Necessary conditions for metrics in integral Bernstein-type inequalities

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Abstract

Let $\mathcal{T}_n$ be the set of all trigonometric polynomials of degree at most $n$. Denote by $\Phi^+$ the class of all functions $\varphi: (0, \infty) \to \mathbb{R}$ of the form $\varphi(u) = \psi(\ln u)$, where $\psi$ is nondecreasing and convex on $(-\infty, \infty)$. In 1979, Arestov extended the classical Bernstein inequality $\|T_n'\|_C \leq n\|T_n\|_C$, $T_n \in \mathcal{T}_n$, to metrics defined by $\varphi \in \Phi^+$:

$$\int_0^{2\pi} \varphi(|T_n'(t)|) \, dt \leq \int_0^{2\pi} \varphi(n|T_n(t)|) \, dt, \quad T_n \in \mathcal{T}_n.$$ 

We study the question whether it is possible to extend the class $\Phi^+$, and prove that under certain assumptions $\Phi^+$ is the largest possible class.

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1. Introduction

Let $\mathcal{T}_n$ be the set of all trigonometric polynomials of degree at most $n$ with complex coefficients. The inequality

$$\|T_n'\|_C \leq n\|T_n\|_C, \quad T_n \in \mathcal{T}_n,$$

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is well known in approximation theory and is called the Bernstein inequality. Inequality (1) turns into equality iff \( T_n(t) = a \cos nt + b \sin nt \), where \( a, b \in \mathbb{C} \). The inequality was stated by Bernstein and Landau for polynomials with real coefficients (for details, see [5, Section 10, pp. 25–26; Section 3.4, p. 527], [8, Ch. 6, Theorems 1.2.4, 1.2.5]) in 1912–1914 and by Riesz for polynomials with complex coefficients ([10], [11, Vol. 2, Ch. 10]) in 1914.

We say that a function \( \varphi \) is increasing on an interval \( I \) if \( \varphi(u_1) \leq \varphi(u_2) \) for all \( u_1 \leq u_2, u_1, u_2 \in I \); \( \varphi \) is convex on \( I \) if \( \varphi(\alpha u_1 + (1 - \alpha)u_2) \leq \alpha \varphi(u_1) + (1 - \alpha)\varphi(u_2) \) for all \( u_1, u_2 \in I \) and \( \alpha \in [0, 1] \); \( \varphi \) is concave on \( I \) if \( -\varphi \) is convex on \( I \).

In 1933, Zygmund \([11, Vol. 2, Ch. 10, (3.25)]\) proved the following statement. If \( \varphi \) is an increasing and convex function on \([0, \infty)\), then

\[
\int_0^{2\pi} \varphi(|T'_n(t)|) \, dt \leq \int_0^{2\pi} \varphi(n|T_n(t)|) \, dt, \quad T_n \in T_n. \tag{2}
\]

For \( \varphi(u) = u^p, \, p \geq 1 \), inequality (2) implies the Bernstein inequality in the space \( L_p \):

\[
\|T'_n\|_p \leq n\|T_n\|_p, \quad T_n \in T_n,
\]

where \( \|f\|_p = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(t)|^p \, dt \right)^{1/p} \).

In 1979, Arestov \([1–3]\) found weaker conditions on functions \( \varphi \) which provide the validity of inequality (2). Before we give Arestov’s result, we introduce some notation \([2,4]\).

We denote by \( \Phi^+ \) the class of functions \( \varphi \) defined on \((0, \infty)\) with the following properties:

(i) \( \varphi \) is locally absolutely continuous;
(ii) \( \varphi \) increases on \((0, \infty)\);
(iii) \( u \varphi'(u) \) increases on \((0, \infty)\).

