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Solving singularly perturbed differential-difference equations arising in science and engineering with Fibonacci polynomials

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ABSTRACT

In this paper, we introduce a method to solve singularly perturbed differential-difference equations of mixed type, i.e., containing both terms having a negative shift and terms having a positive shift in terms of Fibonacci polynomials. Similar boundary value problems are associated with expected first exit time problems of the membrane potential in the models for the neuron. First, we present some preliminaries about polynomial interpolation and properties of Fibonacci polynomials then a new approach implementing a collocation method in combination with matrices of Fibonacci polynomials is introduced to approximate the solution of these equations with variable coefficients under the boundary conditions. Numerical results with comparisons are given to confirm the reliability of the proposed method for solving these equations.

1. Introduction

A singularly perturbed differential-difference equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and involving at least one delay or advance term. Many real life phenomena in biosciences, control theory, economics and engineering can be modeled by differential-difference equations and their systems [1] such as evolutionary biology [2], variational problems in control theory [3,4], describing the motion of the sunflower [5], signal transmission [6] and, depolarization in the Stein model [7]. For a detailed discussion on differential-difference equation one may refer to the books and high level monographs: Bellen [1], Driver [8], Bellman and Cooke [9].

In recent years, both mathematicians and physicists have devoted considerable effort to the study of numerical solutions of these equations, for example, Lange and Miura [10–12] gave asymptotic approaches in the study of the class of boundary value problems for linear second order differential difference equations in which the highest order derivative is multiplied by a small parameter. The effect of small shifts on the oscillatory solution of the problem has been discussed in [10]. In [13], Kadalbajoo and Sharma presented a numerical method based on the finite difference scheme to solve boundary value problems for singularly perturbed differential-difference equations with small shifts of mixed type. In [14], they described a numerical approach based on the finite difference method to solve a mathematical model arising from a model of neuronal variability. In [15], Kadalbajoo and Kumar presented a numerical method based on fitted mess and B-spline technique for singularly perturbed differential-difference equations with a small delay. In [16], Kadalbajoo and Kumar, presented a computational method based on piecewise uniform mess and Quasilinearization process for singularly perturbed nonlinear differential-difference equations with small shifts. In [17], Gulsu presented matrix methods for approximate solution of the second order singularly perturbed delay differential equations.

In the present paper, we introduce a method to solve a singularly perturbed differential-difference equation in the form

$$\varepsilon y''(x) + \gamma(x)y'(x-\tau) + \alpha(x)y(x-\delta) + \omega(x)y(x) + \beta(x)y(x+\eta) = g(x), \quad (1)$$

on $0 < x < 1$, $0 < \varepsilon \ll 1$, subject to the boundary conditions

$$y(x) = \phi(x), \quad -\delta \leqslant x \leqslant 0, \quad y(x) = \psi(x), \quad 1 \leqslant x \leqslant 1+\eta, \quad (2)$$

where $\alpha(x)$, $\omega(x)$, $\beta(x)$, $g(x)$, $\phi(x)$ and $\psi(x)$ are smooth functions, τ, δ and η are the small shifting parameters. Furthermore, we assume that $\gamma(x) \geqslant 0$ throughout the interval [0,1]. For a function $y(x)$ to constitute a smooth solution to the problem (1) it must be continuous in the interval [0,1] and be continuously differentiable in the interval (0,1). If $\gamma(x) = 0$ and the shifts δ, η equal to zero and

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$\alpha(x)+\omega(x)+\beta(x)<0$ on the interval [0,1], then the solution exhibits boundary layers at both the ends of the interval [0,1].

We want to approximate the solution of (1) as follows

$$y(x) \simeq \sum_{n=1}^{N+1} c_n F_n(x), \quad 0 \leqslant a \leqslant x \leqslant b, \tag{3}$$

where c_n, $n = 1,2,\ldots,N+1$ are the unknown Fibonacci coefficients, N is any arbitrary positive integer, and $F_n(x)$, $n = 1, 2, \ldots, N+1$ are the Fibonacci polynomials that we introduced in Section 2. To find a numeric solution in the form (3) of the problem (1), we use the collocation points defined by

$$x_i = \frac{i}{N}, \quad i = 0, 1, 2, \ldots, N. \tag{4}$$

2. Preliminaries

2.1. Polynomial interpolation

Let us consider $n+1$ pairs (x_i, y_i). The problem is to find a polynomial p_k, called the interpolating polynomial, such that

$$p_k(x_i) = c_0 + c_1 x_i + \cdots + c_k x_i^k, \quad i = 0, 1, \ldots, n.$$

The points x_i are called interpolation nodes. If $n \neq k$, the problem is over or under-determined. Let P_n be the $(n+1)-$dimensional subspace of $C[a,b]$ spanned by the functions $1, x, \cdots, x^n$. That is, P_n consists of all polynomials of degree at most n.

Theorem 1 18. *Given $n+1$ distinct nodes $x_0, x_1, \ldots, x_n$ and $n+1$ corresponding values $y_0, y_1, \ldots, y_n$ then there exists a unique polynomial $p_n \in P_n$ such that $p_n(x_i) = y_i$ for $i = 0,1,\ldots,n$.*

If we define

$$l_i \in P_n, \quad l_i = \prod_{\substack{i=0,\\ j\neq i}}^{n} \frac{(x-x_i)}{(x_i-x_j)}, \quad i = 0, 1, \ldots, n,$$

then $l_i(x_j) = \delta_{ij}$. The polynomials $l_i(x)$ are called Lagrange characteristic polynomials. If $f(x_i) = y_i$ for $i = 0,1,\ldots,n$, f being a given function, the interpolating polynomial $p_n(x)$ will be denoted by $p_n f(x)$. Let us introduce a lower triangular matrix X of infinite size, called the interpolation matrix on [a,b] whose entries x_{ij} for $i, j = 0,1,\ldots,n$ represent the points of [a,b], with the assumption that on each row the entries are all distinct. Thus, for any $n \geqslant 0$, the $(n+1)-$th row of X contains $n+1$ distinct values that can be identified as nodes, so that, for a given function f, we can uniquely define an interpolating polynomial $p_n f$ of degree n at those nodes.

