On the Index of Convergence of an Irreducible Boolean Matrix

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ABSTRACT

First we give a new proof of the sharp upper bound for the indices of convergence of \( n \times n \) irreducible Boolean (or nonnegative) matrices with period \( p \). Then we characterize the matrices with the largest index.

INTRODUCTION

Let \( \mathcal{B} = \{0, 1\} \) be the two elements Boolean algebra with the “Boolean addition” \( a + b = \max(a, b) \) and “Boolean multiplication” \( ab = \min(a, b) \) (We define the order \( 0 < 1 \) between the symbols \( 0 \) and \( 1 \).) The matrices over \( \mathcal{B} \) are called “Boolean matrices”. Let \( \mathcal{B}_n \) be the set of \( n \times n \) Boolean matrices; then \( \mathcal{B}_n \) forms a finite multiplicative semigroup of order \( 2^n \).

Boolean matrices are closely related to nonnegative matrices. For a given nonnegative matrix \( A \), we define \( \tilde{A} \) to be the corresponding Boolean matrix obtained by replacing all the positive entries of \( A \) by \( 1 \). Clearly, \( \tilde{A} \) completely determines the zero-nonzero pattern of the nonnegative matrix \( A \) and vice versa. Furthermore, this correspondence preserves matrix addition and multiplication, namely we have \( \tilde{A} + \tilde{B} = \tilde{A} + \tilde{B} \) and \( \tilde{AB} = \tilde{A} + \tilde{B} \) (where the operations on right side are Boolean). From this we see that the study of


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the "combinatorial properties" (those properties which only depend on the zero-nonzero pattern of the matrix) of nonnegative matrices is equivalent to the study of the corresponding properties of Boolean matrices. For example, it is well known that zero-nonzero patterns of powers $A, A^2, \ldots, A^m, \ldots$ (hence the primitivity of $A$, the primitive exponent of $A$, and the index of cyclicity of $A$) depend only on the zero-nonzero pattern of $A$, so we may just study the powers $A, A^2, \ldots, A^m, \ldots$, which is usually more convenient, by making use of the Boolean notation.

In this paper, $0$ and $I$ will denote the Boolean matrices with all elements $0$ and all elements $1$, respectively. $P_n$ will denote the set of $n \times n$ permutation matrices in $\mathbb{B}_n$. $E_{ij}$ will denote the matrix with $(i, j)$ entry $1$ and all the other entries $0$, and $e_i$ will denote the row vector with $r$th component $1$ and all the other components $0$. $A^T$ will denote the transpose of $A$.

Let $A, B \in \mathbb{B}_n$. If there exists some $P \in P_n$ such that $A = P^TBP$, then we say that $A$ and $B$ are permutation similar, denoted by $A \approx B$.

A Boolean matrix $A \in \mathbb{B}_n$ is reducible if

$$A = \begin{bmatrix} B & 0 \\ D & C \end{bmatrix}$$

where $B$ and $C$ are nonvacuous square matrices. $A$ is irreducible if it is not reducible.

Let $A \in \mathbb{B}_n$. The sequence of powers $A, A^2, \ldots$ clearly forms a finite subsemigroup $\langle A \rangle$ of $\mathbb{B}_n$, and then there exists a least positive integer $k = k(A)$ such that $A^k = A^{k+t}$ for some $t > 0$, and there exists a least positive integer $p = p(A)$ such that $A^k = A^{k+p}$. We call the integer $k = k(A)$ the index of convergence of $A$, and the integer $p = p(A)$ the period of convergence of $A$. It is easy to see that $\langle A \rangle = \{ A, \ldots, A^{k+p-1} \}$ is a semigroup of order $k + p - 1$. It is also known from the elements of semigroup theory that $\{ A^k, \ldots, A^{k+p-1} \}$ is a cyclic group with unit $A^e$ (and generator $A^{e+1}$), where $e$ is the unique integer between $k$ and $k + p - 1$ such that $e \equiv 0 \pmod{p}$.

The study of $k(A)$ and $p(A)$ is the central part of the study of powers of Boolean matrices (or powers of nonnegative matrices). Now the properties of $p(A)$ is already clear by the work of Rosenblatt [4, 3]. He has proved that:

(i) If $A$ is irreducible, then $p(A)$ is the greatest common divisor of the distinct lengths of the elementary cycles of the associated digraph $D(A)$.

(ii) If $A$ is reducible, then $p(A)$ is the least common multiple of $p(A_1), \ldots, p(A_m)$, where $A_1, \ldots, A_m$ are the irreducible constituents of $A$.

From (i) and the fact that the greatest common divisor of the lengths of the elementary cycles of $D(A)$ is also equal to the index of cyclicity of $A$ [1], it follows that when $A \in \mathbb{B}_n$ ($n > 1$) is irreducible, then $p(A) = 1$ if and only
if $A$ is primitive, and in this case $k(A)$ is just the primitive exponent $\gamma(A)$—the least positive integer $k$ such that $A^k = J$. So the concept of the index of convergence of an irreducible matrix is a generalization of the concept of the primitive exponent of a primitive matrix.

Since our knowledge about $p(A)$ is quite complete, the main interest in the powers of Boolean matrices will be the study of $k(A)$. Heap and Lynn [2] and Schwarz [5] have studied the upper bounds of $k(A)$. In [2], Heap and Lynn proved that if $A \in \mathcal{B}_n$ is irreducible with period $p$ and $n = pr + s$ where $r = \lfloor n/p \rfloor$ and $0 \leq s \leq p - 1$, then

$$k(A) \leq p(r^2 - 2r + 2) + 2s,$$  \hspace{1cm} \text{(H-L)}$$

and this upper bound is sharp in the case $s = 0$. The upper bound (H-L) is also a generalization of Wielandt's upper bound $\gamma(A) \leq n^2 - 2n + 2$ for the primitive exponents of $n \times n$ primitive matrices [7].

In [5], S. Schwarz improved Heap and Lynn's upper bound and proved that

$$k(A) \leq pw_r + s,$$  \hspace{1cm} \text{(S)}$$

where

$$w_r = \begin{cases} r^2 - 2r + 2, & r > 1, \\ 0, & r = 1, \quad s > 0. \end{cases}$$

Schwarz also gave example to show that in the case $n = 7$ and $p = 2$, his improved upper bound (S) is sharp, but he didn't give examples to show that the upper bound (S) is sharp for all possible cases of $n$ and $p$. Besides, Schwarz proved the upper bound (S) entirely in the language of binary relations [whereas Heap and Lynn proved their upper bound (H-L) mainly by graph theory techniques], and his proof is quite involved. In Section I of this paper, we will use a different approach (our starting point is the "imprimitive normal form" of an irreducible matrix) to consider the index of convergence $k(A)$ of an irreducible Boolean matrix $A$. We will first prove a general property (Lemma 2.1) for characterizing $k(A)$ in terms of the imprimitive normal form of $A$, and then use this characterizing property to give a simple new proof of the upper bound (S), solely in terms of matrix theory. Using this characterizing property, we are also able to construct examples to show that the upper bound (S) is sharp for all possible cases of $n$ and $p$.

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1 In this paper, we usually exclude the trivial case $r = 1, s = 0$ (i.e., $n = p$) in which $A$ is permutation similar to the circulant $E_{12} + E_{23} + \cdots + E_{p-1,p} + E_{p1}$, and hence $k(A) = 1$. 
In Section II of this paper, we will consider the equality case of the bound (S) and will give a complete characterization for those \( n \times n \) irreducible Boolean matrices with period \( p \) whose indices of convergence \( k(A) \) reach the upper bound \( pw_y + s \). An \( n \times n \) irreducible Boolean matrix \( A \) with period \( p \) is called an extreme matrix if its index of convergence \( k(A) \) is the largest possible value \( pw_y + s \). The set of all \( n \times n \) extreme matrices with period \( p \) is denoted by \( L_{n,p} \). We will show that:

(i) If \( r > 1 \), then \( L_{n,p} \) consists of exactly \( 2^s + s \cdot 2^{s-1} \) permutation similar equivalence classes, which we will list explicitly.

(ii) If \( r = 1 \), then \( L_{n,p} \) consists of exactly \( 2^{s-1} \) (or one, in the trivial case \( s = 0 \)) permutation similar equivalence classes, which will also be listed explicitly.

I. A NEW PROOF OF THE SHARP UPPER BOUND

1. Preliminaries

The starting point in our proof is the "imprimitive normal form" of an irreducible matrix [1, Chapter 2, 2.20]: If \( A \) is an irreducible Boolean matrix with period \( p \) (or with index of cyclicity \( p \)), then there exists \( p \in \mathbb{P} \), such that

\[
P P A P ^ T = \begin{bmatrix}
0 & A_1 & 0 & \cdots & 0 \\
A_1 & 0 & A_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & A_{p-1} & 0 \\
A_p & 0 & A_1 & \cdots & 0 \\
\end{bmatrix}
\]  

(1.1)

where the zero blocks along the diagonal are square. If in (1.1) the block \( A_i \) is of the size \( n_i \times n_{i+1} \) \((i = 1, \ldots, p; \) here we read the subscripts mod \( p \)), then we denote the matrix of the form (1.1) by \((n_1, A_1, n_2, \ldots, n_p, A_p, n_1)\). In case the size of each block is unimportant or is clear from the context, we simply denote it by \((A_1, \ldots, A_p)\). Furthermore, for \( p \) matrices \( Z_i \) of size \( n_i \times n_{i+1} \) \((i = 1, \ldots, p)\) and any nonnegative integer \( m \), we define \((Z_1, Z_2, \ldots, Z_p)_m \) to be the block partitioned matrix \([A_{ij}] (1 \leq i, j \leq p)\) with the blocks

\[
A_{ij} = \begin{cases}
Z_i & \text{if } j - 1 \equiv m \pmod{p}, \\
0 & \text{otherwise}.
\end{cases}
\]
It is clear from our notation that \((A_1, \ldots, A_p) = (A_1, \ldots, A_p)_1\).

