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# Whitney Numbers of the Second Kind for the Star Poset

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The integers $W_0, \ldots, W_t$ are called Whitney numbers of the second kind for a ranked poset if $W_k$ is the number of elements of rank $k$. The set of transpositions $T = \{(1, n), (2, n), \ldots, (n-1, n)\}$ generates $S_n$, the symmetric group. We define the star poset, a ranked poset the elements of which are those of $S_n$ and the partial order of which is obtained from the Cayley graph using $T$. We characterize minimal factorizations of elements of $S_n$ as products of generators in $T$ and provide recurrences, generating functions and explicit formulae for the Whitney numbers of the second kind for the star poset.

## 1. INTRODUCTION

Let $(P, \leqslant)$ be a finite poset. By $a \leqslant^* b$ we mean if $a \leqslant c \leqslant b$ then $a = c$ or $b = c$. A partial order is completely defined if all pairs $(a, b)$ for which $a \leqslant^* b$ are given. A poset $P$ is said to be ranked if each element $a \in P$ can be assigned a non-negative integer $\text{rank}(a)$ so that if $a \leqslant^* b$ then $\text{rank}(b) = \text{rank}(a) + 1$.

Let $n$ be a positive integer and let $T$ be the set of transpositions $\{(1, n), (2, n), \ldots, (n-1, n)\}$. We know that $T$ is a minimal generating set for $S_n$, the symmetric group [2]. Let $G(n)$ be a graph the vertex set $V(n)$ of which is $S_n$ and the edge set of which is given by $E(n) = \{e = (p_1, p_2) \mid p_1, p_2 \in S_n \text{ and } p_1\pi = p_2 \text{ for some } \pi \in T\}$. $G(n)$ is called the Cayley graph [1] for $S_n$ using the generating set $T$. $G(n)$ is an undirected, connected graph on $n!$ vertices.

We define a partial order $\leqslant$ on $V(n)$ as follows: for $p_1, p_2 \in V(n)$ we say that $p_1 \leqslant^* p_2$ if there is an edge $e \in E(n)$ where $e = (p_1, p_2)$ and $d(p_1, I) < d(p_2, I)$, where $I$ is the identity in $S_n$ and $d$ is the usual graph distance. This defines a ranked poset where the rank is given by the distance from $I$ in $G(n)$. We denote this poset by $(S_n, \leqslant)$ and call it the *star poset* (the graph with vertex set $\{1, 2, \ldots, n\}$ and edge set $T$, forms a tree called the star graph on $n$ vertices [1]).

The non-negative integers $W_0, W_1, \ldots, W_t$ are called the Whitney numbers of the second kind for a ranked poset if $W_k$ is the number of elements in the poset of rank $k$ [4]. We are primarily interested in determining the Whitney numbers of the second kind for the star poset. In particular, we will first characterize minimal factorizations of elements of $S_n$ in terms of the generators in $T$. We will then find recurrences, generating functions and closed-form formulae for these Whitney numbers.

## 2. THE STAR POSET

Let us denote a permutation $\pi \in S_n$ in the usual manner by listing its image $[\pi(1), \pi(2), \ldots, \pi(n)]$ so that $I = [1, 2, \ldots, n]$. We adopt the convention that composition of permutations is to be done from right to left. Thus, $[1, 3, 2][2, 1, 3] = [3, 1, 2]$. We will adopt the usual notation for cycles. For example, if $\pi$ is the cycle $[3, 4, 2, 1]$ we write $\pi = (1, 3, 2, 4)$. In the representation of a permutation as a product of disjoint cycles we will not include cycles of the form $(j)$ for $\pi(j) = j$.

Before beginning a discussion of properties of the star poset, let us consider an example. The star graph for $T = \{(1, 4), (2, 4), (3, 4)\}$ is shown in Figure 1. The Cayley graph for $S_4$ is shown in Figure 2. Note that there are $4! = 24$ elements, $I$ is the



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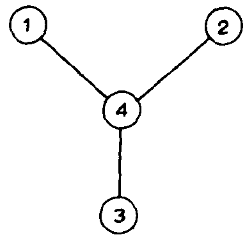


FIGURE 1. Star graph for $\{(1,4),(2,4),(3,4)\}$.

minimal element, and the corresponding Whitney numbers of the second kind are $W_0=1$, $W_1=3$, $W_2=6$, $W_3=9$ and $W_4=5$.

We begin with a few simple lemmas concerning products of generators from $T$.

LEMMA 2.1. *If $\sigma=(a_1,n)(a_2,n)\cdots(a_k,n)$ and if $a_1,a_2,\ldots,a_k$ are all distinct, then $\sigma$ is the cycle $(n,a_k,\ldots,a_1)$ and $\sigma(n)\neq n$.*

LEMMA 2.2. *If $\sigma=(a_1,n)(a_2,n)\cdots(a_k,n)$ and if $a_1,a_2,\ldots,a_{k-1}$ are all distinct and $a_k=a_1$ then $\sigma$ is the single cycle $(a_1,a_{k-1},a_{k-2},\ldots,a_2)$ and $\sigma(n)=n$.*

LEMMA 2.3. *If $\sigma(a_1,n)(a_2,n)\cdots(a_k,n)$, where $a_1,a_2,\ldots,a_{j-1},a_{j+1},\ldots,a_k$ are all distinct, $a_i=a_j\ (i\neq j)$, and $1<j<k$, then $\sigma$ is the product of two disjoint cycles, $\sigma=\lambda\beta$, where $\beta(n)=n$, $\lambda(n)\neq n$, and thus $\sigma(n)\neq n$.*

