Oscillation and Asymptotics for Nonlinear Second-Order Differential Equations

HONG-WU WU, QI-RU WANG AND YUAN-TONG XU*

Department of Mathematics, Sun Yat-Sen (Zhongshan) University
Guangzhou 510275, P.R. China

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Abstract—This paper discusses a class of second-order nonlinear differential equations. By using the generalized Riccati technique and the averaging technique, new oscillation criteria are obtained for all solutions of the equation to be oscillatory. Asymptotic behavior for forced equations is also discussed. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Nonlinear differential equation, Oscillation, Asymptotic behavior, Second order, Generalized Riccati technique, Averaging technique.

1. INTRODUCTION

In this paper, we consider the second-order nonlinear differential equation of the form

\[ [r(t)\Phi(x(t))\varphi(x'(t))]' + c(t)\varphi(x(t)) = 0, \quad t \geq t_0, \]

(1)

where \( r(t), c(t) \in C([t_0, \infty), \mathbb{R}) \), \( \Phi \in C(\mathbb{R}, \mathbb{R}) \), and \( \varphi(s) \) is a real-valued function defined by \( \varphi(s) = |s|^{p-2}s \) with \( p > 1 \) a fixed real number.

In what follows, we shall assume the following conditions hold.

(a) \( r(t) > 0; \)
(b) \( 0 < \Phi(x) \leq \gamma, \) for all \( x \) where \( \gamma \) is a real number.

By a solution of equation (1), we mean a function \( x \in C^1([T_x, \infty), \mathbb{R}) \) for some \( T_x \geq t_0 \) which has the property that \( r(t)\Phi(x(t))\varphi(x'(t)) \in C^1([T_x, \infty), \mathbb{R}) \) and satisfies equation (1) on \([T_x, \infty).\)

A solution of equation (1) is called oscillatory if it has arbitrarily large zeros, otherwise, it is called nonoscillatory. Finally, equation (1) is called oscillatory if all its solutions are oscillatory.

The oscillation problem for (1) and its various particular cases such as the half-linear differential equation

\[ [r(t)\varphi(x'(t))]' + c(t)\varphi(x(t)) = 0 \]

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*Author to whom all correspondence should be addressed at xyt@zsu.edu.cn.

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has been studied extensively in recent years, e.g., see [1–18] and the references cited therein. Note that in 2002, Ayanlar and Tiryaki [1] established oscillation criteria for equation (1) and one of the main results is as follows.

**Theorem A.** Let \( D_0 = \{(t, s) : t > s \geq t_0\} \) and \( D = \{(t, s) : t \geq s \geq t_0\} \). Assume that \( H \in C(D, \mathbb{R}) \) satisfies the following two conditions.

(i) \( H(t, t) = 0 \), for \( t \geq t_0 \), \( H(t, s) > 0 \), for \( t > s \geq t_0 \).

(ii) \( H \) has a continuous and nonpositive partial derivative on \( D \) with respect to the second variable.

Suppose that \( h : D_0 \to \mathbb{R} \) is a continuous function with

\[
-\frac{\partial H(t, s)}{\partial s} = h(t, s)[H(t, s)]^{1/q}, \quad \text{for all } (t, s) \in D_0,
\]

where \( (1/p) + (1/q) = 1 \). If there exists a function \( \rho(t) \in C^1([t_0, \infty), (0, \infty)) \) such that \( \rho'(t) \geq 0 \), for all \( t \geq t_0 \) and

\[
\limsup_{t \to \infty} \left\{ X(t, t_0) - \gamma Y(t, t_0) \right\} = \infty,
\]

where

\[
X(t, t_0) = \frac{1}{H(t, t_0)} \int_{t_0}^{t} H(t, s)\rho(s)c(s) \, ds,
\]

\[
Y(t, t_0) = \frac{1}{H(t, t_0)} \int_{t_0}^{t} \rho(s)r(s) \left[ \frac{1}{p} \left( h(t, s) + [H(t, s)]^{1/p} \frac{\rho'(s)}{\rho(s)} \right) \right]^{p} \, ds,
\]

then equation (1) is oscillatory.