Since \(k(A) = k(PAP^T)\), we might always assume that \(A = (A_1, \ldots, A_p)\) is already in the normal form \((1.1)\). For convenience we also define \(A_{j+p} = A_j\) for all \(j\), and we define

\[
A_i(m) = A_i A_{i+1} \cdots A_{i+m-1}
\]

to be the product of \(m\) successive matrices \([\text{we define } A_i(0) \text{ to be the identity matrix } I_n]\. The following lemma about the powers of the normal form \((A_1, \ldots, A_p)\) can be verified by a simple induction:

**Lemma 1.1.** If \(A = (A_1, \ldots, A_p)\), then

\[
A^m = (A_1(m), \ldots, A_p(m))_m
\]  

(1.2)

The following necessary and sufficient condition for a matrix of the form \((1.1)\) to be irreducible with period \(p\) will be useful:

**Lemma 1.2 [6, Corollary 5.1].** If \(A = (A_1, \ldots, A_p) \in \mathcal{B}_n\) is of the form \((1.1)\), then the following two properties are equivalent:

1. \(A\) is irreducible with period \(p\).
2. Each block \(A_i (1 \leq i \leq p)\) contains no zero row and no zero column, and the product \(A_i(p) = A_1 A_2 \cdots A_p\) is a primitive matrix.

Since \((BA)^{k+1} = B(AB)^k A\), we also have the following:

**Theorem 1.1.** Let \(A, B\), be two \(n \times n\) Boolean matrices containing no zero row and no zero column such that \(AB\) is primitive. Then \(BA\) is also primitive, and their primitive exponents differ at most by 1:

\[|\gamma(BA) - \gamma(AB)| \leq 1.\]

**Corollary 1.1.** If \(A = (A_1, \ldots, A_p)\) is irreducible with period \(p\), then each \(A_i(p) (1 \leq i \leq p)\) is primitive and \(|\gamma(A_i(p)) - \gamma(A_j(p))| \leq 1\) for all \(1 \leq i, j \leq p\).

We would also like to quote Wielandt’s well-known results about primitive exponents for our later use:
THEOREM 1.2 [7]. Let $A$ be an $n \times n$ primitive Boolean matrix, and $\gamma(A)$ its exponent. Then $\gamma(A) \leq n^2 - 2n + 2$, and equality holds if and only if there exists $P \in \mathbb{P}_n$ such that

$$PAP^T = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & 1 & & & 0 \end{bmatrix}$$

(unspecified entries are all zero).

2. The Upper Bound

In this subsection we will use the matrices $A_i(m)$ ($i = 1, \ldots, p$) to study the index of convergence $k(A)$ of the irreducible matrix $A = (A_1, \ldots, A_p)$ with period $p$. Note that if for some integer $m$ we have $A_i(m) = J$ for all $1 \leq i \leq p$, then it is also true that $A_i(t) = J$ for all $t \geq m$, since each $A_1, \ldots, A_p$ contains no zero row and no zero column. Also there does exist such an integer $m$, since $A_i(p)$ is primitive.

**Lemma 2.1.** Let $A = (n_1, A_1, n_2, \ldots, n_p, A_p, n_1) = (A_1, \ldots, A_p)$ be an irreducible Boolean matrix with period $p$. Then the index of convergence $k(A)$ is the least integer $k > 0$ such that $A_i = J$ holds for all $i = 1, 2, \ldots, p$.

**Proof.** If $A_i(k) = J$ for all $i = 1, 2, \ldots, p$, then by Lemma 1.1 we have $A^k = (A_1(k), \ldots, A_p(k))$ and $A^{k+p} = (A_1(k+p), \ldots, A_p(k+p))$. Now $A_i(k) = A_i(k+p) = J$ ($i = 1, \ldots, p$) and $k+p \equiv k \pmod{p}$, so $A^k = A^{k+p}$ and thus $k(A) \leq k$.

Conversely, if $A_i(k-1) \neq J$ for some $j$, then $A^{k-1} = (A_1(k-1), \ldots, A_j(k-1), \ldots, A_p(k-1))$ and $A^{k-1+p} = (A_1(k-1+p), \ldots, A_j(k-1+p), \ldots, A_p(k-1+p))$. Now $A_j(k-1+p) = J = A_j(k-1)$, so $A^{k-1} = A^{k-1+p}$ and hence $k(A) > k-1$. Combining the two parts, we have $k(A) = k$, and the lemma is proved.

The next lemma gives an upper bound of $k(A)$ in terms of the primitive exponents of the primitive matrices $A_i(p)$ ($i = 1, \ldots, p$).

**Lemma 2.2.** Let $A = (A_1, \ldots, A_p)$ be irreducible with period $p$. Suppose $1 \leq i_1 < i_2 < \cdots < i_t < p$ and $\gamma_{i_j} = \gamma(A_{i_j}(p))$ is the primitive exponent of the
primitive matrix $A_{i_j}(p)$ ($1 \leq j \leq t$). Then

$$k(A) \leq p \max(\gamma_{i_1}, \ldots, \gamma_{i_t}) + p - t. \quad (2.1)$$

**Proof.** Let $h = \max(\gamma_{i_1}, \ldots, \gamma_{i_t})$ and $k = ph + p - t$. Since $\gamma_{i_j} = \gamma(A_{i_j}(p))$ and $h \geq \gamma_{i_j}$, we have $A_{i_j}(ph) = [A_{i_j}(p)]^h - I$. Now we consider $A_i(k)$ for $1 \leq l \leq p$. Note that

$$|\{i_1, \ldots, i_t\}| + |\{l, l+1, \ldots, l+p-t\}| = p + 1 > p,$$

so there exist $j$ and $q$ ($1 \leq j \leq t, 0 \leq q \leq p - t$) such that $i_j = l + q \pmod{p}$. Thus

$$A_{i_j} = A_{l+q}$$

and

$$A_i(k) = A_i(q)A_{l+q}(k-q)$$

$$= A_i(q)A_{l+q}(ph)A_{l+q+ph}(k-ph-q)$$

$$= A_i(q)A_{i_j}(ph)A_{l+q+ph}(p-t-q)$$

$$= A_i(q)A_{l+q+ph}(p-t-q) = J$$

for all $l = 1, 2, \ldots, p$. By Lemma 2.1 we have

$$k(A) \leq k = ph + p - t - p \max(\gamma_{i_1}, \ldots, \gamma_{i_t}) + p - t.$$

**Corollary 2.1.** Let $A = (n_1, A_1, \ldots, n_p, A_p, n_1)$ be irreducible with period $p$, and let $m = \min(n_1, n_2, \ldots, n_p)$. Then

$$k(A) \leq p(m^2 - 2m + 3) - 1. \quad (2.2)$$

**Proof.** Assume $m = n_i$ for some $i$. Note that $A_i(p)$ is an $n_i \times n_i$ primitive matrix with exponent $\gamma_i = \gamma(A_i(p)) \leq n_i^2 - 2n_i + 2 - m^2 + 2m - 2$ (Theorem 1.2). So by taking $t = 1$ and $i_1 = i$ in Lemma 2.2 we have $k(A) \leq p\gamma_i + p - 1 \leq p(m^2 - 2m + 3) - 1.$

Now we are ready to prove the upper bound for $k(A)$. 

THEOREM 2.1. Let $A$ be an $n \times n$ irreducible Boolean matrix with period $p$, and let $n = pr + s$, where $r = \lfloor n/p \rfloor$ and $0 \leq s < p - 1$. ($\lfloor x \rfloor$ is the integer part of $x$). Then

$$k(A) \leq p(r^2 - 2r + 2) + s.$$ (2.3)

Proof. Without loss of generality, we may assume that $A$ is in the imprimitive normal form $A = (n_1, A_1, \ldots, n_p, A_p, n_1)$, where $n_1 + \cdots + n_p = n$. Let $m = \min(n_1, \ldots, n_p)$; then $m \leq r$.

Case 1: $m < r - 1$. By Corollary 2.1 we have

$$k(A) \leq p(m - 2m + 3) - 1 < p(r^2 - 4r + 6) - 1 < p(r^2 - 2r + 2) + s.$$

The last inequality follows from the fact that $r > m + 1 \geq 2$.

Case 2: $m = r$. By the fact that $n_1 + \cdots + n_p = n = pr + s$ and $0 \leq s \leq p - 1$, it is easy to see that there exist $p - s$ indices $i_1, \ldots, i_{p-s}$ with $1 \leq i_1 < i_2 < \cdots < i_{p-s} \leq p$ such that $n_{i_j} = n_{i_0} = \cdots = n_{i_{p-s}} = r$. For $j = 1, \ldots, p - s$, $A_{i_j}(p)$ is an $r \times r$ primitive matrix with the exponent $\gamma_{i_j} = \gamma(A_{i_j}(p)) \leq r^2 - 2r + 2$. Taking $t = p - s$ in Lemma 2.2, we then have $k(A) \leq p \max(\gamma_{i_1}, \ldots, \gamma_{i_{p-s}}) + p - (p - s) \leq p(r^2 - 2r + 2) + s$, and this completes the proof.

Next we will show that the upper bound (2.3) is sharp for $r > 1$. The sharp upper bound for $r = 1$ will also be proved.