PROOF. Set $\sigma=\alpha\beta\varepsilon$, where $\alpha=(a_1,n)\cdots(a_{i-1},n)$, $\beta=(a_i,n)\cdots(a_j,n)$, and $\varepsilon=(a_{j+1},n)\cdots(a_k,n)$. By Lemma 1.2, $\beta(n)=n$. By the assumption of distinctness, $\beta$ fixes what $\alpha$ moves and $\alpha$ fixes what $\beta$ moves, so that $\alpha\beta=\beta\alpha$. Setting $\lambda=\alpha\varepsilon$ we have $\sigma=\beta\alpha\varepsilon=\beta\lambda$, where $\beta$ is a cycle (Lemma 2.2), $\lambda$ is a cycle, where $\lambda(n)\neq n$ (Lemma 2.1), and $\beta$ and $\lambda$ are disjoint. Thus, $\sigma=\lambda\beta$. □

LEMMA 2.4. *If $\sigma=(a_1,n)(a_2,n)\cdots(a_k,n)$, $1<r<j<k$, $a_1=a_j$ and $a_r=a_k$ and $\{a_2,\ldots,a_{k-1}\}$ are all distinct, then $\sigma(n)\neq n$, $\sigma$ is a single cycle, and $\sigma$ can be rewritten as a product of the $k-2$ transpositions $\{(a_2,n),\ldots,(a_{k-1},n)\}$.*

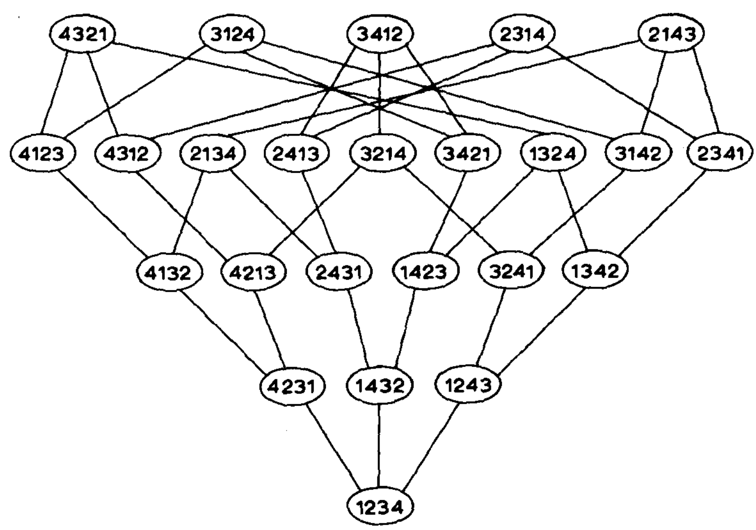


FIGURE 2. The star poset $(S_4,\leqslant)$.

PROOF. Set $\sigma=\rho\delta(a_k, n)$, where $\rho=(a_1, n)\cdots(a_j, n)$ and $\delta=(a_{j+1}, n)\cdots(a_{k-1}, n)$. Then $\rho\delta=\delta\rho$ and $\sigma=(a_{j+1}, n)\cdots(a_{k-1}, n)(a_1, n)\cdots(a_j, n)(a_k, n)$. Then $\sigma=\delta(a_1, n)\alpha\varepsilon$, where $\alpha=(a_2, n)\cdots(a_{r-1}, n)$ and $\varepsilon=(a_r, n)\cdots(a_j, n)(a_k, n)=(a_k, n)\cdots(a_j, n)(a_k, n)$.

Then $\alpha\varepsilon=\varepsilon\alpha$ and

$$\begin{aligned}\sigma&=\delta(a_1, n)(a_k, n)(a_{r+1}, n)\cdots(a_{j-1}, n)(a_j, n)(a_k, n)\alpha\\&=\delta(a_1, n)(a_k, n)(a_{r+1}, n)\cdots(a_{j-1}, n)(a_1, n)(a_k, n)\alpha\\&=\delta(a_1, n)(a_k, n)(b_1, n)\cdots(b_t, n)(a_1, n)(a_k, n)\alpha\end{aligned}$$

where $b_1=a_{r+1}, \ldots, b_t=a_{j-1}$, and $t=j-1-r$. Then

$$\begin{aligned}\sigma&=\delta(n, a_k, a_1)(n, b_t, \ldots, b_1)(n, a_k, a_1)\alpha\\&=\delta(n, a_1, b_t, \ldots, b_1, a_k)\alpha\\&=\delta(a_k, n)(b_1, n)(b_2, n)\cdots(b_t, n)(a_1, n)\alpha.\end{aligned}$$

Finally, by Lemma 2.1, $\sigma(n)\neq n$. □

DEFINITION 2.5. $\sigma$ is in canonical form if $\sigma$ is written in the following way: $\sigma=C_1C_2\cdots C_k$, where the $C_i$ $(i=1, \ldots, k)$ are the non-trivial disjoint cycles of $\sigma$, and:

(1) if $\sigma(n)\neq n$, then $C_1(n)\neq n$ and $C_i(n)=n$ $(i=2, \ldots, k)$;

(2) if $C_1(n)\neq n$ then $C_1=(a_1, n)(a_2, n)\cdots(a_j, n)$, where $a_1, a_2, \ldots, a_j$ are all distinct—thus $C_1=(n, a_j, a_{j-1}, \ldots, a_1)$;

(3) for $i>1$ when $\sigma(n)\neq n$ and for all $i$ when $\sigma(n)=n$, $C_i$ has the form $(a_1, n)(a_2, n)\cdots(a_j, n)(a_1, n)$, where $a_1, a_2, \ldots, a_j$ are distinct—thus $C_i=(a_j, a_{j-1}, \ldots, a_1)$.

DEFINITION 2.6. $M_\sigma=|\{i \mid 1\leq i\leq n \text{ and } \sigma(i)\neq i\}|$.

DEFINITION 2.7. $C_\sigma$ is the number of non-trivial disjoint cycles of $\sigma$.

