{-3}e^{-3N}, \tag{9}$$

and $R = 3G'(N) + 12G(N) = 12H_0^2 + \kappa^2\rho_0a_0^{-3}e^{-3N}$, which can be solved with respect to N as follows,

$$N = -\frac{1}{3}\ln\left(\frac{(R - 12H_0^2)}{\kappa^2\rho_0a_0^{-3}}\right). \tag{10}$$

Eq. (7) takes the following form:

$$0 = 3\left(R - 9H_0^2\right)\left(R - 12H_0^2\right)\frac{d^2F(R)}{d^2R} - \left(\frac{1}{2}R - 9H_0^2\right)\frac{dF(R)}{dR} - \frac{1}{2}F(R). \tag{11}$$

By changing the variable from R to x by $x = \frac{R}{3H_0^2} - 3$, Eq. (11) reduces to the hypergeometric differential equation:

$$0 = x(1-x)\frac{d^2F}{dx^2} + \left(\gamma - (\alpha + \beta + 1)x\right)\frac{dF}{dx} - \alpha\beta F. \tag{12}$$

Here

$$\gamma = -\frac{1}{2}, \qquad \alpha + \beta = -\frac{1}{6}, \qquad \alpha\beta = -\frac{1}{6}. \tag{13}$$

Solution of (12) is given by Gauss' hypergeometric function $F(\alpha, \beta, \gamma; x)$:

$$F(x) = AF(\alpha, \beta, \gamma; x) + Bx^{1-\gamma}F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; x). \tag{14}$$

Here A and B are constants. Thus, we demonstrated that modified $F(R)$ gravity may describe the ΛCDM epoch without the need to introduce the effective cosmological constant.

As an another example, we reconstruct $F(R)$ gravity reproducing the system with non-phantom matter and phantom matter in the Einstein gravity, whose FRW equation is given by

$$\frac{3}{\kappa^2}H^2 = \rho_qa^{-c} + \rho_pa^c. \tag{15}$$

Here ρ_q, ρ_p, and c are positive constants. When a is small as in the early universe, the first term in the r.h.s. dominates and it behaves as the universe described by the Einstein gravity with a matter whose EoS parameter is $w = -1 + c/3 > -1$, that is, non-phantom like. On the other hand, when a is large as in the late universe, the second term dominates and behaves as a phantom-like matter with $w = -1 - c/3 < -1$. Then since $G(N) \equiv g(N)^2 = H^2$, we find

$$G = G_q e^{-cN} + G_p e^{cN},$$
$$G_q \equiv \frac{\kappa^2}{3}\rho_q a_0^{-c}, \qquad G_p \equiv \frac{\kappa^2}{3}\rho_p a_0^{c}. \tag{16}$$

Then since $R = 3G'(N) + 12G(N)$,

$$e^{cN} = \frac{R \pm \sqrt{R^2 - 4(144 - 9c^2)}}{2(12+3c)}, \tag{17}$$

when $c \neq 4$ and

$$e^{cN} = \frac{R}{24G_p}, \tag{18}$$

when $c = 4$. In the following, just for simplicity, we consider $c = 4$ case. In the case, the non-phantom matter corresponding to the first term in the r.h.s. of (15) could be radiation with $w = 1/3$. Then Eq. (7) in this case is given by

$$0 = -6\left(\frac{24G_pG_q}{R} + \frac{R}{24}\right)R\frac{d^2F(R)}{dR^2} + \frac{9}{2}\left(-\frac{24G_pG_q}{R} + \frac{R}{24}\right)\frac{dF(R)}{dR} - \frac{F(R)}{2}. \tag{19}$$

By changing variable R to x by $R^2 = -576G_pG_qx$, we can rewrite Eq. (19) as

$$0 = (1-x)x\frac{d^2F}{dx^2} + \left(\frac{3}{4} + \frac{x}{4}\right)\frac{dF}{dx} - \frac{F}{2}, \tag{20}$$

whose solutions are again given by Gauss' hypergeometric function (14) with

$$\gamma = \frac{3}{4}, \qquad \alpha + \beta + 1 = -\frac{1}{4}, \qquad \alpha\beta = \frac{1}{2}. \tag{21}$$