Definition 1. Let $f(x)$ be defined on $[a,b]$, the modulus of continuity of $f(x)$ on $[a,b]$, $\omega(\delta)$, is defined for $\delta > 0$ by

$$\omega(\delta) = \sup_{|x-y|<\delta} |f(x) - f(y)|.$$

Corollary 1 19. *Let $f \in C[a,b]$ and X be an interpolation matrix on [a,b]. Then*

$$G_n(X) = \|f - p_n f\|_\infty \leqslant 6(1+\Gamma_n(X))\,\omega\!\left(\frac{b-a}{2n}\right),$$

where $\Gamma_n(X)$ denotes the Lebesgue constant of X, defined as

$$\Gamma_n(X) = \left\| \sum_{i=0}^{n} |l_i^{(n)}(x)| \right\|_\infty, \tag{5}$$

and where $l_i^{(n)}(x) \in P_n$ is the i-th characteristic polynomial associated with the $(n+1)-$th row of X.

2.2. The Fibonacci polynomials and properties

Leonardo of Pisa also known as Leonardus Pisanus, or, most commonly, Fibonacci (from “filius Bonacci”), was an Italian mathematician of the 13*th* century.

Fibonacci is best known to the modern world for the spreading of the Hindu-Arabic numerical system in Europe, primarily through the publication in 1202 of his Liber Abaci (Book of Calculation), and for a number sequence named the Fibonacci numbers after him, which he did not discover but used as an example in the Liber Abaci. In Fibonacci’s Liber Abaci book, chapter 12, he posed, and solved, a problem involving the growth of a population of rabbits based on idealized assumptions. The solution, generation by generation, was a sequence of numbers later known as the Fibonacci numbers. The number sequence was known to Indian mathematicians as early as the 6*th* century, but it was Fibonacci’s Liber Abaci that introduced it to the West. In the Fibonacci sequence of numbers, each number is the sum of the previous two numbers, starting with 0 and 1. This sequence begins 0, 1, 1, 2, 3, 5, ... [20].

Definition 2. For any positive real number k, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in N}$ is defined recurrently by

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad n \geqslant 1, \tag{6}$$

with initial conditions

$$F_{k,0} = 0, \quad F_{k,1} = 1. \tag{7}$$

Particular cases of the k-Fibonacci sequence are constructed from the following relations

if $k = 1$, the classical Fibonacci sequence is obtained

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1}, \quad n \geqslant 1,$$

if $k = 2$, the Pell sequence appears:

$$P_0 = 0, \quad P_1 = 1, \quad P_{n+1} = 2P_n + P_{n-1}, \quad n \geqslant 1,$$

if $k = 3$, the following sequence appears:

$$H_0 = 0, \quad H_1 = 1, \quad H_{n+1} = 3H_n + H_{n-1}, \quad n \geqslant 1.$$

If k be a real variable x then $F_{k,n} = F_{x,n}$ and they correspond to the Fibonacci polynomials defined by

$$F_{n+1}(x) = \begin{cases} 1, & n = 0 \\ x, & n = 1, \\ xF_n(x) + F_{n-1}(x), & n > 1 \end{cases} \tag{8}$$

from where the first six Fibonacci polynomials (Fig. 1) are

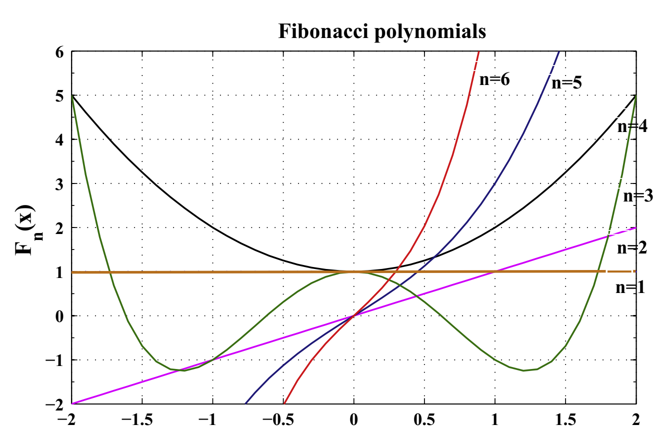

Fig. 1. The behavior of the first six Fibonacci polynomials.

$$F_1(x) = 1,$$
$$F_2(x) = x,$$
$$F_3(x) = x^2 + 1,$$
$$F_4(x) = x^3 + 2x,$$
$$F_5(x) = x^4 + 3x^2 + 1,$$
$$F_6(x) = x^5 + 4x^3 + 3x,$$

and from these expressions, as for the k-Fibonacci numbers we can write

$$F_{n+1}(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} x^{n-2i}, \quad n \geqslant 0, \tag{9}$$

where $\lfloor \frac{n}{2} \rfloor$ denotes the greatest integer in $\frac{n}{2}$.

Note that $F_{2n}(0) = 0$ and $x = 0$ is the only real root, while $F_{2n+1}(0) = 1$ with no real roots. Also for $x = k \in N$ we obtain the elements of the k-Fibonacci sequences [21].

The Fibonacci polynomials have generating function [20]

$$G(x,t) = \frac{t}{1-t^2-tx} = \sum_{n=1}^{\infty} F_n(x)t^n = t + xt^2 + (x^2+1)t^3 + (x^3+2x)t^4 + \cdots.$$

The Fibonacci polynomials are normalized so that

$$F_n(1) = F_n,$$

where the F_n is n*th* Fibonacci number.