DEFINITION 2.8. $\delta_\sigma=\begin{cases}0 & \text{if } \sigma(n)=n,\\ -2 & \text{if } \sigma(n)\neq n.\end{cases}$

LEMMA 2.9. *If $\sigma=t_1t_2\cdots t_j$ $(t_i\in T,\ 1\leq i\leq j)$ is in canonical form, then $j=M_\sigma+C_\sigma+\delta_\sigma$.*

PROOF. If $\sigma(n)\neq n$, $C_\sigma=k$, and $C_1$ is a product of $s$ transpositions, then $C_1$ moves $s+1$ elements. The remaining cycles $C_2, C_3, \ldots, C_k$ are disjoint and are of the form $(a_1, n)(a_2, n)\cdots(a_1, n)$, and there are $k-1=C_\sigma-1$ repetitions of transpositions. Thus, $C_2\cdots C_k$ moves $j-s-C_\sigma+1$ elements and, therefore, $M_\sigma=s+1+j-s-C_\sigma+1=j-\delta_\sigma-C_\sigma$. If $\sigma(n)=n$, then $C_1\cdots C_k$ moves $j-C_\sigma$ elements and $M_\sigma=j-C_\sigma-\delta_\sigma$. □

THEOREM 2.10. *Let $\sigma=t_1t_2\cdots t_m$ $(t_i\in T,\ 1\leq i\leq m)$ be any expression for $\sigma$ as a product of generators from $T$. Then $\sigma$ can be rewritten in the canonical form as $\sigma=t_{i_1}t_{i_2}\cdots t_{i_k}$ $(t_{i_j}\in T,\ 1\leq j\leq k)$, where $k\leq m$, $i_1, i_2, \ldots, i_k$ are distinct and $\{i_1, i_2, \ldots, i_k\}$ is a subset of $\{1, \ldots, m\}$.*

PROOF. Let $\sigma=(a_1, n)(a_2, n)\cdots(a_k, n)$. If $k=1$ then $\sigma$ is a single transposition and thus in canonical form. Now suppose $\sigma=\tau(b, n)$, where $b\in\{1, \ldots, n-1\}$. It suffices to show that if $\tau$ is in canonical form then $\sigma$ can be rewritten in the canonical form. The proof then follows by induction on $k$.

Therefore, suppose $\sigma = \tau(b, n)$, where $\tau = C_1C_2 \cdots C_t$ is the canonical form of $\tau$. Let $M(\tau) = \{x \mid \tau(x) \neq x\}$. If $b$ is not in $M(\tau)$ then $(b, n)$ commutes with $C_2, \ldots, C_t$ and $\sigma = C_1(b, n)C_2 \cdots C_t$. If $C_1(n) = n$ then $C_1(b, n) = (b, n)C_1$ and $\sigma = (b, n)C_1C_2 \cdots C_t$ is in canonical form. If $C_1(n) \neq n$ then $C_1' = C_1(n, b)$ is a cycle, where $C_1'(n) \neq n$ (Lemma 2.1) and $\sigma = C_1'C_2 \cdots C_t$ is again in canonical form.

Now suppose that $b \in M(\tau)$ and, in particular, $b \in M(C_i)$ for $i \geq 2$, or that $b \in M(C_1)$ and $C_1(n) = n$. Then $b$ is not in $M(C_j)$ for $j \neq i$ so that $(b, n)$ commutes with $C_j$. Thus $\sigma = C_1 \cdots C_i(b, n)C_{i+1} \cdots C_t$, where $C_i = (b_1, n) \cdots (b_s, n)(b_1, n)$.

*Case 1.* If $C_1(n) \neq n$ and $b = b_1$, set $C_1' = C_1(b_1, n) \cdots (b_s, n)$. Then $\sigma = C_1'C_2 \cdots C_{i-1}C_{i+1} \cdots C_t$ and is in canonical form.

*Case 2.* If $C_1(n) = n$ and $b \neq b_1$ then, by Lemma 2.4, $C_i(b, n)$ can be rewritten as a single cycle $C_i'$, where $C_i'(n) \neq n$. Then $\sigma = C_i'C_1C_2 \cdots C_{i-1}C_{i+1} \cdots C_t$ is in canonical form (or for $i = 1$, $\sigma = C_1'C_2 \cdots C_t$ is in canonical form).

*Case 3.* If $C_1(n) \neq n$ and $b \neq b_1$ then, by Lemma 2.4, $C_i(b, n)$ can be rewritten as a single cycle $C_i'$, where $C_i'(n) \neq n$. Set $C_1' = C_1C_i'$. Then $C_1'$ is a single cycle, where $C_1'(n) \neq n$ and $\sigma = C_1'C_2 \cdots C_{i-1}C_{i+1} \cdots C_t$ is in canonical form.

Finally, suppose that $b \in M(C_1)$ and $C_1(n) \neq n$. Then $\sigma = C_1(b, n)C_2 \cdots C_t$, where $C_1 = (b_1, n) \cdots (b_s, n)$.

*Case 1.* If $b = b_s$ and $s > 1$, write $C_1' = (b_1, n) \cdots (b_{s-1}, n)$; then $\sigma = C_1'C_2 \cdots C_t$ is in canonical form. If $b = b_s$ and $s = 1$ we have that $\sigma = C_2 \cdots C_t$ is in canonical form.

*Case 2.* If $b \neq b_1 \neq b_s$ then, by Lemma 2.2, $C_1$ can be written as a product of two cycles $C_0C_1'$, where $C_0(n) \neq n$. Then $\sigma = C_0C_1'C_2 \cdots C_t$ is in canonical form.