Let us now study a model where the dominant component is phantom-like one. Such kind of system can be easily expressed in the standard General Relativity when a phantom fluid is considered, where the FRW equation reads $H^2(t) = \frac{\kappa^2}{3}\rho_{\rm ph}$. Here the subscript *ph* denotes the phantom nature of the fluid. As the EoS for the fluid is given by $p_{\rm ph} = w_{\rm ph}\rho_{\rm ph}$ with $w_{\rm ph} < -1$, by using the conservation equation $\dot{\rho}_{\rm ph} + 3H(1+w_{\rm ph})\rho_{\rm ph} = 0$, the solution for the FRW equation $H^2(t) = \frac{\kappa^2}{3}\rho_{\rm ph}$ is well known, and it yields $a(t) = a_0(t_s - t)^{-H_0}$, where a_0 is a constant, $H_0 = -\frac{1}{3(1+w_{\rm ph})}$ and t_s is the so-called Rip time. Then, the solution describes the Universe that ends at the Big Rip singularity in the time t_s. The same behavior can be achieved in $F(R)$ theory with no need to introduce a phantom fluid. Eq. (7) can be solved and the expression for the $F(R)$ that reproduces the solution is reconstructed. The expression for the Hubble parameter as a function of the number of e-folds is given by $H^2(N) = H_0^2 e^{2N/H_0}$. Then, Eq. (7), with no matter contribution, takes the form:

$$R^2\frac{d^2F(R)}{dR^2} + AR\frac{dF(R)}{dR} + BF(R) = 0, \tag{22}$$

where $A = -H_0(1+H_0)$ and $B = \frac{(1+2H_0)}{2}$. This equation is the well-known Euler equation whose solution yields

$$F(R) = C_1R^{m_+} + C_2R^{m_-}, \quad \text{where } m_\pm = \frac{1 - A \pm \sqrt{(A-1)^2 - 4B}}{2}. \tag{23}$$

Thus, the phantom dark energy cosmology $a(t) = a_0(t_s - t)^{-H_0}$ can be also obtained in the frame of $F(R)$ theory and no phantom fluid is needed.

We can consider now the model where the transition to the phantom epoch occurs. It has been pointed out that $F(R)$ could behave as an effective cosmological constant, such that its current observed value is well reproduced. One can reconstruct the model where late-time acceleration is reproduced by an effective cosmological constant and then the phantom barrier is crossed (see Ref. [5] for such reconstruction in the presence of auxiliary scalar). Such transition, which may take place at current time, could be achieved in $F(R)$ gravity. The solution considered can be expressed as:

$$H^2 = H_1\left(\frac{a}{a_0}\right)^m + H_0 = H_1e^{mN} + H_0, \tag{24}$$

where H_1, H_0 and α are positive constants. This solution can be constructed in GR when a cosmological constant and a phantom fluid are included. In the present case, the solution (24) can be achieved just by an $F(R)$ function, such that the transition from non-phantom to phantom epoch is reproduced. Scalar curvature can be written in terms of the number of e-folds again. Then, Eq. (7) takes the form:

$$x(1-x)F''(x) + \left[x\left(-\frac{6+m}{6m}\right) - \frac{1}{3m}\right]F'(x) - \frac{m+4}{m}F(x) = 0, \tag{25}$$

where $x = \frac{1}{3H_0(m+4)}(12H_0 - R)$. Eq. (25) reduces to the hypergeometric differential equation (14), so the solution is given, as in some of the examples studied above, by the Gauss' hypergeometric function (15), whose parameters for this case are given by

$$\gamma = -\frac{1}{3m}, \qquad \alpha + \beta = -\frac{3m+2}{2m}, \qquad \alpha\beta = \frac{m+4}{2m}, \tag{26}$$

and the obtained $F(R)$ gravity produces the FRW cosmology with the late-time crossing of the phantom barrier in the universe evolution.