Put \( \psi(u) = \varphi(e^u) \); that is, \( \varphi(u) = \psi(\ln u) \). Clearly, \( \varphi \) belongs to \( \Phi^+ \) iff the function \( \psi \) is increasing and convex on \((-\infty, \infty)\). For example, all increasing convex functions, the functions \( \ln u, \ln^+ u = \max\{0, \ln u\}, \ln(1 + u^p), \) and \( u^p, \, p > 0 \), belong to \( \Phi^+ \).

We denote by \( P_n \) the set of all algebraic polynomials of degree at most \( n \) with complex coefficients. Let polynomials \( A_n \) and \( P_n \) from \( P_n \) be given by \( A_n(z) = \sum_{k=0}^n \binom{n}{k} \lambda_k z^k \) and \( P_n(z) = \sum_{k=0}^n \binom{n}{k} c_k z^k \). The polynomial

\[
A_n P_n(z) = \sum_{k=0}^n \binom{n}{k} \lambda_k c_k z^k \tag{3}
\]

called the composition of \( A_n \) and \( P_n \) (for details, see [9, Vol. 2, Section 5]). Suppose that \( A_n \) is fixed, then Eq. (3) defines a linear operator on \( P_n \), which we denote by the same symbol \( A_n \). For example, if \( A_n(z) = (1 + e^{i\theta} z)^n, \, \theta \in \mathbb{R}, \) then \( (A_n P_n)(z) = P_n(e^{i\theta} z) \) is the operator of rotation by angle \( \theta \); in particular, \( A_n(z) = (1 + z)^n \) defines the identity operator. The polynomial \( \Delta_n(z) = \frac{n}{2}(1 + z)^n - \frac{n}{2} P_n(z) \) defines the differential operator

\[
(\Delta_n P_n)(z) = z P'_n(z) - \frac{n}{2} P_n(z).
\]

In the sequel, if \( P_n \in P_n \) has degree \( m < n \), then we say that \( z = \infty \) is a zero of \( P_n \) with multiplicity \( n - m \). Let \( P_n^0 \) be the set of all polynomials \( P_n \in P_n \) such that all \( n \) zeros of \( P_n \) lie in the unit disk \(|z| \leq 1\), and let \( P_n^\infty \) be the set of all polynomials \( P_n \in P_n \) such that all zeros of \( P_n \) lie in the domain \(|z| \geq 1\). Furthermore, we say that an operator \( A_n \) belongs to the class
\( \Omega_n^0 \) if \( A_n \mathcal{P}_n^0 \subset \mathcal{P}_n^0 \), and that \( A_n \) belongs to the class \( \Omega_n^\infty \) if \( A_n \mathcal{P}_n^\infty \subset \mathcal{P}_n^\infty \). Using Theorems 151 and 152 from [9, Section 5] (see also [2]), one can easily prove that \( A_n \in \Omega_n^0 \) iff the polynomial \( A_n \in \mathcal{P}_n^0 \), and that \( A_n \in \Omega_n^\infty \) iff the polynomial \( A_n \in \mathcal{P}_n^\infty \). Finally, let \( \Omega_n = \Omega_n^0 \cup \Omega_n^\infty \).

**Theorem A** (Arestov [2]). If \( \varphi \in \Phi^+ \) and \( A_n \in \Omega_n \), then, for all \( P_n \in \mathcal{P}_n \),

\[
\int_0^{2\pi} \varphi(|A_n P_n(e^{it})|) \, dt \leq \int_0^{2\pi} \varphi(C(A_n)|P_n(e^{it})|) \, dt,
\]

where \( C(A_n) = \max\{|\lambda_0|, |\lambda_n|\} \). Equality holds in (4) if and only if \( P_n \) has the form

\[
P_n(z) = az^n, \quad P_n(z) = a, \quad \text{or} \quad P_n(z) = az^n + b \quad (a, b \in \mathbb{C}),
\]

depending on whether

\[
A_n \in \Omega_n^0, \quad A_n \in \Omega_n^\infty, \quad \text{or} \quad A_n \in \Omega_n^0 \cap \Omega_n^\infty.
\]