*Case 3.* If $b = b_1 \neq b_s$ and $s > 1$, set $C_1' = C_1(b, n)$. Then $C_1'(n) = n$ and $C_1'C_2 \cdots C_t$ is in canonical form. If $b = b_1 \neq b_s$ and $s = 1$, then $C_1 = (b, n)$, $C_1(n, b) = I$, and $\sigma = C_2 \cdots C_t$ is in canonical form. □

The rank of an element $\pi$ in the star poset is the distance from $I$ in $G(n)$ where each edge in $G(n)$ corresponds to a generator from $T$. Theorem 2.10 states that every $\pi \in S_n$ can be written in the canonical form of Definition 2.5. The canonical form provides a path from $I$ to $\pi$; we will show that this is a shortest path.

COROLLARY 2.11. *$\sigma \in S_n$ is of rank $k$ in the star poset iff $k = M_\sigma + C_\sigma + \delta_\sigma$.*

PROOF. From Lemma 2.9, we know that since $\sigma$ can be written in the canonical form, $\text{rank}(\sigma) \leq M_\sigma + C_\sigma + \delta_\sigma$. But from Theorem 2.10 we have $\text{rank}(\sigma) \geq M_\sigma + C_\sigma + \delta_\sigma$. □

DEFINITION 2.12. $W_{n,k}$ is the number of permutations of rank $k$ in $(S_n, \leq)$.

Corollary 2.11 provides a characterization of these Whitney numbers; namely

$$W_{n,k} = |\{\sigma \in S_n \mid k = M_\sigma + C_\sigma + \delta_\sigma\}|.$$

REMARK. It is not difficult to determine the largest value of $M_\sigma + C_\sigma + \delta_\sigma$. This number is the largest value of rank($\sigma$) and is the height of the poset $(S_n, \leq)$. This value is also the diameter of the Cayley graph $G(n)$, since there is a color-preserving graph automorphism mapping any $v \in V(n)$ to $I$ [1].

COROLLARY 2.13. *The height of* $(S_n, \leqslant)$ *is given by* $(3n-4)/2$ *for* $n$ *even and* $(3n-3)/2$ *for* $n$ *odd.*

PROOF. We note that $0 \leqslant M_\sigma \leqslant n$, $0 \leqslant C_\sigma \leqslant n/2$ and $\delta_\sigma$ is either 0 or $-2$. Also, $M_\sigma$, $C_\sigma$ and $\delta_\sigma$ cannot be chosen independently. For example, if $M_\sigma = n$ then $\delta_\sigma = -2$ and also $C_\sigma$ can equal $n/2$ only if $n$ is even, at which point $M_\sigma = n$.

For $n$ even, $M_\sigma + C_\sigma + \delta_\sigma$ has the maximum value $(3n-4)/2$ at $M_\sigma = n$, $\delta_\sigma = -2$, and $C_\sigma = n/2$. This can occur if $\sigma$ is a product on $n/2 - 1$ disjoint cycles of length 2 that each fix $n$, and a single cycle of length 2 that moves $n$. $M_\sigma + C_\sigma + \delta_\sigma$ also has the maximum value at $M_\sigma = n-1$, $\delta_\sigma = 0$, and $C_\sigma = n/2 - 1$. This can occur if $\sigma$ is a product of $n/2 - 2$ disjoint cycles of length 2 where $n$ is fixed and a single cycle of length 3 where $n$ is fixed.

For $n$ odd, $M_\sigma + C_\sigma + \delta_\sigma$ has the maximum value of $(3n-3)/2$ at $M_\sigma = n-1$, $\delta_\sigma = 0$, and $C_\sigma = (n-1)/2$. This can occur if $\sigma$ is a product of $(n-1)/2$ disjoint cycles of length 2 that each fix $n$. □

## 3. RECURRENCE RELATIONS

We now turn our attention to establishing a recurrence relation for the numbers $\{W_{n,k}\}$. Our approach is to define an ordinary generating function on $k$ for $\{W_{n,k}\}$ and then establish the recurrence relation, on $n$, among these generating functions.

DEFINITION 3.1. Let $W_n(X)$ be the ordinary generating function for the Whitney numbers of the second kind for the poset $(S_n, \leqslant)$. Then

$$W_n(X) = \sum_{k=0}^{\infty} W_{n,k} X^k.$$

We will establish the recurrence relation on $\{W_n(X)\}$. Since we will have need to express $W_n(X)$ as a sum of other generating functions, it is convenient to make the following definition.

DEFINITION 3.2. Let $Q$ be a subset of $S_n$. Define $T_Q(X)$ to be the generating function

$$T_Q(X) = \sum_{k=0}^{\infty} Y_{n,k} X^k,$$

where $Y_{n,k}$ is the number of elements in $Q$ of rank $k$ in $(S_n, \leqslant)$. Thus $T_{S_n}(X) = W_n(X)$ and if $H$ and $L$ partition $S_n$, we have $T_H(X) + T_L(X) = W_n(X)$.

DEFINITION 3.3. Let $F_i = \{\pi \in S_n \mid \pi(n) = i\}$ for $i = 1, \ldots, n$.

Then $F_1, F_2, \ldots, F_n$ partition $S_n$ and $F_n$ is a subgroup. Let us denote $F_n$ by $H_n$ and define $H_{n-1}$ as the subgroup of $S_{n-1}$ that fixes $n-1$. With these definitions, the first lemma is clear.

LEMMA 3.4. $W_n(X) = T_{F_1}(X) + \cdots + T_{F_{n-1}}(X) + T_{H_n}(X)$.

The next lemma is the first in a series of lemmas that will allow us to work with Lemma 3.4 and eventually result in our recurrence relation.

LEMMA 3.5. $T_{F_1}(X) = T_{F_2}(X) = \cdots = T_{F_{n-1}}(X) = XW_{n-1}(X)$.