Another example with transient phantom behavior in $F(R)$ gravity can be achieved by following the same reconstruction described above. In this case, we consider the following Hubble parameter:

$$H^2(N) = H_0\ln\left(\frac{a}{a_0}\right) + H_1 = H_0N + H_1, \tag{27}$$

where H_0 and H_1 are positive constants. For this model, we have a contribution of an effective cosmological constant, and a term that will produce a superaccelerating phase although no future singularity will take place (compare with earlier model [6] with transient phantom era). The solution for the model (27) can be expressed as a function of time

$$H(t) = \frac{a_0H_0}{2}(t - t_0). \tag{28}$$

Then, the Universe is superaccelerating, but as it can be seen from (28), in spite of its phantom nature, no future singularity occurs. The differential reconstruction equation can be obtained as

$$a_2x\frac{d^2F(x)}{dx} + (a_1x + b_1)\frac{dF(x)}{dx} + b_0F(x) = 0, \tag{29}$$

where we have performed a variable change $x = H_0N + H_1$, and the constant parameters are $a_2 = H_0^2$, $a_1 = -H_0$, $b_1 = -\frac{H_0^2}{2}$ and $b_0 = 2H_0$. Eq. (29) is a kind of the degenerate hypergeometric equation, whose solutions are given by the Kummers series $K(a, b; x)$:

$$F(R) = K\left(-2, -\frac{1}{2H_0}; \frac{R - 3H_0}{12}\right). \tag{30}$$

Hence, such $F(R)$ gravity has cosmological solution with the transient phantom behavior which does not evolve to future singularity.

Let us now consider the case where a future contracting Universe is reconstructed in this kind of models. We study a model where the universe is currently accelerating, then the future contraction of the Universe occurs. The following solution for the Hubble parameter is considered,

$$H(t) = 2H_1(t_0 - t), \tag{31}$$

where H_1 and t_0 are positive constants. For this example, the Hubble parameter (31) turns negative for $t > t_0$, when the Universe starts to contract itself, while for $t \ll t_0$, the cosmology is typically ΛCDM one. Using notations $\tilde{H}_0 = 4H_1 t_0^2$ and $\tilde{H}_1 = 4H_1$ and repeating the above calculation, one gets:

$$F(R) = K\left(-8\tilde{H}_1, -\frac{\tilde{H}_1}{8}; \frac{12\tilde{H}_0 - 3\tilde{H}_1 - R}{12\tilde{H}_1}\right). \tag{32}$$

Hence, the oscillating cosmology (31) that describes the asymptotically contracting Universe with a current accelerated epoch can be found in specific $F(R)$ gravity.

Thus, we explicitly demonstrated that $F(R)$ gravity reconstruction is possible for any cosmology under consideration without the need to introduce the auxiliary scalar. However, the obtained modified gravity has typically polynomial structure with terms which contain positive and negative powers of curvature as in the first such model unifying the early-time inflation and late-time acceleration [7]. As a rule such models do not pass all the local gravitational tests. Some generalization of above cosmological reconstruction is necessary.

3. Cosmological reconstruction of viable $F(R)$ gravity

In this section, we show how the cosmological reconstruction may be applied to viable modified gravity which passes the local gravitational tests. In this way, the non-linear structure of modified $F(R)$ gravity may be accounted for, unlike the previous section where only polynomial $F(R)$ structures may be reconstructed. Let us write $F(R)$ (1) in the following form: $F(R) = F_0(R) + F_1(R)$. Here we choose $F_0(R)$ as a known function like that of GR or one of viable $F(R)$ models introduced in [8], or viable $F(R)$ theories unifying inflation with dark energy [9,10], for example

$$F_0(R) = \frac{1}{2\kappa^2}\left(R - \frac{(R-R_0)^{2n+1} + R_0^{2n+1}}{f_0 + f_1\{(R-R_0)^{2n+1} + R_0^{2n+1}\}}\right). \tag{33}$$