The space \( T_2 \) can be identified with the space \( \mathcal{P}_2 \) by the mapping \( T_2(t) = e^{-int} P_2(e^{it}) \), \( P_2 \in \mathcal{P}_2 \); moreover,

\[
|T_2(t)| = |P_2(e^{it})|, \quad |T_2'(t)| = |(\Delta_2 P_2)(e^{it})|.
\]

Note that \( \Delta_2 \subset \Omega_2 \cap \Omega_2^\infty \) and \( C(\Delta_2) = n \). Hence, inequality (2) is a consequence of Theorem A.

Professor Arestov asked the author whether it is possible to extend the class \( \Phi^+ \) in Theorem A. In this paper we prove that, under certain assumptions, \( \Phi^+ \) is the largest possible class.

2. Main result

We study inequality (4) for the class \( \Phi = \Phi_n \) of functions \( \varphi \) defined on \((0, \infty)\) with the following properties:

(i) \( \varphi \) is continuous on \((0, \infty)\);

(ii) \( \varphi \) increases on \((0, \infty)\);

(iii) for all \( P_n \in \mathcal{P}_n \),

\[
\int_0^{2\pi} \varphi(|P_n(e^{it})|) \, dt < \infty.
\]

An example of the function \( \varphi(u) = -\exp(1/u) \) shows that the third condition cannot be removed.

Now we will introduce a class \( \Phi^- \subset \Phi \) with the property that, for every \( \varphi \in \Phi^- \), inequality (4) is not satisfied (as will be stated in Theorem 1).

**Definition.** Denote by \( \Phi^- \) the set of all functions \( \varphi(u) = \psi(\ln u) \), where \( \varphi \in \Phi \), and there exist points \( v_1 < v_* < v_2 \) and a real number \( k \) such that the function

\[
\psi(v) - k \cdot v
\]

(i) increases on \([v_1, v_*]\) and decreases on \([v_*, v_2]\),

(ii) does not coincide with a constant in any neighborhood of the point \( v_* \).
Remark 1. Let us clarify this definition. Suppose that \( \varphi \not\in \Phi^+ \) and the corresponding function \( \psi \) has a locally absolutely continuous derivative \( \psi' \) everywhere. Then \( \psi''(v_\ast) < 0 \) for some \( v_\ast \). Hence, there exist points \( v_1 < v_\ast < v_2 \) such that
\[
\psi'(v) > \psi'(v_\ast), \quad v \in [v_1, v_\ast],
\]
\[
\psi'(v) < \psi'(v_\ast), \quad v \in [v_\ast, v_2].
\]
Furthermore, for the function \( \psi \) we have the representation
\[
\psi(v) - \psi'(v_\ast)(v - v_\ast) = \psi(v_\ast) + \int_{v_\ast}^{v} (\psi'(\eta) - \psi'(v_\ast)) d\eta.
\]
It follows from (5) that the function \( \psi(v) - \psi'(v_\ast)(v - v_\ast) \) increases on \([v_1, v_\ast]\) and decreases on \([v_\ast, v_2]\). Therefore, \( \varphi \) belongs to \( \Phi^- \).

Thus, if \( \varphi \in \Phi \) has a locally absolutely continuous derivative on \((0, \infty)\), then either \( \varphi \in \Phi^+ \) or \( \varphi \in \Phi^- \).

Remark 2. If \( \psi \) is strictly concave on some interval \([v_1, v_2]\), then \( \varphi \in \Phi^- \).

Remark 3. Let us give two examples of functions from \( \Phi^- \). For the function \( \varphi(u) = u/(1 + u) \), by means of which convergence in measure can be defined [6, Ch. 4, Ex. 4.7.60\(^\circ\)], the corresponding function \( \psi(v) = e^v/(1 + e^v) \) is concave on \([0, \infty)\) and, therefore, \( \varphi \in \Phi^- \).