PROOF. Let $K_i$ $(1 \leq i \leq n-1)$ be the subgroup generated by $T - \{(i, n)\}$. It is not difficult to verify algebraically that $K_i$ is the subgroup that fixes $i$ and that $F_i = K_i(i, n)$ is a right coset of $K_i$. Since $I$ is the only element of $S_n$ with rank 0, and $I$ is not in $F_i$, then $T_{F_i}(X)$ has no constant term. We then write $T_{F_i}(X) = XP(X)$ for some polynomial $P(X)$.

It is clear that $K_i$ is isomorphic to $S_{n-1}$ and, from Theorem 2.10, elements of $K_i$ form a connected subgraph of the Cayley graph, $G(n)$. Now let $\sigma \in F_i$, say $\sigma = \tau(i, n)$, and $\tau' \in (S_{n-1}, \leq)$ where $\tau'$ is the isomorphic image of $\tau$. Then rank$(\sigma)$ in $(S_n, \leq)$ is one more than rank$(\tau')$ in $(S_{n-1}, \leq)$. From this we can conclude that $P(X) = W_{n-1}(X)$ and the lemma follows. □

We now partition $H_n$ into two subsets. As above, let $K_1$ be the subgroup generated by $T - \{(1, n)\}$. It can be shown that $(1, n)K_1(1, n) = H_n$ and since $K_1(1, n) = F_1$ we have $(1, n)F_1 = H_n$. We can interpret this last identity in terms of the Cayley graph as follows: every $f \in F_1$ is adjacent to some $h \in H_n$ by way of a $(1, n)$ edge. Let us then define

$$H_n^+ = \{h \in H_n \mid h(1) = 1\}, \qquad H_n^- = \{h \in H_n \mid h(1) \neq 1\}.$$

This partitions $H_n$. We use this to partition $F_1$ as follows:

$$F_1^+ = (f \in F_1 \mid (1, n)f = h \text{ where } h \in H_n^+\},$$

$$F_1^- = (f \in F_1 \mid (1, n)f = h \text{ where } h \in H_n^-\}.$$

Since $I \in H_n$, it is clear that $I \notin F_1^+$ or $F_1^-$ and $T_{F_1^+}(X)$ and $T_{F_1^-}(X)$ have no constant terms. Let us then write $T_{F_1^+}(X) = XU(X)$ and $T_{F_1^-}(X) = XD(X)$ for polynomials $U(X)$ and $D(X)$. Lemma 3.5 has that $T_{F_1}(X) = XW_{n-1}(X)$, but $T_{F_1}(X) = T_{F_1^+}(X) + T_{F_1^-}(X) = X(U(X) + D(X))$. Thus, we have proved:

LEMMA 3.6. $U(X) + D(X) = W_{n-1}(X)$.

Next we have:

LEMMA 3.7. $T_{H_n}(X) = U(X) + X^2D(X)$.

PROOF. If $f \in F_1^+$, then there is some $h \in H_n^+$ such that $(1, n)f = h$. Now suppose rank$(f) = r$ and consider rank$(h)$. We know $r = M_f + C_f + \delta_f = M_f + C_f - 2$. But then $(1, n)f$ fixes 1 and $n$, while $f$ moves $n$ and 1, which means $M_h = M_f - 2$. From the canonical form of $f$, we know that since $h = (1, n)f$ fixes 1, the canonical form of $h$ contains no $(1, n)$ generator. Since $h(1) = 1$ and $h(n) = n$, then $(1, n)$ is one of the disjoint cycles of $f$, and then $C_h = C_f - 1$. But then rank$(h) = M_h + C_h + \delta_h = (M_f - 2) + (C_f - 1) + 0 = M_f + C_f - 2 - 1 =$ rank$(f) - 1$. Thus rank$(f) =$ rank$(h) + 1$ for $f \in F_1^+$ with $h \in H_n^+$. The coefficient of $X^j$ in the generating function $U(X)$ is the number of elements in $F_1^+$ of rank $j + 1$, which is the number of elements in $H_n^+$ of rank $j$. Thus $U(X) = T_{H_n^+}(X)$.

If $f \in F_1^-$, then there is some $h \in H_n^-$ such that $(1, n)f = h$. As before, suppose rank$(f) = r$ and consider rank$(h)$. We know $r = M_f + C_f + \delta_f = M_f + C_f - 2$. But then $(1, n)f$ moves 1 and fixes $n$, whereas $f$ moves $n$ and 1, which means $M_h = M_f - 1$. From the canonical form of $f$, we know that since $h = (1, n)f$ moves 1 and $f$ moves 1, $C_h = C_f$. But then rank$(h) = M_h + C_h + \delta_h = (M_f - 1) + C_f + 0 = M_f + C_f - 2 + 1 =$ rank$(f) + 1$. The coefficient of $X^j$ in the generating function $D(X)$ is the number of elements in $F_1^-$ of rank $j + 1$, which is the number of elements in $H_n^-$ of rank $j + 2$. Thus $X^2D(X) = T_{H_n^-}(X)$. □

The final lemma before the main theorem in this section relates the generating funtions $U(X)$ and $T_{H_{n-1}}(X)$. Note that $T_{H_{n-1}}(X)$ is the generating function for the set of all permutations in $(S_{n-1}, \leqslant)$ in which $n-1$ is fixed.

LEMMA 3.8. $U(X) = T_{H_{n-1}}(X)$.