Note first, that the equations for the Fibonacci polynomials may be written in matrix form as

$$\mathbf{F}(x) = \mathbf{Z}\mathbf{X}(x), \tag{10}$$

where $\mathbf{F}(x) = [F_1(x), F_2(x), F_3(x), \ldots, F_{N+1}(x)]^T$, $\mathbf{X}(x) = [1, x, x^2, x^3, \ldots, x^N]^T$, and Z is the lower triangular matrix with entrances the coefficients appearing in the expansion of the Fibonacci polynomials in increasing powers of x. If N is odd,

$$\mathbf{Z} = \begin{pmatrix} 1 & 0 & & & \cdots & 0 & & \\ 0 & 1 & 0 & & \cdots & & & \\ 1 & 0 & 1 & 0 & \cdots & & & \\ 0 & 2 & 0 & 1 & 0 & \cdots & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & & & & & 0 \\ 0 & \frac{(\frac{N+1}{2})!}{(\frac{N-1}{2})!} & 0 & \cdots & 0 & N-1 & 0 & 1 \end{pmatrix}_{(N+1)\times(N+1)},$$

if N is even

$$\mathbf{Z} = \begin{pmatrix} 1 & 0 & & & & \cdots & & 0 \\ 0 & 1 & 0 & & & \cdots & & \\ 1 & 0 & 1 & 0 & & \cdots & & \\ 0 & 2 & 0 & 1 & 0 & \cdots & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & & & & & 0 \\ 1 & 0 & \frac{(\frac{N+2}{2})!}{2(\frac{N-2}{2})!} & \cdots & 0 & N-1 & 0 & 1 \end{pmatrix}_{(N+1)\times(N+1)}.$$

Note that in matrix Z the non-zero entrances build precisely the diagonals of the Pascal triangle and the sum of the elements in the same row gives the classical Fibonacci sequence. In addition, matrix $\mathbf{Z}$ is invertible and therefore x^n may be written as a linear combination of Fibonacci polynomials that is given in closed form in the following theorem, which is the version of the Zeckendorf's theorem for the Fibonacci polynomials.

Theorem 2 [21]. *For every integer $n \geqslant 1$, x^{n-1} may be written in a unique way as linear combination of the n first Fibonacci polynomials as*

$$x^{n-1} = \sum_{n=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \left[\binom{n}{i} - \binom{n}{i-1} \right] F_{n-2i}(x),$$

where $\binom{n}{-1} = 0.$

Let us assume that f is a solution of (1). *We would like to interpolate f by*

$$p_N(x) = \sum_{n=0}^{N} f_n B_n(x), \quad N \geqslant m, \quad 0 \leqslant a \leqslant x \leqslant b, \tag{11}$$

such that p_N and f are equal on the nodes $a \leqslant x_0 < x_1 < \cdots < x_n \leqslant b$. Since all $f(x_i)$ values are unknown, we can use (1) to find the interpolation polynomial at the nodes $x_0, x_1, \ldots, x_N$ without knowing $f(x_i)$ values. To do this, we put the interpolation polynomials $p_N(x) = f_0 + f_1 x + \cdots + f_N x^N$ into (1). If p_N equals f on the nodes, then p_N satisfies (1) on the nodes. Hence, we can obtain a system of linear equations depending on $f_0, f_1, \ldots, f_N$. Therefore, we can find the solution of (1) with some errors which are the interpolation and computational errors.

3. Function approximation

Suppose the solution of the differential-difference Eq. (1) expressed in terms of the Fibonacci polynomials such as (3). Then the function defined in the relation (3) can be written in matrix form

$$y(x) \simeq \mathbf{C}\mathbf{F}(x), \tag{12}$$

where $\mathbf{C} = [c_1, c_2, \ldots, c_{N+1}]$. Then from Eq. (10)

$$y(x) \simeq \mathbf{C}\mathbf{Z}\mathbf{X}(x) = \mathbf{X}^T(x)\mathbf{Z}^T\mathbf{C}^T. \tag{13}$$

The differentiation of vector $\mathbf{X}(x)$ in Eq. (10), can be expressed as

$$\mathbf{X}^{(1)}(x) = \mathbf{D}\mathbf{X}(x), \tag{14}$$

where

$$\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 2 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & N & 0 \end{pmatrix}.$$

If we approximate $y(x) \simeq \mathbf{Z}\mathbf{X}(x)$, then for $i \geqslant 2$ (i is the order derivatives) we get

$$\mathbf{X}^{(i)}(x) = \mathbf{D}\mathbf{X}^{(i-1)}(x) = \mathbf{D}^2\mathbf{X}^{(i-2)}(x) = \ldots = \mathbf{D}^i\mathbf{X}(x), \tag{15}$$

and therefore

$$\mathbf{F}^{(i)}(x) = \mathbf{Z}\mathbf{X}^{(i)}(x) = \mathbf{Z}\mathbf{D}^i\mathbf{X}(x), \tag{16}$$

then

$$y^{(i)}(x) \simeq \mathbf{C}\mathbf{F}^{(i)}(x) = \mathbf{C}\mathbf{Z}\mathbf{X}^{(i)}(x) = \mathbf{C}\mathbf{Z}\mathbf{D}^i\mathbf{X}(x) = \mathbf{X}^T(x)\mathbf{D}^{Ti}\mathbf{Z}^T\mathbf{C}^T. \tag{17}$$

By putting $x \to x + v$ in the relation (12) we obtain the matrix form

$$y(x+v) \simeq \mathbf{C}\mathbf{F}(x+v). \tag{18}$$

The relation between the matrix $\mathbf{X}(x+v)$ and $\mathbf{X}(x)$ is

$$\mathbf{X}(x+\nu)=\mathbf{M}_\nu\mathbf{X}(x), \tag{19}$$

where

$$\mathbf{M}_\nu=\begin{pmatrix} \binom{0}{0}\nu^0 & \binom{1}{0}\nu^1 & \binom{2}{0}\nu^2 & \cdots & \binom{N}{0}\nu^N \\ 0 & \binom{1}{1}\nu^0 & \binom{2}{1}\nu^1 & \cdots & \binom{N}{1}\nu^{N-1} \\ 0 & 0 & \binom{2}{2}\nu^0 & \cdots & \binom{N}{2}\nu^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{N}{n}\nu^0 \end{pmatrix}. \tag{20}$$