Let \( C_0(v), v \in [0, 1], \) be the Cantor function [6, Ch. 3, Prop. 3.6.5], and let \( [v] \) denote the integer part of \( v \). The singular function \( \varphi \) defined by \( \varphi(e^v) = C_0(v - [v]) + [v] \) also belongs to \( \Phi^- \).

Remark 4. It is sufficient to consider only one of the following two cases: \( \Lambda_n \in \Omega_n^0 \) or \( \Lambda_n \in \Omega_n^\infty \). Indeed, applying the methods of de Bruijn and Springer [7] and Arestov [3], consider the map \( I = I_n \) on \( \mathcal{P}_n \) defined by
\[
(I P_n)(z) = z^n P_n(1/z), \quad P_n \in \mathcal{P}_n.
\]
It is clear that \( |P_n(e^{it})| = |(I P_n)(e^{-it})|, \ t \in [0, 2\pi], \ P_n \in \mathcal{P}_n, \) and
\[
|\Lambda_n P_n(e^{it})| = |(I (\Lambda_n P_n))(e^{-it})| = |((I \Lambda_n)(I P_n))(e^{-it})|, \quad \Lambda_n \in \Omega_n.
\]
Moreover, the map \( I \) is a bijection of \( \mathcal{P}_n^\infty \) onto \( \mathcal{P}_n^0 \). Therefore, if, say, \( \Lambda_n \in \Omega_n^\infty \), then \( I \Lambda_n \in \Omega_n^0 \). Thus, inequality (7) is valid for an operator \( \Lambda_n \) and a polynomial \( P_n \) iff it is valid for \( I \Lambda_n \) and \( I P_n \).

The polynomial \( \Lambda_n(z) = c(1 + e^{i\theta} z)^n \) defines on \( \mathcal{P}_n \) the operator
\[
(\Lambda_n P_n)(z) = c P_n(e^{i\theta} z), \quad c \in \mathbb{C}, \ \theta \in \mathbb{R}.
\]
For this operator, inequality (4) turns into equality for every \( P_n \in \mathcal{P}_n \), and so operators (6) are excluded from the further consideration.

Theorem 1. If \( \varphi \in \Phi^- \), \( \Lambda_n \in \Omega_n \), and \( \Lambda_n \) is not of the form (6), then there exists a polynomial \( P_n \in \mathcal{P}_n \) such that
\[
\int_0^{2\pi} \varphi \left( |(\Lambda_n P_n)(e^{it})| \right) dt > \int_0^{2\pi} \varphi \left( C(\Lambda_n)|P_n(e^{it})| \right) dt,
\]
where \( C(\Lambda_n) = \max\{ |\lambda_0|, |\lambda_n| \} \).
Proof. In view of Remark 4, it is sufficient to prove the theorem for

\[ A_n \in \Omega_n^0, \quad A_n(z) \neq c(1 + e^{i\theta}z)^n, \quad c \in \mathbb{C}, \quad \theta \in \mathbb{R}. \]  \hfill (8)

Without loss of generality, we can assume that \( \lambda_n = 1 \). We claim that \( |\lambda_0| \leq 1 \) and \( |\lambda_{n-1}| < 1 \). Indeed, by conditions (8), \( A_n \) has \( n \) zeros according to multiplicity \( z_1, \ldots, z_n \) and all the zeros lie on the unit circle. Consequently,

\[ |\lambda_0| = |z_1 \cdots z_n| \leq 1, \quad |\lambda_{n-1}| = \frac{1}{n}(z_1 + \cdots + z_n) \leq 1. \]