Motivated by the idea of Wang [2], in the present paper, we do not require the condition “\( \rho'(t) \geq 0 \)” in [1] and shall establish, by employing the generalized Riccati technique and the integral averaging technique, several new criteria for oscillation of equation (1). We also discuss the asymptotic behavior for forced equations. Our results extend, improve, and unify Theorem A and a number of other existing results and handle the cases which are not covered by known criteria. Finally, several interesting examples are also included to show the versatility of our results.

**2. MAIN RESULTS**

In order to prove our theorems, we use the following well-known inequality which is due to Hardy, Littlewood and Polya [19].

**Lemma 1.** If \( X \) and \( Y \) are nonnegative, then

\[
X^\lambda - \lambda XY^{\lambda-1} + (\lambda - 1)Y^\lambda \geq 0, \quad \lambda > 1,
\]

where the equality holds if and only if \( X = Y \).

**Theorem 1.** In equation (1), suppose that Conditions (a) and (b) hold. Let \( D_0 = \{(t, s) : t > s \geq t_0\} \) and \( D = \{(t, s) : t \geq s \geq t_0\} \). Moreover, suppose that there exist functions \( h \in C(D_0, \mathbb{R}) \), \( h \in C(D_0, \mathbb{R}) \), and \( \rho(t) \in C^1([t_0, \infty), (0, \infty)) \) that satisfy the following three conditions.

(i) \( H(t, t) = 0 \) for \( t \geq t_0 \), \( H(t, s) > 0 \) for \( (t, s) \in D_0 \).

(ii) \( H \) has a continuous and nonpositive partial derivative on \( D_0 \) with respect to the second variable.

(iii)

\[
-\frac{\partial H}{\partial s}(t, s) - H(t, s)\frac{\rho'(s)}{\rho(s)} = h(t, s), (t, s) \in D_0.
\]
If
\[ \limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left[ H(t, s) \rho(s) c(s) - \frac{\gamma \rho(s) r(s) [h(t, s)]^p}{p^p H(p-1)(t, s)} \right] ds = \infty, \] (2)
then equation (1) is oscillatory.

**Proof.** Let \( x(t) \) be a nonoscillatory solution of equation (1). Without loss of generality, we may assume that \( x(t) \neq 0 \) on \( [T_0, \infty) \) for some sufficiently large \( T_0 \geq t_0 \). Now, we define
\[ \omega(t) = \rho(t) \frac{r(t) \Phi(x(t)) \varphi(x'(t))}{\varphi(x(t))}, \quad \text{for } t \geq T_0. \] (3)