Using the procedure similar to the one of second section, one gets the reconstruction equation corresponding to (7)

$$\begin{aligned} 0 = {} & -9G\big(N(R)\big)\big(4G'\big(N(R)\big) + G''\big(N(R)\big)\big)\frac{d^2F_0(R)}{dR^2} \\ & + \left(3G\big(N(R)\big) + \frac{3}{2}G'\big(N(R)\big)\right)\frac{dF_0(R)}{dR} - \frac{F_0(R)}{2} \\ & - 9G\big(N(R)\big)\big(4G'\big(N(R)\big) + G''\big(N(R)\big)\big)\frac{d^2F_1(R)}{dR^2} \\ & + \left(3G\big(N(R)\big) + \frac{3}{2}G'\big(N(R)\big)\right)\frac{dF_1(R)}{dR} \\ & - \frac{F_1(R)}{2} + \sum_i \rho_{i0} a_0^{-3(1+w_i)} \mathrm{e}^{-3(1+w_i)N(R)}. \end{aligned} \tag{34}$$

The above equation can be regarded as a differential equation for $F_1(R)$. For a given $G(N)$ or $g(N)$ (5), if one can solve (7) as $F(R) = \hat{F}(R)$, we also find the solution of (34) as $F_1(R) = \hat{F}(R) - F_0(R)$. For example, for $G(N)$ (9), by using (14), we find

$$\begin{aligned} F_1(R) = {} & AF(\alpha, \beta, \gamma; x) \\ & + Bx^{1-\gamma}F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; x) \\ & - F_0(R). \end{aligned} \tag{35}$$

Here α, β, γ, and x are given by $x = \frac{R}{3H_0^2} - 3$ and (13). Using $F_0(R)$ (33) one has

$$\begin{aligned} F_1(R) = {} & AF(\alpha, \beta, \gamma; x) \\ & + Bx^{1-\gamma}F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; x) \\ & - \frac{1}{2\kappa^2}\left(R - \frac{(R-R_0)^{2n+1} + R_0^{2n+1}}{f_0 + f_1\{(R-R_0)^{2n+1} + R_0^{2n+1}\}}\right), \end{aligned} \tag{36}$$

which describes the asymptotically de Sitter universe. Instead of $x = \frac{R}{3H_0^2} - 3$ and (13), if we choose α, β, γ, and x as $R^2 = -576G_pG_q x$ and in (21), $F_1(R)$ (36) shows the asymptotically phantom universe behavior, where H diverges in future.

One may start from $F_0(R)$ given by hypergeometric function (14) with $x = \frac{1}{3H_0(m+4)}(12H_0 - R)$ and (26). In such a model, there occurs Big Rip singularity. Let $\tilde{F}(R)$ be $F(R)$ again given by hypergeometric function (14) with $x = \frac{R}{3H_0^2} - 3$ and (13):

$$\begin{aligned} \tilde{F}(R) &= \tilde{A}F(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}; \tilde{x}) \\ &\quad + \tilde{B}\tilde{x}^{1-\tilde{\gamma}}F(\tilde{\alpha} - \tilde{\gamma} + 1, \tilde{\beta} - \tilde{\gamma} + 1, 2 - \tilde{\gamma}; \tilde{x}), \\ \tilde{x} &= \frac{R}{3H_0^2} - 3, \qquad \tilde{\gamma} = -\frac{1}{2}, \\ \tilde{\alpha} + \tilde{\beta} &= -\frac{1}{6}, \qquad \tilde{\alpha}\tilde{\beta} = -\frac{1}{6}. \end{aligned} \tag{37}$$

If we choose $F(R) = \tilde{F}(R)$, the ΛCDM model emerges. Then choosing $F_1(R) = \tilde{F}(R) - F_0(R)$, the Big Rip singularity, which occurs in $F_0(R)$ model, does not appear and the universe becomes asymptotically de Sitter space. Hence, the reconstruction method suggests the way to create the non-singular modified gravity models [5,6,11]. Of course, it should be checked that reconstruction term is not large (or it affects only the very early-time/late-time universe) so that the theory passes the local tests as it was before the adding of correction term.