3. The Sharpness of the Upper Bound

For $r > 1$, let

$$W_r = \begin{bmatrix} 0 & 1 & & & \\ 0 & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 1 & 1 & & & 0 \end{bmatrix} \in \mathcal{B}_r \quad \left( W_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right).$$

By Theorem 1.2 we know that $W_r$ is primitive with the primitive exponent $\gamma(W_r) = r^2 - 2r + 2$, i.e., $W_r^{r^2 - 2r + 2} = J$. It can also be verified that

$$W_r^{r^2 - 2r + 1} = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & \vdots & & J_{r-1} \\ \vdots & & & \\ 1 & & & \end{bmatrix}.$$
Example 3.1. Suppose $s = 0 \ (n = pr)$ and $r > 1$. Take the $n \times n$ Boolean matrix $A = (A_1, \ldots, A_p)$ with $A_1 = \cdots = A_{p-1} = I_r$ and $A_p = W_r$. We know that $A$ is irreducible with period $p$ by Lemma 1.2, and it is easy to see that $A_i(p(r^2 - 2r + 2)) = W_r^{r^2 - 2r + 2} = I$ for $i = 1, 2, \ldots, p$ and $A_i(p(r^2 - 2r + 2) - 1) = W_r^{r^2 - 2r + 1} \neq I$. Therefore $k(A) = p(r^2 - 2r + 2)$ by Lemma 2.1.

Before giving the next example, we define two Boolean matrices for $r > 1$ as follows:

\[
X = \begin{bmatrix}
0 & 1 & \cdots & \cdots & 1 \\
0 & 1 & \cdots & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & 0 & \cdots & \cdots & 0 \\
0 & 1 & \cdots & \cdots & 0
\end{bmatrix}_{(r+1) \times (r+1)} , \quad Y = \begin{bmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & 1 & 1
\end{bmatrix}_{(r+1) \times (r+1)}
\]

Then it can be verified that

\[
YX = W_r, \quad (XY)^{r^2 - 2r + 2} = X(YX)^{r^2 - 2r + 2} = X[YX]^{r^2 - 2r + 2}Y = X \begin{bmatrix}
0 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
1 & 1 & \cdots & 1
\end{bmatrix} Y \neq I.
\]

Example 3.2. Suppose $n = pr + s$ with $r > 1$ and $0 < s < p - 1$. Take the $n \times n$ Boolean matrix $A = (A_1, \ldots, A_p)$ with $A_1 = \cdots = A_{s-1} = I_r$ and $A_s = X$, $A_{s+1} = \cdots = A_{p-1} = I_r$, and $A_p = Y$, where $X$ and $Y$ are defined as above. Then $A$ is irreducible with period $p$ by Lemma 1.2 and $A_i(p(r^2 - 2r + 2) + s - 1) = (XY)^{r^2 - 2r + 2} \neq I$, so $k(A) > p(r^2 - 2r + 2) + s - 1$. Combining this with the upper bound $k(A) \leq p(r^2 - 2r + 2) + s$, we get $k(A) = p(r^2 - 2r + 2) + s$.

Examples 3.1 and 3.2 show that the upper bound $p(r^2 - 2r + 2) + s$ is sharp in the case $r > 1$. The following theorem gives the sharp upper bound in the case $r = 1$.

Theorem 3.1. Let $A$ be an $n \times n$ irreducible Boolean matrix with period $p$, and suppose $n = p + s$ with $0 < s < p - 1$. Then $k(A) \leq s$, and this upper bound is sharp.
Proof. Without loss of generality, we may assume that $A$ is in the imprimitive normal form $A = (n_1, A_1, \ldots, n_p, A_p, n_1)$ where $n_1 + \cdots + n_p = n = p + s$. Then among any $s + 1$ numbers $n_i, \ldots, n_{i+s}$ in the set \{ $n_1, \ldots, n_p$ \}, at least one of them is 1. It follows that among the $s$ matrices $A_i, A_{i+1}, \ldots, A_{i+s-1}$ (their sizes involve the $s + 1$ numbers $n_i, n_{i+1}, \ldots, n_{i+s}$), at least one has only one row or only one column, hence equals $J$ by the fact that it contains no zero row and no zero column. Thus $A_i(s) = A_i A_{i+1} \cdots A_{i+s-1} = J$ for $i = 1, \ldots, p$ and $k(A) \leq s$ by Lemma 2.1.

The following example shows that the upper bound $k(A) \leq s$ is sharp: Take $A = (n_1, A_1, \ldots, n_p, A_p, n_1)$ with $n_1 = \cdots = n_s = 2$, $n_{s+1} = \cdots = n_p = 1$, and

\[
A_1 = \cdots = A_{s-1} = I_2, \quad A_s = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A_{s+1} = \cdots = A_{p-1} = I_1, \quad A_p = \begin{bmatrix} 1 & 1 \end{bmatrix}.
\]

Then $A$ is irreducible with period $p$ by Lemma 1.2. Now $A_i(s-1) = I_2 \neq J$, so $k(A) > s - 1$ and therefore $k(A) = s$.

II. CHARACTERIZATION OF IRREDUCIBLE BOOLEAN MATRICES WITH LARGEST INDICES OF CONVERGENCE

4. A First Necessary Condition

The following two lemmas concern the conditions for two matrices of the form (1.1) to be permutation similar.

**Lemma 4.1.** Let $A = (n_1, A_1, n_2, \ldots, n_p, A_p, n_1)$ and $X = (n_1, X_1, n_2, X_2, \ldots, n_p, X_p, n_1)$ be two Boolean matrices with $n_1 + \cdots + n_p = n$. Suppose there exist permutation matrices $Q_1, Q_2, \ldots, Q_p$ (with suitable sizes) such that $X_i = Q_i^T A_i Q_{i+1}$ for $i = 1, \ldots, p-1$ and $X_p = Q_p^T A_p Q_1$. Then $A \simeq X$.

**Proof.** Let $P = \text{diag}[Q_1, Q_2, \ldots, Q_p] \in \mathbb{P}_n$. Then it can be directly verified that $X = PTAP$ and thus $A \simeq X$.

**Lemma 4.2.** Let $A = (n_1, A_1, n_2, A_2, \ldots, n_p, A_p, n_1)$ and $X = (n_1, X_1, n_2, X_2, \ldots, n_p, X_p, n_1)$ be irreducible matrices with period $p$ where $n_1 = n_2 = \cdots = n_s - r + 1$ and $n_{s+1} = \cdots = n_p = r$ for some $s$, $1 \leq s \leq p - 1$. Then $A \simeq X$ if and only if there exist permutation matrices $Q_1, Q_2, \ldots, Q_p$ (with
size \(n_1, \ldots, n_p\) respectively) such that \(X_i = Q_i^T A_i Q_{i+1}^T\) for \(i = 1, \ldots, p - 1\) and \(X_p = Q_p^T A_p Q_1^T\).

Proof. The sufficiency part follows from Lemma 4.1. To prove the necessity, we note that if \(A \approx X\), then the associated digraphs \(D(A)\) and \(D(X)\) are isomorphic. Now \(V(D(A)) = V_1 \cup V_2 \cup \cdots \cup V_p\) and \(V(D(X)) = V_1 \cup \tilde{V}_2 \cup \cdots \cup \tilde{V}_p\), where each subset \(V_i\) or \(\tilde{V}_i\) corresponds to the set of rows of the block \(A_i\) or \(X_i\). Let \(|V_i| = |\tilde{V}_i| = n_i\) and thus \(|V_1| = \cdots = |V_s| = r + 1 = |\tilde{V}_1| = \cdots = |\tilde{V}_s|\) and \(|V_{s+1}| = \cdots = |V_p| = r = |\tilde{V}_{s+1}| = \cdots = |\tilde{V}_p|\). From the hypothesis, both \(A\) and \(X\) are of the form (1.1); therefore we see that the length of any walk from any vector of \(V_i\) or \(\tilde{V}_i\) to any vertex of \(V_j\) or \(\tilde{V}_j\) in \(D(A)\) or \(D(X)\) is congruent to \(j - i\) modulo \(p\). These subsets \(V_1, \ldots, V_p\) or \(\tilde{V}_1, \ldots, \tilde{V}_p\) are called the imprimitive parts of the vertex sets of the strong digraphs \(D(A)\) or \(D(X)\). Since any isomorphism \(\sigma\) between \(D(A)\) and \(D(X)\) maps an imprimitive part of \(D(A)\) to an imprimitive part of \(D(X)\) with the same size and \(\sigma\) keeps the cyclic order of these imprimitive parts, we see that \(\sigma\) must map \(V_i\) to \(\tilde{V}_i\) for \(i = 1, \ldots, p\). It follows that if \(X = P^T A P\), then the permutation matrix \(P\) must be of the form \(P = \text{diag}(Q_1, \ldots, Q_p)\), and thus \(X_i = Q_i^T A_i Q_{i+1}^T\) for \(i = 1, \ldots, p - 1\) and \(X_p = Q_p^T A_p Q_1^T\). This completes the proof. 

The following corollary is a direct consequence of the above lemma.

**Corollary 4.1.** Suppose \(A = (n_1, A_1, \ldots, n_p, A_p, n_1)\), \(X = (n_1, X_1, \ldots, n_p, X_p, n_1)\) where both \(A\) and \(X\) are irreducible matrices with period \(p\) and \(n_1 = \cdots = n_s = r + 1, n_{s+1} = \cdots = n_p = r\) for some \(1 \leq s \leq p - 1\). Then:

(i) For each \(i, m\) with \(1 \leq i \leq p\), \(A_i(m)\) is permutation equivalent to \(X_i(m)\) (i.e. there exist permutation matrices \(P, Q\) with suitable sizes such that \(A_i(m) = PX_i(m)Q\)).