The last inequality turns into equality only if \( z_1 = \cdots = z_n = e^{i\theta} \) for some \( \theta \in \mathbb{R} \), but then \( A_n \) is an operator of the form (6) and we do not consider such operators. Consequently, under our assumptions, \( C(A_n) = \max\{|\lambda_0|, |\lambda_n|\} = 1 \), and we must prove that there exists a polynomial \( P \in \mathcal{P}_n \) such that

\[ \int_0^{2\pi} \varphi(|A_n P(e^{it})|) \, dt - \int_0^{2\pi} \varphi(|P(e^{it})|) \, dt > 0. \]  \hfill (9)

Suppose that \( \varphi(u) = \psi(\ln u) \), points \( v_1 < v_* < v_2 \) and a constant \( k \) satisfy conditions (i) and (ii) of the definition of the class \( \Phi^* \). Consider the function

\[ \tilde{\varphi}(u) = \varphi(u) - k \cdot \ln u = \psi(\ln u) - k \cdot \ln u, \]

and set \( u_1 = e^{v_1}, u_2 = e^{v_2}, u_* = e^{v_*} \). Clearly, \( \tilde{\varphi} \) increases on \([u_1, u_*]\), decreases on \([u_*, u_2]\), and does not coincide with a constant in any neighborhood of the point \( u_* \).

Let us construct a polynomial \( P \in \mathcal{P}_n \) that satisfies (9) in the form

\[ P(z) = mz^{n-1}(z - a), \quad a \in (0, 1), \quad m > 0. \]

We have \( A_n P(z) = m(z^n - \lambda_{n-1}az^{n-1}) = mz^{n-1}(z - \lambda_{n-1}a) \),

\[ \int_0^{2\pi} |A_n P(e^{it})| \, dt = \int_0^{2\pi} m |e^{it} - \lambda_{n-1}a| \, dt = \int_0^{2\pi} m |e^{it} - |\lambda_{n-1}|a| \, dt. \]

Let \( Q(e^{it}) = m(e^{it} - |\lambda_{n-1}|a) \); then inequality (9) is equivalent to the inequality

\[ \int_0^{2\pi} \left[ \psi\left(|Q(e^{it})|\right) - \psi\left(|P(e^{it})|\right) \right] \, dt > 0. \]  \hfill (10)

Let us compare \( |P(e^{it})|^2 \) and \( |Q(e^{it})|^2 \) on the interval \([0, 2\pi]\). We have

\[ |P(e^{it})|^2 = m^2(1 + a^2 - 2a \cos t), \]
\[ |Q(e^{it})|^2 = m^2 \left(1 + |\lambda_{n-1}|^2a^2 - 2|\lambda_{n-1}|a \cos t \right), \]  \hfill (11)

and, consequently,

\[ \frac{1}{m^2}\left(|P(e^{it})|^2 - |Q(e^{it})|^2\right) = 1 + a^2 - 2a \cos t - 1 - |\lambda_{n-1}|^2a^2 + 2|\lambda_{n-1}|a \cos t \]
\[ = a(1 - |\lambda_{n-1}|)(a + |\lambda_{n-1}|a - 2 \cos t). \]  \hfill (12)
Let \( t_0 = \arccos \left( (a + |\lambda_{n-1}|a)/2 \right) \). Evidently, \( t_0 \in (0, \pi) \), and it can be verified easily that

\[
|P(e^{it})| = |Q(e^{it})| = m \sqrt{1 - |\lambda_{n-1}|a^2}.
\] (13)

It follows from (11) that the absolute values \(|P(e^{it})|\) and \(|Q(e^{it})|\) are even functions of \( t \) that are increasing on \([0, \pi]\); by (12),

\[
|Q(e^{it})| > |P(e^{it})|, \quad t \in [0, t_0), \quad \text{and} \quad |Q(e^{it})| < |P(e^{it})|, \quad t \in (t_0, \pi].
\] (14)

Thus, we conclude that the values \(|Q(e^{it})|\) and \(|P(e^{it})|\) belong to the interval \([|P(1)|, |P(-1)|]\) for all \( t \in [0, 2\pi] \) and

\[
|P(1)| = m(1 - a), \quad |P(-1)| = m(1 + a).
\] (15)