From Eqs. (13), (18) and (19), we have

$$y(x+\nu)\simeq\mathbf{CZM}_\nu\mathbf{X}(x)=\mathbf{X}^T(x)\mathbf{M}_\nu^T\mathbf{Z}^T\mathbf{C}^T. \tag{21}$$

From Eq. (18)

$$y^{(i)}(x+\nu)\simeq\mathbf{CF}^{(i)}(x+\nu), \tag{22}$$

therefore from Eqs. (16), (19) and (22) we have

$$y^{(i)}(x+\nu)\simeq\mathbf{X}^T(x)\mathbf{M}_\nu^T\mathbf{D}^{Ti}\mathbf{Z}^T\mathbf{C}^T. \tag{23}$$

4. Method of solution

By substituting Eqs. (13), (17), (21) and (23) in (1)

$$\varepsilon\mathbf{X}^T(x)\mathbf{D}^{T2}\mathbf{Z}^T\mathbf{C}^T+\gamma(x)\mathbf{X}^T(x)\mathbf{M}_{-\tau}^T\mathbf{D}^T\mathbf{Z}^T\mathbf{C}^T+\alpha(x)\mathbf{X}^T(x)\mathbf{M}_{-\delta}^T\mathbf{Z}^T\mathbf{C}^T + \omega(x)\mathbf{X}^T(x)\mathbf{Z}^T\mathbf{C}^T+\beta(x)\mathbf{X}^T(x)\mathbf{M}_\eta^T\mathbf{Z}^T\mathbf{C}^T=g(x),$$

or

$$\left\{\varepsilon\mathbf{X}^T(x)\mathbf{D}^{T2}\mathbf{Z}^T+\gamma(x)\mathbf{X}^T(x)\mathbf{M}_{-\tau}^T\mathbf{D}^T\mathbf{Z}^T+\alpha(x)\mathbf{X}^T(x)\mathbf{M}_{-\delta}^T\mathbf{Z}^T + \omega(x)\mathbf{X}^T(x)\mathbf{Z}^T+\beta(x)\mathbf{X}^T(x)\mathbf{M}_\eta^T\mathbf{Z}^T\right\}\mathbf{C}^T=g(x), \tag{24}$$

Then by substituting collocation points (4) into Eq. (24), we have

$$\left\{\mathbf{EXD}^{T2}\mathbf{Z}^T+\mathbf{\Gamma XM}_{-\tau}^T\mathbf{D}^T\mathbf{Z}^T+\mathbf{AXM}_{-\delta}^T\mathbf{Z}^T+\mathbf{\Omega XZ}^T+\mathbf{BXM}_\eta^T\mathbf{Z}^T\right\}\mathbf{C}^T=\mathbf{G}, \tag{25}$$

that

$$\mathbf{E}=\begin{bmatrix}\varepsilon & 0 & \cdots & 0\\ 0 & \varepsilon & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \varepsilon\end{bmatrix},\quad \mathbf{\Gamma}=\begin{bmatrix}\gamma(x_0) & 0 & \cdots & 0\\ 0 & \gamma(x_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \gamma(x_N)\end{bmatrix},$$

$$\mathbf{A}=\begin{bmatrix}\alpha(x_0) & 0 & \cdots & 0\\ 0 & \alpha(x_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \alpha(x_N)\end{bmatrix},$$

$$\mathbf{\Omega}=\begin{bmatrix}\omega(x_0) & 0 & \cdots & 0\\ 0 & \omega(x_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \omega(x_N)\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}\beta(x_0) & 0 & \cdots & 0\\ 0 & \beta(x_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \beta(x_N)\end{bmatrix},$$

$$\mathbf{X}=\begin{bmatrix}\mathbf{X}^T(x_0)\\ \mathbf{X}^T(x_1)\\ \vdots\\ \mathbf{X}^T(x_N)\end{bmatrix}=\begin{bmatrix}1 & x_0 & x_0^2 & \cdots & x_0^N\\ 1 & x_1 & x_1^2 & \cdots & x_1^N\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & x_N & x_N^2 & \cdots & x_N^N\end{bmatrix},\quad \mathbf{G}=\begin{bmatrix}g(x_0)\\ g(x_1)\\ \vdots\\ g(x_N)\end{bmatrix}.$$

Hence, Eq. (25) can be written in the form

$$\mathbf{WC}^T=\mathbf{G},\quad \mathbf{W}=[w_{p,q}],\ p,q=1,\cdots,N+1, \tag{26}$$

where

$$\mathbf{W}=\mathbf{EXD}^{T2}\mathbf{Z}^T+\mathbf{\Gamma XM}_{-\tau}^T\mathbf{D}^T\mathbf{Z}^T+\mathbf{AXM}_{-\delta}^T\mathbf{Z}^T+\mathbf{\Omega XZ}^T+\mathbf{BXM}_\eta^T\mathbf{Z}^T. \tag{27}$$

For the boundary conditions (2), first we consider them as follows

$$y(0)=\phi(0),\quad y(1)=\psi(1),$$

then we obtain the corresponding matrix forms for these conditions, by means of the relations (13), as

$$\mathbf{X}^T(0)\mathbf{Z}^T\mathbf{C}^T=\phi(0),$$
$$\mathbf{X}^T(1)\mathbf{Z}^T\mathbf{C}^T=\psi(1),$$

briefly, the matrix form for conditions (2) is

$$\mathbf{U}_i\mathbf{C}^T=[\lambda_i],\quad i=1,2, \tag{28}$$

that

$$\begin{cases}\mathbf{U}_1=\mathbf{X}^T(0)\mathbf{Z}^T, & \mathbf{U}_2=\mathbf{X}^T(1)\mathbf{Z}^T,\\ \lambda_1=\phi(0), & \lambda_1=\psi(1).\end{cases}$$