PROOF. Let $A = \{\pi \in S_n \mid \pi(n) = n,\ \pi(1) = 1\}$ and $H_{n-1} = \{\pi \in S_{n-1} \mid \pi(n-1) = n-1\}$. Define a function $Z: A \to H_{n-1}$ by $Z(\pi_1) = \pi_2$, where if $\pi_1 = [1, a_2, \ldots, a_{n-1}, n]$ then $\pi_2 = [a_2 - 1, \ldots, n-1]$. $A$ is a subgroup of $S_n$, $Z$ is a group isomorphism and $U(X)$ is a generating function for $A$. To complete the proof, we must show that rank is preserved under this isomorphism. Let $\sigma \in A$, and $\sigma = C_1 C_2 \cdots C_t$ be its canonical form. Since $\sigma(1) = 1$, 1 is not in any of these cycles and $Z(\sigma) = Z(C_1)Z(C_2)\cdots Z(C_t)$. But then $M_{C_i} + C_{C_i} + \delta_{C_i} = M_{Z(C_i)} + C_{Z(C_i)} + \delta_{Z(C_i)}$. Thus $\text{rank}(\sigma) = \text{rank}(Z(\sigma))$ and we are done.

THEOREM 3.9. *The polynomials $W_n(X)$ are defined by the following recurrence:*

$$W_n(X) = [(n-1)X + 1]W_{n-1}(X) + (X^2 - 1)(n-2)XW_{n-2}(X) \qquad \text{for } n \geqslant 3,$$
$$W_1 = 1, \qquad W_2 = 1 + X.$$

PROOF. Starting with Lemma 3.4 we have

$$\begin{aligned}
W_n(X) &= T_{F_1}(X) + \cdots + T_{F_{n-1}}(X) + T_{H_n}(X) \\
&= (n-1)XW_{n-1}(X) + T_{H_n}(X) && \text{(Lemma 3.5)} \\
&= (n-1)XW_{n-1}(X) + U(X) + X^2D(X) && \text{(Lemma 3.7)} \\
&= (n-1)XW_{n-1}(X) + U(X) + X^2(W_{n-2}(X) - U(X)) && \text{(Lemma 3.6)} \\
&= (n-1)XW_{n-1}(X) + X^2W_{n-2}(X) + U(X)(1 - X^2) \\
&= (n-1)XW_{n-1}(X) + X^2W_{n-2}(X) + T_{H_{n-1}}(X)(1 - X^2) && \text{(Lemma 3.8)} \\
&= (n-1)XW_{n-1}(X) + X^2W_{n-2}(X) + (W_{n-1}(X) - (n-2)XW_{n-2}(X))(1 - X^2) \\
& && \text{(Lemma 3.5)} \\
&= [(n-1)X + 1]W_{n-1}(X) + (X^2 - 1)(n-2)XW_{n-2}(X). \quad \square
\end{aligned}$$

## 4. GENERATING FUNCTIONS

In this section we will solve the recurrence relation on $\{W_n(X)\}$ given by Theorem 3.9 and thus obtain a generating function for these Whitney numbers. We then examine the coefficients of $W_n(X)$ and exhibit an explicit formula for $W_{n,k}$ involving binomial coefficients and Stirling numbers of the first kind.

Let $G(Y)$ be the exponential generating funtion for $\{W_n(X)\}$; i.e. define

$$G(Y) = \sum_{k=1}^{\infty} \frac{W_k(X)Y^{k-1}}{(k-1)!}.$$

Using standard techniques, it is not difficult to derive the differential equation

$$G'(Y)(1 - XY) = (1 + X + XY(X^2 - 1))G(Y).$$

Integrating, we obtain our generating function.

THEOREM 4.1. *The exponential generating function for $\{W_n(X)\}$ is given by*

$$G(Y) = (1 - XY)^{-X^{-1}}e^{Y - X^2Y}.$$

THEOREM 4.2. *The generating function for $\{W_{n,k}\}$ is given by*

$$W_n(X)=\sum_{k=0}^{n-1}\binom{n-1}{k}(1-X^2)^{n-1-k}X^k\prod_{j=0}^{k-1}(X+j+1)$$

*where, for $k=0$, we define $\prod_{j=0}^{-1}(X+j+1)=1$.*

PROOF. Note that $(\delta^nG/\delta Y^n)|_{Y=0}=W_n(X)$. By induction, one can show that

$$\frac{\delta^nG}{\delta Y}=\sum_{k=0}^{n-1}\binom{n-1}{k}(1-X^2)^{n-1-k}X^k\prod_{j=0}^{k-1}(X+j+1)[e^{Y-X^2Y}(1-XY)^{-X-1-k}],$$

from which the result follows directly. □

Recall that the Stirling numbers of the first kind $\{s(n, k)\}$ are the coefficients in the polynomial expansion of $X(X-1)\cdots(X-n+1)$ [3]. In particular,

$$X(X-1)(X-2)\cdots(X-n+1)=\sum_{k=0}^{n}s(n,k)X^k.$$

Thus

$$X\prod_{j=0}^{k-1}(X+j+1)=(-1)^{k+1}\sum_{j=0}^{k+1}s(k+1,j)(-1)^jX^j,$$

and using

$$(1-X^2)^{n-1-k}=\sum_{t=0}^{n-1-k}\binom{n-1-k}{t}(-1)^tX^{2t}$$

$W_n(X)$ can be rewritten as follows:

FORMULA 4.3.

$$\sum_{k=0}^{n-1}\sum_{t=0}^{n-1-k}\sum_{j=0}^{k+1}\binom{n-1}{k}\binom{n-1-k}{t}s(k+1,j)(-1)^{t+k+j+1}X^{2t+k-1+j}.$$

By examining coefficients, we arrive at a formula for $W_{n,u}$.