Gauss' hypergeometric function $F(\alpha, \beta, \gamma; x)$ is defined by

$$F(\alpha, \beta, \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)n!}z^n. \tag{38}$$

Since

$$\begin{aligned} \alpha_0, \beta_0 &= \frac{-3m - 2 \pm \sqrt{m^2 - 20m + 4}}{4m} < 0, \\ \tilde{\alpha}, \tilde{\beta} &= \frac{-1 \pm 5}{12}, \end{aligned} \tag{39}$$

when R is large, $F_1(R)$ behaves as $F_1(R) \sim R^{(3m+2+\sqrt{m^2-20m+4})/4m}$. In spite of the above expression, since the total $F(R) = F_0(R) + F_1(R)$ is given by $\tilde{F}(R)$ (37), the Big Rip type singularity does not occur. The asymptotic behavior of $F_1(R)$ cancels the large R behavior in $F_0(R)$ suggesting the way to present the non-singular cosmological evolution.

We now consider the case that H and therefore G oscillate as

$$G(N) = G_0 + G_1 \sin\left(\frac{N}{N_0}\right), \tag{40}$$

with positive constants G_0, G_1, and N_0. Let the amplitude of the oscillation is small but the frequency is large:

$$G_0 \gg \frac{G_1}{N_0}, \quad N_0 \gg 1. \tag{41}$$

When $G_1 = 0$, we obtain de Sitter space, where the scalar curvature is a constant $R = 12G_0$. Writing $G(N)$ as

$$G = \frac{R}{6} - G_0, \tag{42}$$

by using (7), one arrives at general relativity:

$$F(R) = c_0(R - 6G_0). \tag{43}$$

Instead of (42), using an arbitrary function $\tilde{F}$, if we write

$$G = G_0 + \tilde{F}(R) - \tilde{F}(12G_0), \tag{44}$$

we obtain a general $F(R)$ gravity, which admits de Sitter space solution. When $G_1 \neq 0$, under the assumption (41), one may identify $F(R)$ in (43) with $F_0(R)$. We now write $G(N)$ and the scalar curvature R as

$$G(N) = \frac{R}{6} - G_0 + \frac{G_1}{N_0} g(N), \quad R = 12G_0 + \frac{3G_1}{N_0} r(N), \tag{45}$$

with adequate functions $g(N)$ and $r(N)$. Then since $R = 6g'(N)g(N) + 12g(N)^2$ and from (41), we find

$$g(N) = -\left(N_0 \sin\frac{N}{N_0} + \frac{1}{2}\cos\frac{N}{N_0}\right),$$
$$r(N) = 4N_0 \sin\frac{N}{N_0} + \cos\frac{N}{N_0}. \tag{46}$$

By assuming

$$F(R) = c_0\left(R - 6G_0 + \frac{G_1^2}{N_0^3} f(R)\right), \tag{47}$$

and identifying

$$F_1(R) = \frac{c_0 G_1^2}{N_0^3} f(R), \tag{48}$$

from (34), one obtains

$$0 = G_0 \frac{df}{dr} - \sin\left(\frac{N}{N_0}\right) + o\left(\frac{G_1}{N_0}, N_0\right), \tag{49}$$

which can be solved as

$$f(R) = -\frac{1}{2G_0}\left(\cos^{-1} r \mp r\sqrt{1 - r^2}\right). \tag{50}$$

Then at least perturbatively, one can construct a model which exhibits the oscillation of H.

Before going further, let us find $F(R)$ equivalent to the Einstein gravity with a perfect fluid with a constant EoS parameter w, where H behaves as

$$\frac{3}{\kappa^2} H^2 = \rho_0 e^{-3(w+1)}. \tag{51}$$

Then

$$G(N) = \frac{\kappa^2 \rho_0}{3} e^{-3(w+1)},$$
$$R(N) = (1 - 3w)\kappa^2 \rho_0 e^{-3(w+1)}, \tag{52}$$

which could be solved as

$$N = -\frac{1}{3(w+1)} \ln \frac{R}{(1 - 3w)\kappa^2 \rho_0}. \tag{53}$$

Therefore Eq. (7) has the following form:

$$0 = \frac{3(1+w)}{1-3w} R^2 \frac{d^2F(R)}{dR^2} - \frac{1+3w}{2(1-3w)} R \frac{dF(R)}{dR} - \frac{F(R)}{2}, \tag{54}$$

whose solutions are given by a sum of powers of R

$$F(R) = F_+ R^{n_+} + F_- R^{n_-}. \tag{55}$$

Here $F_\pm$ are constants of integration and $n_\pm$ are given by

$$n_\pm = \frac{1}{2}\left\{\frac{7+9w}{6(1+w)} \pm \sqrt{\left(\frac{7+9w}{6(1+w)}\right)^2 + \frac{2(1-3w)}{3(1+w)}}\right\}. \tag{56}$$

If $w > -1/3$, the universe is decelerating but if $-1 < w < -1/3$, the universe is accelerating as in the quintessence scenario.