Finally, to obtain the solution of Eq. (1) under the conditions (2) by replacing the rows of matrix $\mathbf{U}$ and λ by the last 2 rows of the matrices $\mathbf{W}$ and $\mathbf{G}$, respectively we obtain

$$\widetilde{\mathbf{W}}\mathbf{C}^T=\widetilde{\mathbf{G}}. \tag{29}$$

So we have the new augmented matrix as

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]=\begin{bmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,N+1} & ; & g(x_0)\\ w_{2,1} & w_{2,2} & \cdots & w_{2,N+1} & ; & g(x_1)\\ \vdots & \vdots & \ddots & \vdots & ; & \vdots\\ w_{N-1,1} & w_{N-1,2} & \cdots & w_{N-1,N+1} & ; & g(x_{N-2})\\ u_{1,1} & u_{1,2} & \cdots & u_{1,N+1} & ; & \lambda_1\\ u_{2,1} & u_{2,2} & \cdots & u_{2,N+1} & ; & \lambda_2\end{bmatrix}. \tag{30}$$

However, we do not have to replace the last rows. For example, if the matrix $\mathbf{W}$ is singular, then the rows that have the same factor or all zeros are replaced.

If rank $\widetilde{\mathbf{W}}$ =rank $[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]=N+1$, then we can write $\mathbf{C}^T=\widetilde{\mathbf{W}}^{-1}\widetilde{\mathbf{G}}$.

Thus, the matrix $\mathbf{C}$ (thereby the unknown Fibonacci coefficients) is uniquely determined. Also the Eq. (1) with the conditions (2) has a unique solution. This solution is given by the truncated Fibonacci series (10). However, when $|\widetilde{\mathbf{W}}|=0$, if rank $\widetilde{\mathbf{W}}$ =rank $[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]<N+1$, then we may find a particular solution. Otherwise if rank $\widetilde{\mathbf{W}}\neq$rank $[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]<N+1$, then it is not a solution.

5. Accuracy of the solution and error analysis

In this section, accuracy of the solution and error analysis are briefly discussed. In the following lines the main Theorem of this section will be provided.

Theorem 3. *Let the solution of (1) actually computed by the Fibonacci series solution $p_N(x)$ and $y=f(x)$ be the exact solution. Let the coefficient matrix of (29) be $\overline{\widetilde{\mathbf{W}}}=\widetilde{\mathbf{W}}+\delta\mathbf{W}$ where $\delta\mathbf{W}$ represents the computational error. Let $\mathbf{X}(x)$ and $\mathbf{Z}$ be the matrices which defined in (10). if $\|\overline{\widetilde{\mathbf{W}}}^{-1}\|_F\|\delta\mathbf{W}\|_F\leqslant 1$ and $f\in C^\infty[a,b]$ or $m\geqslant N$, then*

Table 1
Numerical results of solutions $y(x)$ of Eq. (31) whit $N = 40$, $\delta = 0.03$ and $\eta = 0.07$.

x_i	ε_1				
	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}
0.0	1.0	1.0	1.0	1.0	0.999999979854
0.1	0.69558017976599	0.31397326001301	−0.31776109561642	−1.0684034111123	−1.66134008978458
0.2	0.44611286444032	−0.17843681958622	−1.04303246589512	−1.7097594871377	−1. 96173686101883
0.3	0.24640550112903	−0.51918672380764	−1.43275817241829	−1.9071841620307	−1. 99554651877723
0.4	0.09227824677644	−0.73759076734276	−1.62606579525988	−1.9643194569452	−1. 99896885284652
0.5	−0.01952340050835	−0.85283655016632	−1.69381198785957	−1. 9716191887973	−1.99780610752198
0.6	−0.09139717435577	−0.87566080053667	−1.66432524832696	−1. 9476058728924	−1.99159057190782
0.7	−0.12493525931487	−0.80933126223854	−1.53463678575838	−1. 8739686353386	−1.96699049672896
0.8	−0.12095924664145	−0.65002317020772	−1.27150722748743	−1. 6844287984922	−1.87034216199662
0.9	−0.07953958846018	−0.38662964399653	−0.80319013398318	−1. 2058053810300	−1.49070890677460
1.0	6.76880773653e-12	−5.74225111905e-11	6.76006806089e-10	−2. 7236907840e-08	5.56380147235e-04

Table 2
Numerical results of the absolute error functions e_N of Eq. (31) whit $\delta = 0.03$ and $\eta = 0.07$.

ε_1	Method of [24]		Present method			
	$N = 100$	$N = 300$	N=500	$N = 10$	$N = 20$	$N = 40$
2^{-1}	6.1000e−07	7.0000e−08	3.0000e−08	6.7447e−10	9.2654e−12	7.3890e−12
2^{-2}	6.7400e−06	7.5000e−07	2.7000e−07	8.3798e−07	8.6129e−11	1.8112e−11
2^{-3}	5.0780e−05	5.6500e−06	2.0300e−06	5.2773e−04	1.2193e−09	3.2607e−10
2^{-4}	2.9686e−04	3.3200e−05	1.1960e−05	7.7633e−02	2.7995e−06	3.2170e−07
2^{-5}	1.6272e−03	1.8578e−06	6.7020e−04	1.6786e+00	3.9755e−03	3.7673e−05

$$
\begin{aligned}
|p_N(x) - f(x)| \leqslant{}& \frac{r\|\widetilde{\mathbf{C}}\|_F \|\overline{\overline{\mathbf{W}}}^{-1}\|_F}{1 - r\|\overline{\overline{\mathbf{W}}}^{-1}\|_F} \|\mathbf{Z}\|_F \|\mathbf{X}(b-a)\|_F \\
&+ \left| \frac{f^{(N+1)}(\xi_x)}{(N+1)!} \prod_{i=0}^{N} (x - x_i) \right|,
\end{aligned}
$$

where r is the highest value of $|\delta\mathbf{W}|_F$ *and* $\widetilde{C}$ *is the solution of* (29).