THEOREM 4.4. *The Whitney numbers of the second kind for the star poset are given as follows. Let*

$$L=\min\{n-1,u+1\},\qquad T_k=\min\left\{0,\left\lceil\frac{u-2k+1}{2}\right\rceil\right\},$$

$$S_k=\min\left\{n-1-u,\left\lceil\frac{u+1-k}{2}\right\rceil\right\}.$$

*Then*

$$W_{n,u}=\sum_{k=0}^{L}\sum_{t=T_k}^{S_k}\binom{n}{k}\binom{n-k}{t}s(k+1,t)(-1)^{t+u},$$

*where $\lceil\ \rceil$ is the greatest integer function.*

PROOF. We consider the coefficients of $X^u$, using Formula 4.3, where $u\geqslant 0$ is fixed, and we let $u=2t+k-1+j$. By Formula 4.3, we know that since $X^k$ is a factor in $X^u$, we have $k\leqslant u+1$. By the first summand in Formula 4.3, $k\leqslant n-1$. Thus, $k\leqslant L=\min\{u+1,n-1\}$ and every coefficient of $X^k$ with $0\leqslant k\leqslant L$ will contribute to the coefficient of $X^u$ (for appropriate values of $j$ and $t$). For $k$ fixed, we have $2t+j=u-k+1$. But $j\geqslant 0$ so that $2t\leqslant u-k-1$ and $t\leqslant \lceil (u+1-k)/2\rceil$. From the second summand, $t\leqslant n-1-k$, from which we have $t\leqslant S_k=\min\{n-1-u, \lceil (u+1-k)/2\rceil\}$. Again, for $k$ fixed, $j=u-k+1-2t$ and $j\leqslant k+1$ by the third summand. Thus, $u+k+1-2t\leqslant k+1$, $(u-2k)/2\leqslant t$ and $\lceil (u-2k+1)/2\rceil\leqslant t$. From the second summand, we have $0\leqslant t$. Thus, $t\geqslant T_k=\max\{0, \lceil (u-2k+1)/2\rceil\}$. The theorem then follows by setting $j=u-k+1-2t$. □

## 5. Vertical Generating Functions

In this section, we consider another generating function for these Whitney numbers. Here we examine the exponential generating function, on $n$, for $\{W_{n,k}\}$ and use Theorem 3.9 to establish a recurrence, on $k$, among these generating funtions.

From Theorem 3.9, we have

$$W_n(X)=[(n-1)X+1]W_{n-1}(X)+(X^2-1)(n-2)XW_{n-2}(X) \qquad \text{for } n\geqslant 3,$$
$$W_1=1, \qquad W_2=1+X.$$

Equating coefficients on the left and right sides of the above recurrence, it is not difficult to show that, for all $n\geqslant 1$, the following holds:

Recurrence 5.1:

$$W_{n,0}=1,$$
$$W_{n,1}=n-1,$$
$$W_{n,2}=(n-1)(n-2)$$

and, for $k\geqslant 3$,

$$W_{n,k}=W_{n-1,k}+(n-1)W_{n-1,k-1}-(n-2)W_{n-2,k-1}+(n-2)W_{n-2,k-3}.$$

Definition 5.2. The vertical generating function for $\{W_{n,k}\}$ is the exponential generating function defined by

$$H_k(X)=\sum_{n=0}^{\infty}\frac{W_{n+1,k}X^n}{n!}.$$

Once again employing standard techniques, one can show that $\{H_k(X)\}$ satisfies the differential equation of Lemma 5.3.

Lemma 5.3. *The vertical generating functions $\{H_k(X)\}$ satisfy the differential equation*

$$H_k'(X)=H_k(X)+XH_{k-1}'(X)+H_{k-1}(X)-XH_{k-1}(X)+XH_{k-3}(X),$$
$$H_0(X)=e^X, \qquad H_1(X)=Xe^X, \qquad H_2(X)=X^2e^X.$$

Now let $\{P_k(X)\}$ be a collection of functions that satisfy the following differential equation:

Equation 5.4:

$$P_k'(X)=P_{k-1}(X)+XP_{k-1}'(X)+XP_{k-3}(X)$$

where $P_0(X)=1$, $P_1(X)=X$ and $P_2(X)=X^2$.

Setting $H_k(X) = P_k(X)e^X$, the differential equation of Lemma 5.3 is satisfied. By the initial conditions and simple integration, one can see that $\{P_k(X)\}$ are all polynomials and that $P_k(X)$ is of degree $k$.

Definition 5.5. The forward difference of the sequence $\{a_k\}$ is the sequence $\{a_{k+1} - a_k\}$ and is denoted $\{\Delta a_k\}$. The $j$th forward difference of $\{a_k\}$ is denoted as $\{\Delta^j a_k\}$ and is recursively defined as $\{\Delta\Delta^{j-1} a_k\}$.

Theorem 5.6. *The $(k+1)$th forward difference of the sequence $\{W_{n,k}\}_{n=0}^{\infty}$ is $\{0\}$.*

Proof. For any sequence $\{a_i\}$ with exponential generating function $G(X) = \sum_{i=0}^{\infty}(a_i X^i/i!)$, the generating function for $\{\Delta a_i\}$ is $G'(X) - G(X)$. Thus, the generating function for $\{\Delta^j a_k\}$ is

$$\frac{d^j G(X)}{dX} - \frac{d^{j-1}G(X)}{dX}.$$

For our generating function $H_k(X) = P_k(X)e^X$, we have $H'(X) - H(X) = P_k'(X)e^X$ where $P_k'(X)$ is a polynomial of degree at most $k-1$. After $k+1$ such differences, $d^{k+1}P_k(X)/dX = 0$ and the result follows. □

We will now provide a formula for the coeffiients of $\{P_k(X)\}$ and thus have a formula for the vertical generating functions. Let

$$P_k(X) = \sum_{i=0}^{k} d_{k,i} X^i.$$

By equating coefficients on the left and right hand sides of Equation 5.4, we obtain the following recurrence:

Recurrence 5.7:

(i) $$d_{k,1} = d_{k-1,0} \qquad (i = 0);$$

(ii) $$d_{k,i+1} = d_{k-1,i} + \frac{d_{k-3,i-1}}{i+1} \qquad (1 \leqslant i \leqslant k-2);$$