By using the solution (14), which mimics ΛCDM model, and the solution (55), one may consider the following model:

$$F(x) = \left\{A F(\alpha, \beta, \gamma; x) + B x^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; x)\right\} \times \frac{e^{\lambda(\frac{R}{R_1} - \frac{R_1}{R})}}{e^{\lambda(\frac{R}{R_1} - \frac{R_1}{R})} + e^{-\lambda(\frac{R}{R_1} - \frac{R_1}{R})}} + F_+ R^{n_+} + F_- R^{n_-}. \tag{57}$$

Here R_1 is a constant which is sufficiently small compared with the curvature R_0 in the present universe. On the other hand, we choose a positive constant λ to be large enough. We also choose $F_\pm$ to be small enough so that only the first term dominates when $R \gg R_1$. Note that the factor $\frac{e^{\lambda(\frac{R}{R_1} - \frac{R_1}{R})}}{e^{\lambda(\frac{R}{R_1} - \frac{R_1}{R})} + e^{-\lambda(\frac{R}{R_1} - \frac{R_1}{R})}}$ behaves as step function when λ is large:

$$\lim_{\lambda \to +\infty} \frac{e^{\lambda(\frac{R}{R_1} - \frac{R_1}{R})}}{e^{\lambda(\frac{R}{R_1} - \frac{R_1}{R})} + e^{-\lambda(\frac{R}{R_1} - \frac{R_1}{R})}} = \theta(R - R_1) \equiv \begin{cases} 1 & \text{when } R > R_1, \\ 0 & \text{when } R < R_1. \end{cases} \tag{58}$$

Then in the early universe and in the present universe, only the first term dominates and the ΛCDM universe could be reproduced. In the future universe where $R \ll R_1$, the factor $\frac{e^{\lambda(\frac{R}{R_1} - 1)}}{e^{\lambda(\frac{R}{R_1} - 1)} + e^{-\lambda(\frac{R}{R_1} - 1)}}$ decreases very rapidly and the second terms in (57) dominate. Then if $w > -1/3$, the universe decelerates again but if $-1 < w < -1/3$, the universe will be accelerating as in the quintessence scenario.

Thus, we explicitly demonstrated that the viable $F(R)$ gravity may be reconstructed so that any requested cosmology may be realized after the reconstruction. Moreover, one can use the viable $F(R)$ gravity unifying the early-time inflation with late-time acceleration (and manifesting the radiation/matter dominance era between accelerations) and passing local tests in such a scheme. The (small) correction term $F_1(R)$ can be always constructed so that it slightly corrects (if necessary) the cosmological bounds being relevant only at the very early/late universe. This scenario opens the way to extremely realistic description of the universe evolution in $F(R)$ gravity consistent with local tests and cosmological bounds.

4. Reconstruction of modified gravity with extra scalar

We now consider the reconstruction of $F(R)$-gravity coupled with a scalar field, whose action is given by

$$S = \int d^4x \sqrt{-g}\left(\frac{F(R)}{2\kappa^2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) + \mathcal{L}_{\text{matter}}\right). \tag{59}$$

(For some cosmological solutions in such theory, see [13].) Let us redefine the scalar field as $\phi = \phi(\varphi)$,

$$S = \int d^4x\sqrt{-g}\left(\frac{F(R)}{2\kappa^2} - \frac{\omega(\varphi)}{2}\partial_\mu\varphi\partial^\mu\varphi - \tilde{V}(\varphi) + \mathcal{L}_{\text{matter}}\right). \tag{60}$$

Here

$$\omega(\varphi) \equiv \left(\frac{d\phi(\varphi)}{d\varphi}\right)^2, \qquad \tilde{V}(\varphi) \equiv V\big(\phi(\varphi)\big). \tag{61}$$

If ϕ only depends on the time-coordinate t or e-folding N, we may choose $\varphi = t$ or $\varphi = N$.