Now, we choose parameters \( m \) and \( a \) such that

\[
\left| P(e^{it}) \right| = \left| Q(e^{it}) \right| = u_*, \quad \text{and} \quad \left[ |P(1)|, |P(-1)| \right] \subset [u_1, u_2].
\] (16)

This can be done the following way. Let \( a_k \) be a sequence such that \( a_k \to +0, \ k \to \infty \). Define \( m_k \) by

\[
m_k \sqrt{1 - |\lambda_{n-1}|a_k^2} = u_*.
\]

Then \( m_k \to u_*, \ k \to \infty \). Therefore,

\[
m_k(1 - a_k) \to u_* > u_1, \quad \text{and} \quad m_k(1 + a_k) \to u_* < u_2.
\]

Thus we can take \( a = a_k \) and \( m = m_k \) for a sufficiently large value of \( k \).

Combining (14) and (16), we conclude that

\[
\begin{align*}
&u_1 \leq |P(1)| < |P(e^{it})| < |Q(e^{it})| < u_*, \quad t \in (0, t_0), \\
&u_* < |Q(e^{it})| < |P(e^{it})| < |P(-1)| \leq u_2, \quad t \in (t_0, \pi).
\end{align*}
\] (17)

It remains to verify inequality (10) for the constructed polynomial \( P \). By the well-known Jensen formula (see, for example, [9, Section 3, Problem 175]),

\[
\begin{align*}
\int_0^{2\pi} \ln |P(e^{it})| \, dt &= \int_0^{2\pi} \ln |m(e^{it} - a)| \, dt = 2\pi \ln m, \\
\int_0^{2\pi} \ln |Q(e^{it})| \, dt &= \int_0^{2\pi} \ln |m(e^{it} - |\lambda_{n-1}|a)| = 2\pi \ln m.
\end{align*}
\]

Thus,

\[
\begin{align*}
&\int_0^{2\pi} \left[ \varphi (\left| Q(e^{it}) \right|) - \varphi (\left| P(e^{it}) \right|) \right] \, dt \\
&= \int_0^{2\pi} \left[ \varphi (\left| Q(e^{it}) \right|) - \varphi (\left| P(e^{it}) \right|) - k \ln |Q(e^{it})| + k \ln |P(e^{it})| \right] \, dt \\
&= 2 \int_0^{\pi} \left[ \tilde{\varphi}(\left| Q(e^{it}) \right|) - \tilde{\varphi}(\left| P(e^{it}) \right|) \right] \, dt \\
&= 2 \int_0^{t_0} \left[ \tilde{\varphi}(\left| Q(e^{it}) \right|) - \tilde{\varphi}(\left| P(e^{it}) \right|) \right] \, dt + 2 \int_{t_0}^{\pi} \left[ \tilde{\varphi}(\left| Q(e^{it}) \right|) - \tilde{\varphi}(\left| P(e^{it}) \right|) \right] \, dt.
\end{align*}
\]
Relations (17) yield that the last expression is greater than 0. This completes the proof of the theorem. □

**Corollary 1.** For any $\varphi \in \Phi^-$, there exists $T_n \in T_n$ such that

$$\int_0^{2\pi} \varphi \left(|T_n'(t)|\right) \, dt > \int_0^{2\pi} \varphi \left(n|T_n(t)|\right) \, dt.$$ 

For smooth functions $\varphi \in \Phi$, Arestov’s theorem and Theorem 1 give the necessary and sufficient conditions on $\varphi$ for validity of inequality (4).

**Corollary 2.** Suppose that an operator $A_n \in \Omega_n$ is not of the form (6) and a function $\varphi \in \Phi$ has a locally absolutely continuous derivative. Then inequality (4) is valid if and only if $\varphi \in \Phi^+$. The proof immediately follows from Theorem A, Remark 1, and Theorem 1.

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**References**