(ii) For each \(i\) with \(1 \leq i \leq p\), \(A_i(p) \approx X_i(p)\).

The following theorem gives our first necessary condition for \(A \in L_{n,p}\).

**Theorem 4.1.** Let \(n, p\) be positive integers with \(n = rp + s\), where \(0 \leq s \leq p - 1\) and \(r = \lceil n/p \rceil > 1\). Suppose \(A \in L_{n,p}\). Then there exist positive integers \(n_1, \ldots, n_p\) and matrices \(A_1, \ldots, A_p\) such that the following conditions are satisfied:

(i) \(A \approx (n_1, A_1, \ldots, n_p, A_p, n_1)\) and \(A_i(pw_i + s - 1) \neq I\), where \(J\) is a matrix whose entries are all 1.

(ii) \(n_1 = \cdots = n_s = r + 1\) and \(n_{s+1} = \cdots = n_p = r\).

(iii) Each matrix \(A_i(p)\) (\(i = 1, \ldots, p\)) is primitive with exponents \(\gamma(A_i(p)) = w_i + 1 = r^2 - 2r + 3\) for \(1 \leq i \leq s\) and \(\gamma(A_i(p)) = w_i = r^2 - 2r + 2\) for \(s + 1 \leq j \leq p\).
Proof.  (i): $A \in L_{n,p}$, so $A \in \mathbb{B}_n$ is irreducible with period $p$, and $A$ is permutation similar to an imprimitive normal form $(n_1, A_1, \ldots, n_p, A_p, n_1)$ for some $n_1, \ldots, n_p$ and $A_1, \ldots, A_p$. Now $A \in L_{n,p}$, so $k(A) = pw_r + s$. By Lemma 2.1 we may assume, without loss of generality, that $\gamma A_j(pw_r + s - 1) \neq J$.

(ii): First we show that all $n_i \geq r$ $(i = 1, \ldots, p)$. Suppose on the contrary that $n_j < r - 1$ for some $j$ with $1 \leq j \leq p$. Then $\gamma(A_j(p)) \leq w_{n_j} \leq w_{r-1}$ and $k(A) \leq p\gamma(A_j(p)) + p - 1 < p(w_{r-1} + 1) \leq pw_r$ by Lemma 2.2. This contradicts the hypothesis that $A \in L_{n,p}$, so $n_i \geq r$ for $i = 1, \ldots, p$. Now for $1 \leq i \leq s$, $A_1(pw_r + s - 1) = A_1(i - 1)A_i(pw_r)A_{pw_r + i}(s - i) \neq J$. Therefore $A_j(pw_r) = (A_j(p))^{w_r} \neq J$, and thus the primitive matrix $A_j(p)$ has to have the order $n_i \geq r + 1$ $(1 \leq i \leq s)$. Combining the three conditions $n_i \geq r$ $(i = 1, \ldots, p)$, $n_i \geq r + 1$ $(i = 1, \ldots, s)$, and $\sum_{i=1}^p n_i = rp + s$, we obtain that $n_1 = \cdots = n_s = r + 1$ and $n_s + 1 = \cdots = n_p = r$.

(iii): From the above proof, we have $(A_j(p))^{w_r} \neq J$ for $1 < i < s$; thus $\gamma(A_j(p)) \geq w_r + 1$ for $1 \leq i \leq s$. On the other hand, $\gamma(A_j(p)) \leq w_{n_j} = w_r$ for $s + 1 \leq j \leq p$. Now $|\gamma(A_j(p)) - \gamma(A_j(p))| \leq 1$ for all $1 \leq i, j \leq p$ by Theorem 1.1, so we must have $\gamma(A_j(p)) = w_r + 1$ for $1 < i < s$ and $\gamma(A_j(p)) = w_r$ for $s + 1 \leq j \leq p$. This completes the proof.

5. The Case $s = 0$ $(r > 1)$

In this subsection we will characterize the extreme matrices in $L_{n,p}$ in the case $s = 0$ $(n = rp)$ and $r > 1$. Recall the notation

$$W_r = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix} \in \mathbb{B}_r, \quad \left( r > 1, \ W_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right).$$

First we prove the following lemma:

**Lemma 5.1.** Let $X, Y \in \mathbb{B}_r$ such that both $XY$ and $YX$ are primitive with the exponents $\gamma(XY) = \gamma(YX) = w_r$. Then one of $X$ and $Y$ is a permutation matrix and $(X,Y) \simeq (I_r, W_r)$—i.e.,

$$\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \simeq \begin{bmatrix} 0 & I_r \\ W_r & 0 \end{bmatrix}.$$  

**Proof.** Since $\gamma(XY) = \gamma(YX) = w_r$, there exist $P_1, Q_1 \in \mathbb{P}_r$ such that $XY = P_1^T W_r P_1$ and $YX = Q_1^T W_r Q_1$ (Theorem 1.2). Taking $\tilde{X} = P_1^T X Q_1$ and
\( Y = Q_1^T Y P_1^T \), we obtain \( X Y = W_r \) and \( Y X = W_r \). The second equality implies that every row of \( W_r \) is a sum of some rows of \( X \), so \( X \) contains rows \( e_2, \ldots, e_s \), and another row \( \alpha \) with \( e_1 < \alpha \leq e_1 + e_0 \). Now suppose \( \bar{X} \notin P_r \) and \( \alpha = e_1 + e_2 \) and \( \bar{X} = R W_r \) for some \( R \in P_r \). Using the fact \( Y X = W_r \) again, we see either

\[
\bar{Y} R = I_r \quad \text{or} \quad \bar{Y} R = \begin{bmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \end{bmatrix} = I_r + E_{r1}.
\]

If \( \bar{Y} R = I_r + E_{r1} \), then \( \bar{X} \bar{Y} = R W_r (I_r + E_{r1}) R^T \) will contain \( r + 2 \) nonzero entries and hence \( \bar{X} \bar{Y} \neq W_r \), a contradiction. This shows that \( \bar{Y} R = I_r \) and \( \bar{Y} = R^T \in P_r \). Thus one of \( \bar{X} \) and \( \bar{Y} \), hence one of \( X \) and \( Y \), is a permutation matrix.

Now \( (X, Y) \approx (P_1^T X Q_1, Q_1^T Y P_1) \approx (\bar{X}, \bar{Y}) \) by Lemma 2.2. If \( \bar{X} = Q \in P_r \), then \( \bar{Y} = Q^T W_r \) and \( (\bar{X}, \bar{Y}) \approx (I_r, W_r) \) by Lemma 2.2. If \( \bar{Y} = P \in P_r \), then \( X = P^T W_r \) and \( (\bar{X}, \bar{Y}) \approx (W_r, I_r) \approx (I_r, W_r) \). Thus in any case \( (X, Y) \approx (\bar{X}, \bar{Y}) \approx (I_r, W_r) \), and the proof is completed.

Theorem 5.1 tells us that in the case \( s = 0 \) \( (r > 1) \), the extreme matrices in \( L_{n,p} \) is unique up to permutation similarity.
6. **The Case** \( p = 2, s = 1 \ (r > 1) \)

In this subsection we will characterize the extreme matrices in the case \( p = 2 \) and \( s = 1 \ (n = 2r + 1 > 3) \).

**Lemma 6.1.** Let \( r > 1; \) let \( X_{(r+1) \times r} \) and \( Y_{r \times (r+1)} \) be Boolean matrices each of which contains no zero row and no zero column. Suppose \( YX = W \), \( X \) contains a row \( e_1 \), and \( Y \) contains a column \( e_1^T \). Then one of the following three assertions holds:

(i) there exists \( Q \in \mathbb{P}_{r+1} \) such that

\[
X = QH_1, \quad Y = Y_1Q^T, \tag{6.1}
\]

(ii) there exists \( Q \in \mathbb{P}_{r+1} \) such that

\[
X = QH_1, \quad Y = Y_2Q^T, \tag{6.2}
\]

(iii) there exists \( Q \in \mathbb{P}_{r+1} \) such that

\[
X = QH_2, \quad Y = Y_1Q^T, \tag{6.3}
\]

where

\[
H_1 = \begin{bmatrix}
0 & 1 \\
& \ddots \\
& & \ddots & 1 \\
& & 1 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}_{(r+1) \times r},
\]

\[
H_2 = \begin{bmatrix}
0 & 1 \\
& \ddots \\
& & \ddots & 1 \\
& & 1 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix}_{(r+1) \times r},
\]

\[
Y_1 = \begin{bmatrix}
1 \\
& \ddots \\
& & 1 \\
1 & 1 \\
1 & 1
\end{bmatrix}_{r \times (r+1)}, \quad Y_2 = \begin{bmatrix}
1 \\
& \ddots \\
& & 1 \\
1 & 0
\end{bmatrix}_{Y \times (r+1)}.
\]
Proof. By the hypothesis $YX = W_r$, we know that every row of $W_r$ is a
sum of some rows of $X$ and so is the row $e_i$ ($i = 2, \ldots, r$). This shows that $X$
contains rows $e_2, e_3, \ldots, e_r$. Also $X$ contains a row $e_1$ by hypothesis. Therefore
$r$ rows of the matrix $X$ are already determined.