Then the equations corresponding to the first and second FRW equations have the following form

$$\begin{aligned} 0 = {} & -\frac{F(R)}{2} + 3\big(H^2 + HH'\big)F'(R) \\ & - 18\big(4H^3H' + H^2(H')^2 + H^3H''\big)F''(R) \\ & + \kappa^2\left(\frac{H^2\omega(\varphi)(\varphi')^2}{2} + \tilde{V}(\varphi)\right) \\ & + \sum_i \kappa^2\rho_{i0}a_0^{-3(1+w_i)}\mathrm{e}^{-3(1+w_i)N}, \end{aligned} \tag{62}$$

$$\begin{aligned} 0 = {} & \frac{F(R)}{2} - \big(3H^2 + HH'\big)F'(R) \\ & + 6\big(16H^3H' - 4H^2(H')^2 - H(H')^3 \\ & - 4H^2H'H'' - H^3H'''\big)F''(R) \\ & - 36\big(4H^3H' + H^2(H')^2 + H^3H''\big) \\ & \times \big((H')^2 + HH'' + 4HH'\big)F'''(R) \\ & + \kappa^2\left(\frac{H^2\omega(\varphi)(\varphi')^2}{2} - \tilde{V}(\varphi)\right) \\ & + \sum_i \kappa^2 w_i\rho_{i0}a_0^{-3(1+w_i)}\mathrm{e}^{-3(1+w_i)N}, \end{aligned} \tag{63}$$

which can be rewritten as

$$\begin{aligned} \kappa^2\omega(\varphi)(\varphi')^2 = {} & \Big\{-2HH'F'(R) + 6\big(-4H^3H' + 7H^2(H')^2 \\ & + H(H')^3 + 4H^2H'H'' \\ & + H^3H''' + 3H^3H''\big)F''(R) \\ & - 36\big(4H^3H' + H^2(H')^2 + H^3H''\big) \\ & \times \big((H')^2 + HH'' + 4HH'\big)F'''(R) \\ & + \sum_i \kappa^2(1 + w_i)\rho_{i0}a_0^{-3(1+w_i)}\mathrm{e}^{-3(1+w_i)N}\Big\}\frac{1}{H^2}, \end{aligned} \tag{64}$$

$$\begin{aligned} 2\kappa^2\tilde{V}(\varphi) = {} & F(R) + \big(-6H^2 - 4HH'\big)F'(R) \\ & + 6\big(28H^3H' - H^2(H')^2 - H(H')^3 \\ & - 4H^2H'H'' + H^3H''' - 3H^3H''\big)F''(R) \\ & + 36\big(4H^3H' + H^2(H')^2 + H^3H''\big) \\ & \times \big((H')^2 + HH'' + 4HH'\big)F'''(R) \\ & + \sum_i \kappa^2(1 - w_i)\rho_{i0}a_0^{-3(1+w_i)}\mathrm{e}^{-3(1+w_i)N}. \end{aligned} \tag{65}$$

Then if we consider the model given by an adequate function $K = K(\varphi)$,

$$\begin{aligned} \kappa^2\omega(\varphi) = {} & \Big\{-2K(\varphi)K'(\varphi)F'\big(6K(\varphi)K'(\varphi) + 12K(\varphi)^2\big) \\ & + 6\big(-4K(\varphi)^3K'(\varphi) + 7K(\varphi)^2K'(\varphi)^2 \\ & + K(\varphi)K'(\varphi)^3 + 4K(\varphi)^2K'(\varphi)K''(\varphi) + K(\varphi)^3K'''(\varphi) \\ & + 3K(\varphi)^3K''(\varphi)\big)F''\big(6K(\varphi)K'(\varphi) + 12K(\varphi)^2\big) \\ & - 36\big(4K(\varphi)^3K'(\varphi) + K(\varphi)^2K'(\varphi)^2 + K(\varphi)^3K''(\varphi)\big) \\ & \times \big(K'(\varphi)^2 + K(\varphi)K''(\varphi) + 4K(\varphi)K'(\varphi)\big) \\ & \times F'''\big(6K(\varphi)K'(\varphi) + 12K(\varphi)^2\big) \\ & + \sum_i \kappa^2(1 + w_i)\rho_{i0}a_0^{-3(1+w_i)}\mathrm{e}^{-3(1+w_i)\varphi}\Big\}\frac{1}{K(\varphi)^2}, \end{aligned} \tag{66}$$