Proof. By considering $\|\overline{\overline{\mathbf{W}}}^{-1}\|_F \|\delta\mathbf{W}\|_F \leqslant 1$ and $f \in C^\infty[a,b]$ or $m \geqslant N$, from Cauchy-Schwarz inequalities,Theorem 1 and Corollary 1, we obtain that

$$
\begin{aligned}
|p_N(x) - y| &= |p_N(x) - p_N f(x) + p_N f(x) - y| \\
&\leqslant |p_N(x) - p_N f(x)| + |p_N f(x) - y| \\
&\leqslant |p_N(x) - p_N f(x)| + \left| \frac{f^{(N+1)}(\xi_x)}{(N+1)!} \prod_{i=0}^{N} (x - x_i) \right|,
\end{aligned}
$$

and

$$
\begin{aligned}
|p_N(x) - p_N f(x)| &= \left| \sum_{n=0}^{N} \tilde{c}_n F_n - \sum_{n=0}^{N} c_n F_n \right| = \left| \sum_{n=0}^{N} (\tilde{c}_n - c_n) F_n \right| \\
&\leqslant \|[\tilde{c}_0 - c_0 \;\; \tilde{c}_1 - c_1 \ldots \tilde{c}_N - c_N]\|_F \|\mathbf{F}\|_F \\
&\leqslant \|\overline{\overline{\mathbf{W}}}^{-1}\|_F \|\delta\mathbf{W}\|_F \|\widetilde{\mathbf{C}}\|_F \|\mathbf{Z}\|_F \|\mathbf{X}(b-a)\|_F \\
&= \|(\overline{\overline{\mathbf{W}}} - \delta\mathbf{W})^{-1}\|_F \|\delta\mathbf{W}\|_F \|\widetilde{\mathbf{C}}\|_F \|\mathbf{Z}\|_F \|\mathbf{X}(b-a)\|_F \\
&\leqslant \|\overline{\overline{\mathbf{W}}}^{-1}\|_F \|(1 - \overline{\overline{\mathbf{W}}}^{-1}\delta\mathbf{W})^{-1}\|_F \|\delta\mathbf{W}\|_F \|\widetilde{\mathbf{C}}\|_F \|\mathbf{Z}\|_F \|\mathbf{X}(b-a)\|_F \\
&\leqslant \frac{\|\overline{\overline{\mathbf{W}}}^{-1}\|_F \|\delta\mathbf{W}\|_F \|\widetilde{\mathbf{C}}\|_F}{1 - \|\overline{\overline{\mathbf{W}}}^{-1}\delta\mathbf{W}\|_F} \|\mathbf{Z}\|_F \|\mathbf{X}(b-a)\|_F \\
&\leqslant \frac{\|\overline{\overline{\mathbf{W}}}^{-1}\|_F \|\delta\mathbf{W}\|_F \|\widetilde{\mathbf{C}}\|_F}{1 - \|\overline{\overline{\mathbf{W}}}^{-1}\|_F \|\delta\mathbf{W}\|_F} \|\mathbf{Z}\|_F \|\mathbf{X}(b-a)\|_F \\
&\leqslant \frac{r\|\widetilde{\mathbf{C}}\|_F \|\overline{\overline{\mathbf{W}}}^{-1}\|_F}{1 - r\|\overline{\overline{\mathbf{W}}}^{-1}\|_F} \|\mathbf{Z}\|_F \|\mathbf{X}(b-a)\|_F, \quad \|\overline{\overline{\mathbf{W}}}^{-1}\|_F \|\delta\mathbf{W}\|_F \leqslant 1. \quad \square
\end{aligned}
$$

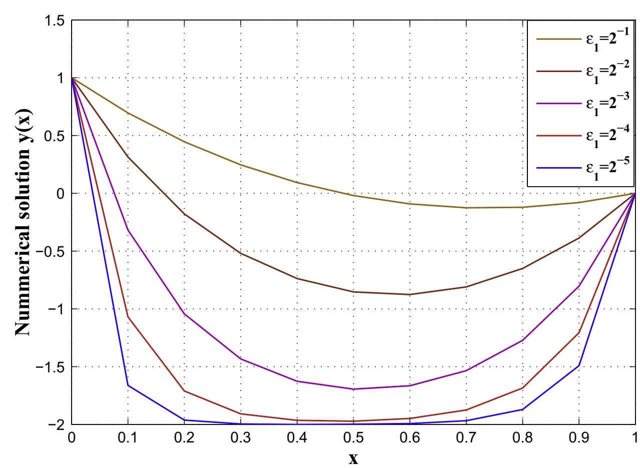

Fig. 2. Numerical solution $y(x)$ of Eq. (31) whit $N = 40$, $\delta = 0.03$ and $\eta = 0.07$.

We can easily check the accuracy of the method. Since the truncated Fibonacci series (10) is an approximate solution of (1), when the function $y_N(x)$, and its derivatives, are substituted in Eq. (1), the resulting equation must be satisfied approximately; that is, for $x = x_c \in [0,1]$, $c = 0,1,2,\ldots$

$$
\begin{aligned}
e_N(x_c) = {}& |\varepsilon y_N''(x_c) + \gamma(x) y_N'(x_c - \tau) + \alpha(x_c) y_N(x_c - \delta) + \omega(x_c) y_N(x_c) \\
&+ \beta(x_c) y_N(x_c + \eta) - g(x_c)| \cong 0,
\end{aligned}
$$

and $e(x_c) \leqslant 10^{-l_c}$. If $\max 10^{-l_c} = 10^{-l}$ is prescribed, then the truncation limit N is increased until the difference $e(x_c)$ at each of the points becomes smaller than the prescribed 10^{-l} [22]. On the other hand, the error can be estimated by the function

$$
\begin{aligned}
e_N(x) = {}& |\varepsilon y_N''(x) + \gamma(x) y_N'(x - \tau) + \alpha(x) y_N(x - \delta) + \omega(x) y_N(x) \\
&+ \beta(x) y_N(x + \eta) - g(x)| \cong 0.
\end{aligned}
$$

If $e_N(x) \to 0$, when N is sufficiently large enough, then the error decreases.