Let $\alpha$ be the remaining row of $X$. We will show that $\alpha \geq e_2$. Suppose this
is not the case; then the 2nd column of $X$ is some unit vector $e_i^T$ (of
dimension $r + 1$), and thus $(i$th column of $Y) = (2$nd column of $W_r) = e_i^T + e_r^T$.
By hypothesis, $Y$ also contains another column, say the $j$th column ($j \neq i$),
which is equal to $e_i^T$. So $(1$st row of $Y) \geq e_i + e_j$, and $e_\alpha = (1$st row of
$W_r) \geq (i$th row of $X) + (j$th row of $X)$. This implies that $(i$th row of $X) = (j$th row of $X) = e_2$, a contradiction. Thus we have proved that $\alpha \geq e_2$.
Since we must also have $\alpha \leq \text{some row of } W_r$, it follows that $e_2 \leq \alpha \leq e_1 + e_2$
and therefore $\alpha = e_2$ or $\alpha = e_1 + e_2$.

Case 1: $\alpha = e_2$. Then the rows of $X$ are $e_1, e_2, e_3, \ldots, e_r$, and so
there exists $Q \in \mathbb{P}_{r+1}$ such that $X = QH_1$. By the fact $YX = (YQ)H_1 = W_r$, we
see that

$$YQ = \begin{bmatrix}
* & 1 & \cdots & *

\end{bmatrix},
\begin{bmatrix}
* \\
\cdots \\
* & 1 & *
\end{bmatrix},$$

where the $2 \times 2$ submatrix

$$C = \begin{bmatrix}
* & * \\
* & *
\end{bmatrix}$$

(consisting of the four corners of $YQ$) contains no zero row and no zero
column. Also note that $Y$ (hence $YQ$) must contain a column $e_i^T$. So the last
row of $C$ has a zero entry and

$$C \in \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\}.$$

Let

$$R = \begin{bmatrix}
0 & 1 & 1 \\
1 & \cdots & 1 \\
1 & 1 & 0
\end{bmatrix} \in \mathbb{P}_{r+1};$$
then \( YQ \in \{ Y_1, Y_1R, Y_2, Y_2R, R \} \). Thus we see that one of the following four conditions holds:

\[
\begin{align*}
X &= QH_1, & Y &= Y_1Q^T, \\
X &= QH_1, & Y &= Y_1RQ^T, \\
X &= QH_1, & Y &= Y_2Q^T, \\
X &= QH_1, & Y &= Y_2RQ^T.
\end{align*}
\]

(6.1) (6.1') (6.2) (6.2')

But \( R = R^T \) and \( RH_1 = H_1 \), so we can write (6.1') as \( X = QR^TH_1 = Q_1H_1 \), \( Y = Y_1RQ^T = Y_1Q_1^T \) with \( Q_1 = QR^T \), and write (6.2') as \( X = Q_1H_1, Y = Y_2Q_1^T \) with \( Q_1 = QR^T \). Thus (6.1) or (6.2) holds.

Case 2: \( \alpha = e_1 + e_2 \). Then the rows of \( X \) are \( e_1, e_2, \ldots, e_r, e_1 + e_2 \), and so there exists \( Q \in \mathbb{P}_{r+1} \) such that \( X = QH_2 \). Now \( Y \) (hence \( YQ \)) contains a column \( e_1^T \) and contains no zero column, so the equation \( YX = (YQ)H_2 = W_r \) implies that \( YQ = Y_1 \), or \( Y = Y_1Q^T \). Thus in this case (6.3) holds and the proof is completed.

**Lemma 6.2.** Let \( r > 1 \), and let \((A_1)_{r(\times)(r+1)}, \) and \((A_2)_{r(\times)(r+1)}\) be Boolean matrices such that both \( A_1A_2 \) and \( A_2A_1 \) are primitive with the exponents \( \gamma(A_2A_1) = w_r \) and \( \gamma(A_1A_2) = w_r + 1 \). Let

\[
M_1 = \begin{bmatrix} 0 & H_1 \\ Y_1 & 0 \end{bmatrix} = (H_1, Y_1), \quad M_2 = (H_1, Y_2), \quad M_3 = (H_2, Y_1) \in \mathbb{B}_{2r+1},
\]

where \( H_1, H_2, Y_1, Y_2 \) are as defined in Lemma 6.1. Then there exist \( P \in \mathbb{P}_r \) and \( Q \in \mathbb{P}_{r+1} \) such that

\[
(QTA_1P^T, PA_2Q) \in \{ M_1, M_2, M_3 \}.
\]

(6.4)

**Proof.** By hypothesis \( A_2A_1 \in \mathbb{B}_r \) and \( \gamma(A_2A_1) = w_r \), so there exists \( P \in \mathbb{P}_r \) such that \( P(A_2A_1)P^T = W_r \). Let \( X = A_1P^T, Y = PA_2; \) then \( YX = W_r, \gamma(XY) = w_r + 1 \). But

\[
(XY)^{w_r} = X(YX)^{w_r - 1} = X(W_r)^{w_r - 1}Y = X \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & \vdots & & 0 \\ \vdots & & & \vdots \\ 1 & & & 1 \end{bmatrix} Y,
\]
so $\gamma(XY) = w_r + 1 \Rightarrow (XY)^{w_r} \neq J \Rightarrow$

$$X \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & 1 \end{bmatrix} Y \neq J$$

$\Rightarrow$ \begin{itemize}
\item $X$ contains a row $e_1$ and $Y$ contains a column $e^T$. (Note that since both $A_1 A_2$ and $A_2 A_1$ are primitive, both $\Lambda_1$ and $\Lambda_2$, and hence both $X$ and $Y$, contain no zero row and no zero column.) Thus $X$ and $Y$ satisfy all the hypotheses of Lemma 6.1. Therefore there exists $Q \in \mathbb{P}_{r+1}$ such that $(Q^TX, YQ) \in \{M_1, M_2, M_3\}$, namely, $(Q^T A_1 P^T, PA_2 Q) \in \{M_1, M_2, M_3\}$. ■
\end{itemize}

The next theorem exhibits all the extreme matrices of $L_{n,p}$ in the case $p = 2$ and $s = 1$ ($n = 2r + 1$ and $r > 1$).

**Theorem 6.1.** Suppose $p = 2$ and $s = 1$ ($n = 2r + 1$, $r > 1$), and $M_1 = (H_1, Y_1)$, $M_2 = (H_1, Y_2)$, $M_3 = (H_2, Y_3)$ are three $n \times n$ Boolean matrices as defined in Lemma 6.2. Then a Boolean matrix $A \in L_{n,p}$ if and only if $A$ is permutation similar to one of $M_1$, $M_2$, and $M_3$.

**Proof.** By [6, Corollary 5.1], we see that $M_1$, $M_2$, $M_3$ are all irreducible with period 2. Note that $Y_1 H_1 = Y_2 H_1 = Y_1 H_2 = W_r$, so

$$(H_1 Y_1)^{w_r} = H_1 W_r^{w_r} Y_1 = H_1 \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & 1 \end{bmatrix} Y_1 \neq J.$$ 

Similarly $(H_1 Y_2)^{w_r} \neq J$ and $(H_2 Y_1)^{w_r} \neq J$. So $k(M_i) \geq 2w_r + 1$ for $i = 1, 2, 3$ by Lemma 2.1. On the other hand, we have the upper bound $k(M_i) \leq 2w_r + 1$, so $k(M_i) = 2w_r + 1$ and thus $M_i \in L_{n,p}$ for $i = 1, 2, 3$. This proves the sufficiency part of the theorem.

Conversely, suppose $A \in L_{n,p}$; then $A$ is permutation similar to some matrix $(n_1, A_1, n_2, A_2, n_1)$ with $n_1 = r + 1$, $n_2 = r$, $\gamma(A_1 A_3) = w_r + 1$, and $\gamma(A_2 A_1) = w_r$. By Lemma 6.2, there exist $P \in \mathbb{P}_r$ and $Q \in \mathbb{P}_{r+1}$ such that

$$(Q^T A_1 P^T, PA_2 Q) \in \{M_1, M_2, M_3\};$$

hence $A \cong (A_1, A_2) \cong (Q^T A_1 P^T, PA_2 Q) \in \{M_1, M_2, M_3\}$ and $A$ is permutation similar to one of $M_1$, $M_2$, and $M_3$. ■
By using Corollary 4.1(i), we can easily verify that each pair of the matrices \( M_1, M_2, \) and \( M_3 \) are not permutation similar (actually \( M_2 \approx M_3^T \)). Therefore we have the following:

**Theorem 6.2.** If \( p = 2 \) and \( n = 2r + 1 \) \((r > 1)\), then there are exactly three permutation similar classes of extreme matrices in \( L_{n,p} \); each of them contains one and only one matrix in the set \( \{ M_1, M_2, M_3 \} \).  

7. **The Case of General \( p \) and \( 1 \leq s \leq p - 1 \) \((r > 1)\)**  

In this subsection, we will exhibit all the matrices in \( L_{n,p} \) for general \( p \) and \( 1 \leq s \leq p - 1 \) (we still assume that \( r > 1 \)). In particular, we will show that there are exactly \( 2^s + s \cdot 2^{s-1} \) extreme matrices in \( L_{n,p} \) up to permutation similarity (or up to the isomorphism of their associated digraphs).

The following lemmas give us further necessary conditions for \( A \in L_{n,p} \).

**Lemma 7.1.** Suppose \( A = (n_1, A_1, \ldots, n_p, A_p, n_1) \), where \( A_1, \ldots, A_p \) and \( n_1, \ldots, n_p \) satisfy all the three conditions (i)-(iii) in Theorem 4.1. Let \( X = A_s \cdots A_{p-1} \), \( Y = A_p A_1 \cdots A_{s-1} \). Then there exist \( P \in \mathbb{P}_r \) and \( Q \in \mathbb{P}_{r+1} \) such that \((Q^T XP^T, PYQ) \in \{ M_1, M_2, M_3 \}\).