$$\begin{aligned} 2\kappa^2\tilde{V}(\varphi) = {} & F\big(6K(\varphi)K'(\varphi) + 12K(\varphi)^2\big) \\ & + \big(-6K(\varphi)^2 - 4K(\varphi)K'(\varphi)\big) \\ & \times F'\big(6K(\varphi)K'(\varphi) + 12K(\varphi)^2\big) \\ & + 6\big(28K(\varphi)^3K'(\varphi) - K(\varphi)^2K'(\varphi)^2 - K(\varphi)K'(\varphi)^3 \\ & - 4K(\varphi)^2K'(\varphi)K''(\varphi) + K(\varphi)^3K'''(\varphi) \\ & - 3K(\varphi)^3K''(\varphi)\big)F''\big(6K(\varphi)K'(\varphi) + 12K(\varphi)^2\big) \\ & + 36\big(4K(\varphi)^3K'(\varphi) + K(\varphi)^2K'(\varphi)^2 + K(\varphi)^3K''(\varphi)\big) \\ & \times \big(K'(\varphi)^2 + K(\varphi)K''(\varphi) + 4K(\varphi)K'(\varphi)\big) \\ & \times F'''\big(6K(\varphi)K'(\varphi) + 12K(\varphi)^2\big) \\ & + \sum_i \kappa^2(1 - w_i)\rho_{i0}a_0^{-3(1+w_i)}\mathrm{e}^{-3(1+w_i)\varphi}, \end{aligned} \tag{67}$$

we find a solution which is given by

$$H(N) = K(N), \qquad \varphi = N. \tag{68}$$

Hence, it is demonstrated that reconstruction can be extended to the case when modified gravity couples with some scalar field. Note that extra scalar may be necessary in the situation when some of cosmological bounds (for instance, cosmological perturbations theory which is extremely complicated in $F(R)$ gravity, for a review, see [12]) cannot be passed within only modified gravity. So far, this is not the case and modified gravity which passes local tests and cosmological bounds is available [8–10].

5. Discussion

In summary, we developed general scheme for cosmological reconstruction of modified $F(R)$ gravity in terms of e-folding (or redshift) without use of auxiliary scalar in intermediate calculations. Using this method, it is possible to construct the specific modified gravity which contains any requested FRW cosmology. The number of $F(R)$ gravity examples is found where the following background evolution may be realized: ΛCDM epoch, deceleration with subsequent transition to effective phantom superacceleration leading to Big Rip singularity, deceleration with transition to transient phantom phase without future singularity, oscillating universe. It is important that all these cosmologies may be realized only by modified gravity without use of any dark components (cosmological constant, phantom, quintessence, etc.).

It is shown that our method may be applied to viable $F(R)$ gravities which pass local tests and unify the early-time inflation with late-time acceleration. In this case, the additional reconstruction may be made so that correction term is not large and it is relevant only in the very early/very late universe. Hence, the purpose of such additional reconstruction is only to improve the cosmological predictions if the original theory does not pass correctly the precise observational cosmological bounds. For instance, in this way it is possible to formulate the modified gravity without finite-time future singularity. It is also demonstrated that the reconstruction scheme may be generalized for the case of modified gravity with scalar field.

The present reconstruction formulation shows that even if specific realistic modified gravity does not pass correctly some cosmological bounds (for instance, does not lead to correct cosmological perturbations structure) it may be improved with eventually desirable result. Hence, the successful development of such method adds very strong argument in favour of unified gravitational alternative for inflation, dark energy and dark matter.

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