Table 3
Numerical results of solutions $y(x)$ of Eq. (32) whit $N = 50$ and $\tau = 0.1\varepsilon$.

x_i	ε				
	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}
0.0	1.0	1.0	0.99999999999500	1.00000003634000	0.99993683468200
0.1	0.89630178502614	0.80261129133847	0.66440505720568	0. 51862944395896	0.43129561737203
0.2	0.83143613800497	0.70554791377722	0.56688785036387	0. 48206352114951	0.45839325060721
0.3	0.79629708095000	0.66906936874565	0.56081122842601	0. 51667390634065	0.50480298925441
0.4	0.78420533815444	0.66997817748539	0.59172341359393	0. 56609112235291	0.55638735717875
0.5	0.79028657883512	0.69480661684964	0.63958509113851	0. 62215430641732	0.61324312579171
0.6	0.81101084262321	0.73581587094255	0.69722010286483	0. 68406223835104	0.67590621897141
0.7	0.84385151050755	0.78864723371787	0.76227917735977	0. 75217477306654	0.74498108422492
0.8	0.88703295117699	0.85094379760531	0.83424478636987	0. 82707588834401	0.82127759305864
0.9	0.93934395652325	0.92154230084522	0.91331679496839	0. 90943600303432	0.90648773257302
1.0	1.00000000000977	1.00000000001608	1.00000000181929	0. 99999637899937	0.99841228558729

Table 4
Numerical results of the absolute error functions $e_N(x)$ of Eq. (32) whit $\tau = 0.1\varepsilon$.

ε	Method of [26]		Present method		
	$N = 8$	$N = 32$	N=256	$N = 32$	$N = 50$
2^{-1}	0.06590116	0.06635201	0.06614488	1.0824e−03	1.0823e−03
2^{-2}	0.04331625	0.04270446	0.04239243	1.3355e−03	1.3357e−03
2^{-3}	0.02354121	0.02374059	0.02355647	1.4221e−03	1.4262e−03
2^{-4}	0.01304051	0.01266778	0.01240349	1.4444e−03	1.4519e−03
2^{-5}	0.00861776	0.00625837	0.00637484	4.0806e−02	2.0459e−03

6. Numerical examples

In this section, three examples are given to certify the convergence and error bound of the presented method.

Since the exact solutions of some problems for different values of τ, δ and η are not known, the maximum absolute errors for these examples are calculated using the following double mesh principle [23]

$$e_N = \max_{0 \leqslant i \leqslant N} |y_i^N - y_{2i}^{2N}|.$$

In this regard, we have presented with Tables and Figures, the values of the approximate solutions $y_N(x)$, and the absolute error function e_N at the selected points of the given interval. All results are computed by using a program written in the Matlab.

Example 1. Consider the following singularly perturbed differential-difference equation with mixed shift from [14,24,25]

$$\varepsilon y''(x) + 0.25y(x-\delta) - y(x) + 0.25y(x+\eta) = 1, \tag{31}$$

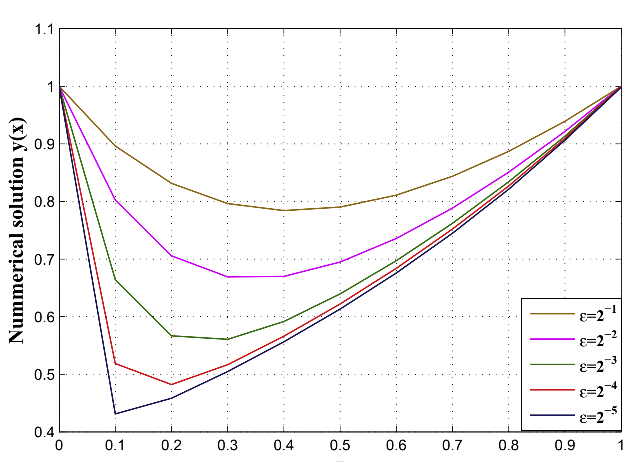

Fig. 3. Numerical solution $y(x)$ of Eq. (32) whit $N = 50$ and $\tau = 0.1\varepsilon$.

on [0,1], under the boundary conditions

$$\begin{cases} y(x) = 1, & -\delta \leqslant x \leqslant 0 \\ y(x) = 0, & 1 \leqslant x \leqslant 1+\eta \end{cases},$$

where $\varepsilon = \varepsilon_1^2$.

For this example, we have $\alpha(x) = \beta(x) = 0.25$,

$$\omega(x) = -1,\ g(x) = 1,\ \phi(x) = 1,\ \psi(x) = 0.$$

From Eq. (25), the fundamental matrix equation of the problem is

$$\left\{\mathbf{EXD}^{T2}\mathbf{Z}^T + \mathbf{AXM}_{-\delta}^T\mathbf{Z}^T + \mathbf{\Omega XZ}^T + \mathbf{BXM}_{\eta}^T\mathbf{Z}^T\right\}\mathbf{C}^T = \mathbf{G}.$$

Table 1 shows the numerical solutions of Eq. (31) by the presented method for $N = 40$, $\delta = 0.03$ and $\eta = 0.07$. Table 2 gives the comparison of the result of the maximum error e_N obtained by the present method and the method of [24] for different values of N. Fig. 2. displays the numerical solution $y(x)$ of the present method for $N = 40$, $\delta = 0.03$ and $\eta = 0.07$.

Example 2. Consider the following singularly perturbed delay differential equation with left layer [26]

$$\varepsilon y''(x) + y'(x-\tau) - y(x) = 0, \tag{32}$$

on [0,1], under the boundary conditions $y(0) = 1$ and $y(1) = 1$.