**Proof.** By conditions (i)-(iii) in Theorem 4.1, we see that \( X \) is of the size \((r+1) \times r\) and \( Y \) is of the size \( r \times (r+1) \), and \( \gamma(XY) = w_r \), \( \gamma(YX) = w_r + 1 \). Therefore the result of this lemma follows directly from Lemma 6.2. ■

**Lemma 7.2.** Suppose \( \tilde{A} = (n_1, \tilde{A}_1, \ldots, n_p, \tilde{A}_p, n_1) \), where \( \tilde{A}_1, \ldots, \tilde{A}_p \) and \( n_1, \ldots, n_p \) satisfy all the three conditions (i)-(iii) in Theorem 4.1. Then there exists \( A = (n_1, A_1, \ldots, n_p, A_p, n_1) \) such that:

(i) \( A \approx \tilde{A} \);
(ii) \( A_1, \ldots, A_p \) also satisfy conditions (i)-(iii) in Theorem 4.1;
(iii) \((A_2 \cdots A_{p-1}, A_p A_1 \cdots A_{s-1}) \in \{ M_1, M_2, M_3 \}\);
(iv) \( A_1 \ldots A_{s-1} \) contains a row \( e_r \).

**Proof.** By Lemma 7.1, there exist \( P \in \mathbb{P}_r \) and \( Q \in \mathbb{P}_{r+1} \) such that \((Q^T \tilde{A}_s \cdots \tilde{A}_{p-1} P, P \tilde{A}_p \tilde{A}_1 \cdots \tilde{A}_{s-1} Q) \in \{ M_1, M_2, M_3 \}\). Let \( \Lambda_{s-1} = \tilde{A}_{s-1} Q \), \( \Lambda_s = Q^T \tilde{A}_s \), \( A_{p-1} = A_{p-1} P^T \), \( A_p = P \tilde{A}_p \), and \( A_i = \tilde{A}_i \) for all other \( i \), and let \( A = (n_1, A_1, \ldots, n_p, A_p, n_1) \). Then (i) is satisfied by Lemma 4.1, and (ii), (iii) are satisfied by the definition of each \( A_i \).
Finally, by the definition of $M_1$, $M_2$, and $M_3$, there exist $i$ and $j$ such

$$1 \leq i, j \leq 2. \quad (7.1)$$

$$i, j \text{ are not both } 2, \quad (7.2)$$

and

$$A_1(pw_r + s - 1) = A_1(s - 1)(H_iY_j)^{w_r} = A_1(s - 1)H_iW^{w_r - 1}Y_j$$

$$= A_1(s - 1)H_i \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & \vdots & & 1 \\ \vdots & & & \vdots \\ 1 & & & 1 \end{bmatrix} Y_j \neq J, \quad (7.3)$$

so $A_1 \cdots A_{s - 1} = A_1(s - 1)$ contains a row $e_r$ and (iv) is established.

Next we study condition (iii) of Lemma 7.2 in the following lemmas.

**Lemma 7.3.** Let $A_{s+1}, \ldots, A_{p-1} \in B_r$ and $(A_s)_{(r+1) \times r}$ be Boolean matrices. Suppose $A_sA_{s+1} \cdots A_{p-1} \in \{H_1, H_2\}$, where $H_1$ and $H_2$ are as defined in Lemma 6.1. Then all the matrices $A_{s+1}, \ldots, A_{p-1}$ are permutation matrices.

**Proof.** Since $H_1$ (or $H_2$) contains the rows $e_1, \ldots, e_r$, each of which is a sum of some rows of $A_{p-1}$, it follows that the rows of $A_{p-1}$ are $e_1, \ldots, e_r$ and thus $A_{p-1} \in P_r$. By repeatedly using this argument we obtain that all the matrices $A_{s+1}, \ldots, A_{p-1}$ are permutation matrices.

**Lemma 7.4.** Let $D_1 (= A_1, \ldots, A_{s-1})$ and $D_2 (= A_s)$ be Boolean matrices with sizes $(r + 1) \times (r + 1)$ and $r \times (r + 1)$, each of them containing no zero row and no zero column. Suppose $D_1$ contains a row $e_r$. Let $F_1$, $F_2$, and $F_3$ be the following $(r + 1) \times (r + 1)$ Boolean matrices:

$$F_1 = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & 1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & 1 \end{bmatrix},$$

$$F_3 = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \vdots & 1 \end{bmatrix}. \quad (7.4)$$
Then we have:

(i) If $D_2 D_1 = Y_1$, then there exists $Q \in P_{r+1}$ such that

$$D_2 Q^T = Y_1$$

and

$$Q D_1 \in \{ I_{r+1}, F_1 \}.$$  \hspace{1cm} (7.5)

(ii) If $D_2 D_1 \in \{ Y_1, Y_2 \}$, then either

$$D_2 Q^T = Y_1, \quad Q D_1 \in \{ I_{r+1}, F_1, F_2, F_3 \} \quad \text{for some} \quad Q \in P_{r+1} \quad (7.6)$$

or

$$D_2 Q^T = Y_2, \quad Q D_1 \in \{ I_{r+1}, F_3 \} \quad \text{for some} \quad Q \in P_{r+1}. \quad (7.7)$$

Proof. (i): If $D_2 D_1 = Y_1$, then every row of $Y_1$ is a sum of some rows $D_1$. Now $Y_1$ contains rows $e_1, \ldots, e_r$, so $D_1$ must contain rows $e_1, \ldots, e_{r-1}$. By the hypothesis, $D_1$ also contains a row $e_r$. Let $\alpha$ be the remaining row of $D_1$; then $e_{r+1} < \alpha < e_r + e_{r+1}$. If $\alpha = e_{r+1}$, then $Q D_1 = I_{r+1}$ for some $Q \in P_{r+1}$ and $D_2 D_1 = D_2 Q^T = Y_1$. If $\alpha = e_r + e_{r+1}$, then $Q D_1 = F_1$ for some $Q \in P_{r+1}$, and $D_2 D_1 = (D_2 Q^T) F_1 = Y_1$. Since $D_2 Q^T$ contains no zero column, we must have $D_2 Q^T = Y_1$ and this proves (i).

(ii): If $D_2 D_1 \in \{ Y_1, Y_2 \}$, then by using a similar argument we see that $D_1$ contains rows $e_2, e_3, \ldots, e_r$. Let $\alpha, \beta$ be the other two rows of $D_1$. Then one of them, say $\alpha$, satisfies $e_1 < \alpha < e_1 + e_{r+1}$, and the other one, $\beta$, satisfies $e_{r+1} < \beta < e_r + e_{r+1}$. Thus $Q D_1 \in \{ I_{r+1}, F_1, F_2, F_3 \}$ for some $Q \in P_{r+1}$. In case $Q D_1 \in \{ I_{r+1}, F_3 \}$, we have $(D_2 Q^T) (Q D_1) \in \{ Y_1, Y_2 \} \Rightarrow D_2 Q^T \in \{ Y_1, Y_2 \}$. In case $Q D_1 \in \{ F_1, F_2 \}$, then $(D_2 Q^T) (Q D_1) \in \{ Y_1, Y_2 \} = D_2 Q^T = Y_1$. Thus either (7.6) or (7.7) is true, and this proves (ii).

Lemma 7.5. Suppose $X_1, \ldots, X_k \in B_{r+1}$ with $X_1 X_2 \cdots X_k \in \{ I_{r+1}, F_j \}$ ($j = 1$ or 3). Then there exist $P_1, \ldots, P_{k-1} \in P_{r+1}$ and $Z_i \in \{ I_{r+1}, F_j \}$ ($i = 1, \ldots, k$) such that

$$X_1 = Z_1 P_1, \quad X_2 = P_1^T Z_2 P_2, \ldots, \quad X_k = P_{k-1}^T Z_k.$$  \hspace{1cm} (7.8)

Proof. We give the proof for the case $j = 1$ (the case $j = 3$ is similar). If $X_1 \cdots X_k = I_{r+1}$, then all $X_i$'s are permutation matrices and we can take $Z_1 = \cdots = Z_k = I_{r+1}$ and $P_1 = X_1, \quad P_2 = X_1 X_2, \ldots, \quad P_{k-1} = X_1 \cdots X_{k-1}$ to get (7.8). If $X_1 \cdots X_k = F_1$, then the rows of $X_k$ are either $e_1, \ldots, e_{r+1}, e_r$, or $e_1, \ldots, e_r, e_{r+1}$, in which case $X_k = P_{k-1}$, or $e_1, \ldots, e_r, e_{r+1}$, in which case $X_k = P_{k-1}^T F_1$ (for some $P_{k-1} \in P_{r+1}$). Let $X_{k-1} = X_{k-1} P_{k-1}^T$. If $X_k = P_{k-1}$, then $X_1 \cdots
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\[ X_{k-2} \bar{X}_{k-1} = F_1 \] and we may thus use induction on \( k \). If \( X_k = P^T_{k-1} F_1 \), then

\[(X_1 \cdots X_{k-2} \bar{X}_{k-1})F_1 = F_1 \] and thus \( X_1 \cdots X_{k-2} \bar{X}_{k-1} \in \{ I_{r+1}, F_1 \} \). Therefore in this case we can also use induction to get (7.8).

Obviously, the converse of Lemma 7.5 is also true.