(iii) $$d_{k,k} = d_{k-1,k-1} \qquad (i = k-1);$$

with

$$d_{0,0} = 1, \quad d_{0,n} = 0 \quad \text{for } n \geqslant 1;$$

$$d_{1,0} = 0, \quad d_{1,1} = 1, \quad d_{1,n} = 0 \quad \text{for } n \geqslant 2;$$

and

$$d_{2,0} = 0, \quad d_{2,1} = 0, \quad d_{2,2} = 1, \quad d_{2,n} = 0 \quad \text{for } n \geqslant 3.$$

Since $d_{0,0} = 1$, it is clear from (iii) that $d_{k,k} = 1$. Setting $i = k-2$ in (ii) we have $d_{k,k-1} = d_{k-1,k-2} + d_{k-3,k-3}/(k-1) = d_{k-1,k-2} + 1/(k-1)$ and

$$d_{k,k-1} = \sum_{r=0}^{k-3} \frac{1}{r+2} \qquad \text{for } k \geqslant 3.$$

Note that for $k = 2$ and $k = 1$ we have $d_{k,k-1} = 0$. In a similar manner, setting $i = k-3$ in (ii) again, we obtain

$$d_{k,k-2} = \sum_{r=3}^{k-3} \frac{1}{r+1} \sum_{s=0}^{r-3} \frac{1}{s+2} \qquad \text{for } k \geqslant 6.$$

Note that for $2 \leqslant k \leqslant 5$, we have $d_{k,k-2} = 0$ (for $k = 2$ we have $d_{2,0} = 0$ by the initial conditions and for $3 \leqslant k \leqslant 5$ we have $d_{k,k-2} = d_{k-1,k-2} + d_{k-3,k-4}/(k-2)$, $d_{1,0} = 0$, and $d_{k-3,k-4} = 0$). Continuing in this manner, we have the following:

THEOREM 5.8. *The coefficients of the polynomials $\{P_k(X)\}$, $\{d_{k,j}\}$, are given by the following:*

$$d_{k,k} = 1,$$

$$d_{k,k-j} = \sum_{\lambda_1=3(j-1)}^{k-3} \frac{1}{\lambda_1 - j + 3} \sum_{\lambda_2=3(j-2)}^{\lambda_1-3} \frac{1}{\lambda_2 - j + 4} \cdots \sum_{\lambda_j=3(j-j)}^{\lambda_{j-1}-3} \frac{1}{\lambda_j - j + j + 2} \qquad \text{for } k \geqslant 3j$$

*and*

$$d_{k,k-j} = 0 \qquad \text{for } k < 3j.$$

REMARK. The formula for the coeffiients of $\{P_k(X)\}$ can be expressed by the recursive formula below:

$$d_{k,k-j} = \begin{cases} 1 & j = 0, \\ 0 & k < 3j, \\ \displaystyle\sum_{r=3(j-1)}^{k-3} \frac{d_{r,r-j+1}}{r-j+3}, & 0 \leqslant 3j \leqslant k. \end{cases}$$

## 6. TABLES AND FORMULAE

We conclude this paper by providing a table of values for $\{W_{n,k}\}$, examples of vertical generating functions and explicit formulae for $W_{n,k}$ for small values of $k$.

REMARK. Table 1 is the motivation for calling the $\{H_k(X)\}$ 'vertical generating functions', since each column defines a generating function.

TABLE 1
Whitney numbers $\{W_{n,k}\}$

| $n$ \ $k$ | 0 | 1 | 2 | 3 | 4 | 5 | | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | | | |
| 2 | 1 | 1 | | | | | | | | | |
| 3 | 1 | 2 | 2 | 1 | | | | | | | |
| 4 | 1 | 3 | 6 | 9 | 5 | | | | | | |
| 5 | 1 | 4 | 12 | 30 | 44 | 26 | 3 | | | | |
| 6 | 1 | 5 | 20 | 70 | 170 | 250 | 169 | 35 | | | |
| 7 | 1 | 6 | 30 | 135 | 460 | 1110 | 1689 | 1 254 | 340 | 15 | |
| 8 | 1 | 7 | 42 | 231 | 1015 | 3430 | 8379 | 13 083 | 10 408 | 3409 | 315 |

TABLE 2
Vertical generating functions $\{H_k(X)\}$

$H_0(X) = e^X$
$H_1(X) = Xe^X$
$H_2(X) = X^2e^X$
$H_3(X) = (X^3 + \frac{1}{2}X^2)e^X$
$H_4(X) = (X^4 + \frac{5}{6}X^3)e^X$
$H_5(X) = (X^5 + \frac{26}{24}X^4)e^X$
$H_6(X) = (X^6 + \frac{154}{120}X^5 + \frac{3}{24}X^4)e^X$

TABLE 3
Explicit formulae

| | |
|---|---|
| $W_{n,0}=$ | $C(n-1,0)$ |
| $W_{n,1}=$ | $C(n-1,1)$ |
| $W_{n,2}=$ | $C(n-1,2)$ |
| $W_{n,3}=$ | $C(n-1,2)+3!\,C(n-1,3)$ |
| $W_{n,4}=$ | $5C(n-1,3)+4!\,C(n-1,4)$ |
| $W_{n,5}=$ | $26C(n-1,4)+5!\,C(n-1,5)$ |
| $W_{n,6}=$ | $3C(n-1,4)+154C(n-1,5)+6!\,C(n-1,6)$ |

REMARK. By examining coefficients, each vertical generating function provides a formula for an infinite collection of $\{W_{n,k}\}$, where $k$ is fixed (Table 3). A small shift in notation allows one to express these formulae in a convenient form:

$$\text{Let } C(n,k)=\begin{cases}\binom{n}{k} & \text{provided that } n \geqslant k,\\ 0 & \text{otherwise.}\end{cases}$$

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