The exact solution is given by

$$y(x) = \frac{(1-e^{m_2})e^{m_1x} + (e^{m_1}-1)e^{m_2x}}{e^{m_1} - e^{m_2}},$$

where

$$m_1 = \frac{-1-\sqrt{1+4(\varepsilon-\tau)}}{2(\varepsilon-\tau)},$$

$$m_2 = \frac{-1+\sqrt{1+4(\varepsilon-\tau)}}{2(\varepsilon-\tau)}.$$

From Eq. (25), the fundamental matrix equation of the problem is

$$\left\{\mathbf{EXD}^{T2}\mathbf{Z}^T + \mathbf{\Gamma XM}_{-\tau}^T\mathbf{D}^T\mathbf{Z}^T + \mathbf{\Omega XZ}^T\right\}\mathbf{C}^T = \mathbf{G}.$$

Table 5
Numerical results of solutions $y(x)$ of Eq. (33) whit $N = 50$ and $\delta = 0.03$.

x_i	ε_2				
	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}
0.0	1.0	1.0	1.0	1.0	1.00000000119
0.1	0.68928280368843	0.35360348454404	−0.13387111798733	−0.68044249534044	−1.09341843304189
0.2	0.43852625642647	−0.09527308924917	−0.71429338024020	−1.15063826721254	− 1.30866474354169
0.3	0.24036107012340	−0.39887966129200	−1.00884749225583	−1.28216981877591	− 1.33079629859325
0.4	0.08897916543394	−0.59241558951874	−1.15307999985931	−1.31882798727055	− 1.33307181934275
0.5	−0.02003649940738	−0.69810805676882	−1.21285463959644	−1. 32845492478751	−1.33330522856314
0.6	−0.08984073154749	−0.72776082117429	−1.21437793625311	−1. 32841936989625	−1.33332209470295
0.7	−0.12241729180674	−0.68406506161786	−1.15555945988132	−1. 31659258918668	−1.33318073257283
0.8	−0.11863510066228	−0.56081692187819	−1.00355880522384	−1. 26212024255499	−1.33022746240777
0.9	−0.07827104899219	−0.34205503297055	−0.67659424906950	−1. 02535850750906	−1.26910287703900
1.0	6.42597086653e−13	−4.0616399132e−12	9.29166077412e−10	2.3984343755e−07	3.89882221872e−04

Table 6
Numerical results of the absolute error functions e_N of Eq. (33) whit $\delta = 0.03$.

ε_2	Method of [24]			Present method		
	$N = 100$	$N = 300$	N=500	$N = 10$	$N = 20$	N=50
2^{-1}	0.000008	0.000001	0.000000	2.9136e−09	5.8078e−12	7.3761e−13
2^{-2}	0.000158	0.000018	0.000006	1.8517e−06	7.4724e−11	5.3242e−11
2^{-3}	0.000820	0.000092	0.000033	5.6407e−04	5.0619e−10	4.5740e−10
2^{-4}	0.004075	0.000462	0.000166	4.1316e−02	2.6620e−06	1.0399e−07
2^{-5}	0.013863	0.001661	0.000602	5.1069e−01	5.3073e−03	5.6208e−04

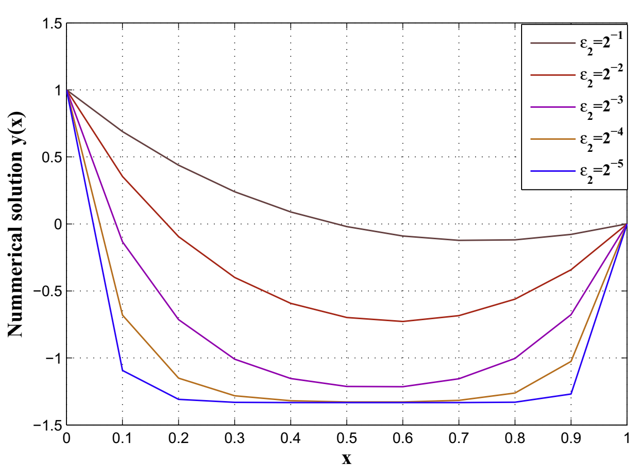

Fig. 4. Numerical solution $y(x)$ of Eq. (33) whit $N = 50$ and $\delta = 0.03$.

Table 3 shows the numerical solutions of Eq. (32) by the presented method for $N = 50$ and $\tau = 0.1\varepsilon$. Table 4 gives the comparison of the result of the maximum error $e_N(x) = \|y_N(x) - y(x)\|_\infty = \max\{|y_N(x) - y(x)|, \quad 0 \leqslant x \leqslant 1\}$ obtained by the present method and the method of [26] for different values of N. Fig. 3. displays the numerical solution $y(x)$ of the present method for $N = 50$ and $\tau = 0.1\varepsilon$.

Example 3. Consider the following singularly perturbed delay differential equation with left layer[14]

$$\varepsilon y''(x) + 0.25y(x - \delta) - y(x) = 1, \tag{33}$$

on [0,1], under the boundary conditions $y(x) = 1, -\delta \leqslant x \leqslant 0$ and $y(1) = 0$ and $\varepsilon = \varepsilon_2^2$.

From Eq. (25), the fundamental matrix equation of the problem is

$$\left\{\mathbf{EXD}^{T2}\mathbf{Z}^T + \mathbf{AXM}^T_{-\delta}\mathbf{Z}^T + \mathbf{\Omega XZ}^T\right\}\mathbf{C}^T = \mathbf{G}.$$

Table 5 shows the numerical solutions of Eq. (33) by the presented method for $N = 50$ and $\delta = 0.03$. Table 6 gives the comparison of the result of the maximum error e_N obtained by the present method and the method of [14] for different values of N. Fig. 4. displays the numerical solution $y(x)$ by the present method for $N = 50$ and $\delta = 0.03$.

7. Conclusion

In this paper, we have worked out a computational method for approximate solution of singularly perturbed differential-difference equations, based on the expansion of the solution as a series of Fibonacci polynomials. This expansion, besides the collocation method has been used for transforming these equations to a linear system of algebraic equations that can be solved easily. To obtain the best approximating solution of the system, we take more forms from the Fibonacci expansion of functions, that is, the truncation limit N must be chosen to be large enough. We have given here only a few values although the solutions can be computed at desired number of uniform points. It can be observed from the tables that this method approximates the exact solution very well with a small number of sampling points. This shows the efficiency and accuracy of the present method. Illustrative examples are given to demonstrate the validity and applicability of proposed method.

Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.rinp.2013.08.001.

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