**Lemma 7.6.** Suppose \( X_1, \ldots, X_k \in \mathbb{B}_{r+1} \) with \( X_1 X_2 \cdots X_k \in \{ I_{r+1}, F_1, F_2, F_3 \} \). Then there exist \( P_1, \ldots, P_{k-1} \in \mathcal{P}_{r+1} \) and \( Z_i \in \{ I_{r+1}, F_1, F_2, F_3 \} \) \((i = 1, \ldots, k)\) such that (7.8) holds and \( Z_1, \ldots, Z_k \) also satisfy the following conditions:

(i) At most one of the \( Z_i \)'s is \( F_2 \).

(ii) If \( Z_{i_1} = F_{j_1} \) and \( Z_{i_2} = F_{j_2} \) with \( i_1 < i_2 \), then \( j_1 < j_2 \).

**Proof.** If \( X_1 \cdots X_k \in \{ I_{r+1}, F_1, F_2, F_3 \} \), then by the fact that each row of \( X_1 \cdots X_k \) is a sum of some rows of \( X_k \), we obtain that \( X_k = P^T_{k-1} Z_k \) for some \( P_{k-1} \in \mathcal{P}_{r+1} \) and \( Z_k \in \{ I_{r+1}, F_1, F_2, F_3 \} \). Let \( \bar{X}_{k-1} = X_{k-1} P^T_{k-1} \); then

\[ X_1 \cdots X_k = (X_1 \cdots X_{k-2} \bar{X}_{k-1})Z_k \in \{ I_{r+1}, F_1, F_2, F_3 \} \].

If \( Z_k = I_{r+1} \), then

\[ X_1 \cdots X_k \in \{ I_{r+1}, F_1, F_2, F_3 \} \] and we can use induction on \( k \). If \( Z_k = F_3 \), then also \( X_1 \cdots X_{k-2} \bar{X}_{k-1} \in \{ I_{r+1}, F_1, F_2, F_3 \} \) and we can again use induction on \( k \). If \( Z_k = F_1 \) or \( Z_k = F_2 \), then we can verify that \( X_1 \cdots X_{k-2} \bar{X}_{k-1} \in \{ I_{r+1}, F_1 \} \). By using the conclusion of Lemma 7.5 we see that there exist \( P_1, \ldots, P_{k-1} \in \mathcal{P}_{r+1} \) and \( Z_1, \ldots, Z_{k-1} \in \{ I_{r+1}, F_1 \} \) (together with above \( P_{k-1} \) and \( Z_k \)) such that (7.8) is true. From the above argument we also see that if \( Z_k = F_1 \) or \( F_2 \), then \( Z_i \neq F_2 \) and \( Z_i \neq F_3 \) for \( 1 \leq i \leq k - 1 \). Thus by induction we see that in the sequence \( Z_1, \ldots, Z_k \), no \( F_2, F_3 \) precedes \( F_1 \) and no \( F_2, F_3 \) precedes \( F_2, F_3 \); therefore (i) and (ii) hold and the proof is completed.

Note that the converse of Lemma 7.6 is also true.

In order to give a characterization of the matrices in \( L_{n_p} \) in the case for general \( p \) and \( 1 \leq s \leq p - 1 \) \((n = pr + s \text{ and } r > 1)\), we define some subsets in \( \mathbb{B}_n \).

**Definition 7.1.** We define:

\[ T_1 = \{(A_1, \ldots, A_p) | A_i \in \{ I_{r+1}, F_1 \} \text{ for } 1 \leq i \leq s - 1, \quad A_s = H_2, \quad A_{s+1} = \cdots = A_{p-1} = I_r, \quad \text{and } A_p = Y_1 \}, \]

\[ T_2 = \{(A_1, \ldots, A_p) | \begin{array}{l}
(1) \quad A_i \in \{ I_{r+1}, F_1, F_2, F_3 \} \text{ for } 1 \leq i \leq s - 1, \\
(2) \quad \text{the sequence } Z_1 = A_1, \ldots, Z_i = A_i, \ldots, Z_{s-1} = A_{s-1} \text{ satisfies conditions (i) and (ii) of Lemma 7.6,} \\
(3) \quad A_s = H_1, \quad A_{s+1} = \cdots = A_{p-1} = I_r, \quad \text{and } A_p = Y_1 \} \}, \]
and define $T = T_1 \cup T_2 \cup T_3$.

It is clear that $T_1$, $T_2$, and $T_3$ are pairwise disjoint (by looking at $A_s$ and $A_p$). If $A = (A_1, \ldots, A_p) \in T$, then $A$ is irreducible with period $p$ by [6, Corollary 5.1]. Now $A_s(p) = A_s \cdots A_p A_1 \cdots A_{s-1} \in \{H_1 Y_1, H_1 Y_2, H_2 Y_1\}$ (since $A \in T$); therefore $A_s(p) = H_1 Y_j$ for some $1 \leq i, j \leq 2$, but $i, j$ are not both equal to 2. Also, by (the converse of) Lemma 7.5 and 7.6 or by a direct calculation, we see that $A_1 A_2 \cdots A_{s-1} \in \{I_{r+1}, F_1, F_2, F_3\}$, which means that $A_1 A_2 \cdots A_{s-1}$ contains a row $e_r$. So

$$A_1(p w_r + s - 1) = A_1 A_2 \cdots A_{s-1}(A_s(p))^{w_r}$$

$$= A_1 A_2 \cdots A_{s-1} H_i Y_j^{w_r - 1} Y_j$$

$$= A_1 A_2 \cdots A_{s-1} H_i W_r^{w_r - 1} Y_j$$

$$= A_1 A_2 \cdots A_{s-1} H_i \left[\begin{array}{cccc}0 & 1 & \cdots & 1 \\1 & \vdots & \ddots & 1 \\ \vdots & \ddots & \ddots & \vdots \\1 & \cdots & \cdots & 1 \end{array}\right]_{Y_j \neq J},$$

because $A_1 A_2 \cdots A_{s-1} H_i$ contains a row $e_i$ and $Y_j$ contains a column $e_j^T$. This shows that $A \in T \Rightarrow k(A) = p w_r + s \Rightarrow A \in I_{n,p}$ and hence $T \subseteq I_{n,p}$.

Now let us compute $|T_1|$, the cardinality of $T$. It is clear that $|T_1| = 2^{s - 1}$ and $|T_3| = 2^{s - 1}$. To compute $|T_2|$, we consider the following counting problem.

**Lemma 7.7.** Let $X = \{a, b, c, e\}$ be a set with four elements. Let $f_k$ be the number of the $k$-permutations (with repetitions allowed) of elements of $X$ satisfying the following conditions:

(i) $b$ appears at most once.
(ii) $c$ does not precede $a$ or $b$, and $b$ does not precede $a$ (or $b$).

Then $f_k = (k + 1) \cdot 2^k$.

**Proof.** Suppose $e$ appears in $k - i$ places and the remaining $i$ places are for $a$, $b$, and $c$ ($i = 0, \ldots, k$). There are exactly $\binom{k}{k-i}$ possibilities for those
e’s. Now for the remaining i places, if b does not appear, then there are i + 1 possibilities, since the number of a’s may equal to 0, 1, . . . , i. If b appears once, then there are i possible places for b, and then all the places preceding b (in the remaining i places not occupied by e) are for a’s, and all the places following b are for c’s. Thus

\[
f_k = \sum_{i=0}^{k} \binom{k}{i} (i + 1 + i) = \sum_{i=0}^{k} \binom{k}{i} + 2 \sum_{i=0}^{k} \binom{i}{i}
\]

\[= 2^k + k \cdot 2^k = (k + 1) \cdot 2^k.\]

By using this result we obtain that \(|T_2| = f_{s-1} = s \cdot 2^{s-1}\) and thus \(|T| = |T_1| + |T_2| + |T_3| = 2^s + s \cdot 2^{s-1}\). The next theorem shows that each pair of the matrices in the set \(T = T_1 \cup T_2 \cup T_3\) are not permutation similar with each other.

**Theorem 7.1.** If \(A, A^* \in T\) and \(A \neq A^*\), then \(A \neq A^*\).

**Proof.** Let \(A = (n_1, A_1, \ldots, n_p, A_p, n_1) \in T\) and \(A^* = (n_1, A_1^*, \ldots, n_p, A_p^*, n_1) \in T\); then \(n_1 - \cdots - n_s - r + 1\) and \(n_{s+1} - \cdots - n_p - r\) by the definition of \(T\). Suppose \(A \cong A^*\) but \(A \neq A^*\). By Corollary 4.1 we see that each \(A_i\) is permutation equivalent to \(A_i^*\) (denoted by \(A_i \sim A_i^*\)), and thus \(A_s \sim A_s^*\) and \(A_p \sim A_p^*\). This means that both \(A\) and \(A^* \in T_j\) for some \(j = 1, 2, 3\). But by assumption \(A \neq A^*\), so \(A_i \neq A_i^*\) for some \(i = 1, \ldots, s - 1\). Now \(A_1 \neq A_1^*\) and \(A_s \sim A_s^*\), so that the only possibility is \(\{A_i, A_i^*\} = (F_1, F_3)\) and \(A, A^* \in T_2\). Suppose \(A_i = F_1\) and \(A_i^* = F_3\); then \(A_i^* \cdots A_{s-1}^*A_s = F_3H_1 = H_1\) and \(A_i \cdots A_{s-1}A_s = F_1H_1\) or \(F_2H_1\). But both \(F_1H_1\) and \(F_2H_1\) are equal to \(H_2\), so \(A_i \cdots A_{s-1}A_s = H_2\), and it is not permutation equivalent to \(A_i^* \cdots A_s^* = H_1\). This contradicts \(A \cong A^*\) (by Corollary 4.1) and completes the proof of the theorem. □

Finally, we prove our main result for the characterization of the (extreme) matrices in \(L_{n,p}\) in the case \(1 \leq s \leq p - 1\) and \(r > 1\).

**Theorem 7.2.** Suppose \(n = rp + s\) with \(r > 1\) and \(1 \leq s \leq p - 1\). Then a Boolean matrix \(A \in L_{n,p}\) if and only if \(A\) is permutation similar to one of the matrices in \(T = T_1 \cup T_2 \cup T_3\).

**Proof.** The sufficiency part follows from the fact that \(T \subseteq L_{n,p}\). Now suppose \(A \in L_{n,p}\). Then by Lemma 7.2, \(A\) is permutation similar to some matrix \((n_1, A_1, \ldots, n_p, A_p, n_1)\) such that all the conditions (i)–(iii) of Theorem
4.1 and (iii), (iv) of Lemma 7.2 are satisfied. By Lemma 7.2(iii) there exist $i, j$ with $1 \leq i, j \leq 2$ and not both $i, j = 2$ such that $A_i \cdots A_{p-1} = H_i$ and $A_p A_i \cdots A_{s-1} = Y_j$. By Lemma 7.3, the matrices $A_{s+1} = Q_{s+1}, \ldots, A_{p-1} = Q_{p-1}$ are all permutation matrices and $A_s = H_i Q_{p-1}^T \cdots Q_{s+1}^T$.

**Case 1**: If $i = 2$, then $j = 1$, $H_i = H_2$, and $Y_j = Y_1$. By Lemma 7.4(i) and Lemma 7.5, there exist $Q_i, P_1, \ldots, P_{s-2} \in \mathbb{P}_{r+1}$ and $Z_i \in \{I_{r+1}, F_1\}$ ($i = 1, \ldots, s-1$) such that $A_p = Y_1 Q_i$, $A_1 = Q_i^T Z_i P_1$, $A_2 = P_1^T Z_2 P_2, \ldots, A_{s-1} = P_{s-2}^T Z_{s-1}$. Thus $A = (A_1, \ldots, A_p) = (Q_i^T Z_i P_1, P_1^T Z_2 P_2, \ldots, P_{s-2}^T Z_{s-1}, H_2 Q_{p-1}^T \cdots Q_{s+1}^T, Q_{s+1}, \ldots, Q_{p-1}, Y_1 Q) \in (Z_1, Z_2, \ldots, Z_{s-1}, H_2, I_r, \ldots, I_r, Y_1) \in T_1$.

**Case 2**: If $i = 1$, then $H_i = H_1$. Let $D_1 = A_1 \cdots A_{s-1}$ and $D_2 = A_p$; then $D_2 D_1 = Y_j \in \{Y_1, Y_2\}$. By Lemma 7.4(ii), we see that either (7.6) or (7.7) is true.

**Subcase 2.1**: (7.7) is true. Then by Lemma 7.5, there exist $Q_i, P_1, \ldots, P_{s-2} \in \mathbb{P}_{r+1}$ and $Z_i \in \{I_{r+1}, F_1, F_2, F_3\}$ ($i = 1, \ldots, s-1$) such that $A_p = Y_2 Q_i$, $A_1 = Q_i^T Z_i P_1$, $A_2 = P_1^T Z_2 P_2, \ldots, A_{s-1} = P_{s-2}^T Z_{s-1}$. Thus

$$A = (A_1, \ldots, A_p) = (Q_i^T Z_i P_1, P_1^T Z_2 P_2, \ldots, P_{s-2}^T Z_{s-1}, H_2 Q_{p-1}^T \cdots Q_{s+1}^T, Q_{s+1}, \ldots, Q_{p-1}, Y_2 Q) \approx (Z_1, Z_2, \ldots, Z_{s-1}, H_1, I_r, \ldots, I_r, Y_2) \in T_3.$$

**Subcase 2.2**: (7.6) is true. Then by Lemma 7.6, there exist $Q_i, P_1, \ldots, P_{s-2} \in \mathbb{P}_{r+1}$ and $Z_i \in \{I_{r+1}, F_1, F_2, F_3\}$ ($i = 1, \ldots, s-1$), where $Z_1, \ldots, Z_{s-1}$ satisfy conditions (i) and (ii) of Lemma 7.6, such that $A_p = Y_1 Q_i$, $A_1 = Q_i^T Z_i P_1$, $A_2 = P_1^T Z_2 P_2, \ldots, A_{s-1} = P_{s-2}^T Z_{s-1}$. Thus

$$A = (A_1, \ldots, A_p) = (Q_i^T Z_i P_1, P_1^T Z_2 P_2, \ldots, P_{s-2}^T Z_{s-1}, H_1 Q_{p-1}^T \cdots Q_{s+1}^T, Q_{s+1}, \ldots, Q_{p-1}, Y_1 Q) \approx (Z_1, Z_2, \ldots, Z_{s-1}, H_1, I_r, \ldots, I_r, Y_1) \in T_2.$$

Combining the above cases, we obtain the desired result.

Now we state our final result of this subsection in the following theorem.

**Theorem 7.3**: If $n = pr + s$ with $0 \leq s \leq p - 1$ and $r > 1$, then there are exactly $2^r + s \cdot 2^{s-1}$ permutation similar classes of extreme matrices in $L_{n,p}$, each of which contains one and only one matrix in the set $T$. 
8. The Case $r = 1$ and $1 \leq s \leq p - 1$

In this final subsection, we solve the remaining (easy) case $r = 1$ and $1 \leq s \leq p - 1$. Recall that in the case $r = 1$, the sharp upper bound for indices of convergence is $k(A) \leq s$ (Theorem 3.1). Thus for $r = 1$ the set of extreme matrices is $L_{n,p} = \{ A \in B_n \mid A \text{ is irreducible with period } p \text{ and } k(A) = s \}$.

Now suppose $n = p + s$ $(1 \leq s \leq p - 1)$ (we exclude the trivial case $n = p$). Let

$$F = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in B_2;$$

then $F$ is an idempotent: $F^2 = F$. Let

$$E = \{(A_1, \ldots, A_p) \mid A_i \in \{ I_2, F \} \text{ for } i = 1, \ldots, s - 1, \ A_1 = I_{2 \times 1}, A_{s+1} = \cdots = A_{p-1} = I_{1 \times 1}, A_p = I_{1 \times 2}\};$$

then $|E| = 2^{s-1}$ and any matrix in $E$ is irreducible with period $p$ [6, Corollary 5.1]. For any $A = (A_1, \ldots, A_p) \in E$, since $A_1A_2 \cdots A_{s-1} \in \{ I_2, F \}$, then $A_1(s - 1) \neq J$ and thus $k(A) = s$. This shows that $A \in E \Rightarrow A \in L_{n,p}$ and therefore $E \subseteq L_{n,p}$.

By Corollary 4.1(i), we also see that if $A = (A_1, \ldots, A_p) \sim A^* = (A_1^*, \ldots, A_s^*)$ and $A, A^* \in E$, then $A_i \sim A_i^*$ for $1 \leq i \leq s - 1$ and hence $A_i = A_i^*$ $(i = 1, \ldots, s - 1)$ by the definition of the set $E$. Thus for $A, A^* \in E$, $A \sim A^* \Rightarrow A = A^*$, which means that no pair of the matrices in the set $E$ are permutation similar with each other.

**Theorem 8.1.** If $n = p + s$ $(1 \leq s \leq p - 1)$, then a Boolean matrix $A \in L_{n,p}$ if and only if $A$ is permutation similar to one of the matrices in $E$.

**Proof.** The sufficiency part follows from the fact that $E \subseteq L_{n,p}$. Now suppose $A \in L_{n,p}$. Then $A$ is permutation similar to some matrix $(n_1, A_1, \ldots, n_p, A_p, n_1)$ such that $n_1 = \cdots = n_s = 2$, $n_{s+1} = \cdots = n_p = 1$, and $A_i(s - 1) \neq J$. Since each $A_i$ contains no zero row and no zero column, we have $A_s = I_{2 \times 1}$, $A_{s+1} = \cdots = A_{p-1} = I_{1 \times 1}$, and $A_p = I_{1 \times 2}$. Now $J \neq A_i(s - 1) = A_1A_2 \cdots A_{s-1} \in B_2$ and $A_i(s - 1)$ contains no zero row and no zero column, so $A_i(s - 1) \sim I_2$ or $A_i(s - 1) \sim F$. By the proof of Lemma 7.5
we see that there exist \( Q, P, P_1, \ldots, P_{s-2} \in \mathbb{P}_2 \) and \( Z_i \in \{ I_2, F \} \) \( (i = 1, \ldots, s - 1) \) such that \( A_1 = Q^T Z_1 P_1 \), \( A_2 = P_1^T Z_2 P_2 \), \( \ldots \), \( A_{s-2} = P_{s-3}^T Z_{s-2} P_{s-2} \) and \( A_{s-1} = P_{s-2}^T Z_{s-1} P \). Note that \( A_s = J_{2 \times 1} = P^T J_{2 \times 1} \), so

\[
A \cong (A_1, \ldots, A_p) = (Q^T Z_1 P_1, P_1^T Z_2 P_2, \ldots, P_{s-3}^T Z_{s-2} P_{s-2}, P_{s-2}^T Z_{s-1} P, P^T J_{2 \times 1}, I_{1 \times 1}, \ldots, I_{1 \times 1}, I_{1 \times 2})
\]

This proves the necessity.

**Theorem 8.2.** Suppose \( n = p + s \) with \( 1 \leq s \leq p - 1 \). Then there are exactly \( 2^{s-1} \) permutation similar classes of extreme matrices in \( L_{n,p} \), each of which contains one and only one matrix in the set \( E \).

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