Then, in view of (a), (b), and making use of equation (1), we obtain that for \( t \geq T_0 \),
\begin{align*}
\omega'(t) &= -\rho(t) c(t) + \frac{\rho'(t)}{\rho(t)} \omega(t) - (p - 1) \left[ \Phi(x(t)) \rho(t) r(t) \right]^{1-q} |\omega(t)|^q \\
&\leq -\rho(t) c(t) + \frac{\rho'(t)}{\rho(t)} \omega(t) - (p - 1) \left[ \gamma \rho(t) r(t) \right]^{1-q} |\omega(t)|^q,
\end{align*}
(4)
where \( 1/p + 1/q = 1 \). We multiply (4), with \( t \) replaced by \( s \), by \( H(t, s) \) and integrate from \( T \geq T_0 \) to \( t > T \), which in view of (i)--(iii) leads to
\begin{align*}
&\int_T^t H(t, s) \rho(s) c(s) ds \\
&\leq -\int_T^t H(t, s) \omega'(s) ds + \int_T^t H(t, s) \frac{\rho'(s)}{\rho(s)} \omega(s) ds \\
&\quad - \int_T^t (p - 1) H(t, s) \left[ \gamma \rho(s) r(s) \right]^{1-q} |\omega(s)|^q ds \\
&= -H(t, s) \omega(s) \left[ t - \int_T^t (p - 1) H(t, s) \left[ \gamma \rho(s) r(s) \right]^{1-q} |\omega(s)|^q ds \\
&\quad - \int_T^t \left[ -\frac{\partial H}{\partial s}(t, s) - H(t, s) \frac{\rho'(s)}{\rho(s)} \right] \omega(s) ds \\
&= H(t, T) \omega(T) - \int_T^t (p - 1) H(t, s) \left[ \gamma \rho(s) r(s) \right]^{1-q} |\omega(s)|^q ds - \int_T^t h(t, s) \omega(s) ds \\
&\leq H(t, T) \omega(T) - \int_T^t (p - 1) H(t, s) \left[ \gamma \rho(s) r(s) \right]^{1-q} |\omega(s)|^q ds + \int_T^t [h(t, s) \omega(s)] ds.
\end{align*}
(5)
Fix \( t > T \), and set
\[ X = \left[ (p - 1) H(t, s) \left[ \gamma \rho(s) r(s) \right]^{1-q} \right]^{1/q} |\omega(s)|, \quad \lambda = q > 1, \]
and
\[ Y = q^{-p} \left[ (p - 1) H(t, s) \left[ \gamma \rho(s) r(s) \right]^{1-q} \right]^{(1-p)/q} |h(t, s)|^{p-1}. \]
Then, by Lemma 1, we obtain
\[ |h(t, s) \omega(s)| - (p - 1) H(t, s) \left[ \gamma \rho(s) r(s) \right]^{1-q} |\omega(s)|^q \leq \frac{\gamma \rho(s) r(s) [h(t, s)]^p}{p^p H(p-1)(t, s)}. \]
So, from (5), for \( t > T \), we obtain
\[ \int_T^t H(t, s) \rho(s) c(s) ds \leq H(t, T) \omega(T) + \int_T^t \frac{\gamma \rho(s) r(s) [h(t, s)]^p}{p^p H(p-1)(t, s)} ds. \]
From Assumption (ii), it is easy to see
\[
\int_{T}^{t} \left[ H(t,s)\rho(s)c(s) - \frac{\gamma \rho(s)r(s)}{p^p H^{(p-1)}(t,s)} h(t,s)^p \right] ds \leq H(t,T)|\omega(T)| \leq H(t,t_0)|\omega(t_0)|. \tag{6}
\]
We use the above inequality for \( T = T_0 \) to obtain
\[
\int_{T_0}^{t} \left[ H(t,s)\rho(s)c(s) - \frac{\gamma \rho(s)r(s)}{p^p H^{(p-1)}(t,s)} h(t,s)^p \right] ds = \int_{T_0}^{T} \left[ H(t,s)\rho(s)c(s) - \frac{\gamma \rho(s)r(s)}{p^p H^{(p-1)}(t,s)} h(t,s)^p \right] ds + \int_{T_0}^{t} \left[ H(t,s)\rho(s)c(s) - \frac{\gamma \rho(s)r(s)}{p^p H^{(p-1)}(t,s)} h(t,s)^p \right] ds \\
\leq H(t,t_0) \int_{T_0}^{t} \rho(s)c(s) ds + H(t,t_0)|\omega(T_0)|.
\]
Therefore,
\[
\limsup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{T_0}^{t} \left[ H(t,s)\rho(s)c(s) - \frac{\gamma \rho(s)r(s)}{p^p H^{(p-1)}(t,s)} h(t,s)^p \right] ds \\
\leq \int_{T_0}^{T_0} \rho(s)c(s) ds + |\omega(T_0)| < \infty.
\]
We obtain a contradiction to condition (2).

This completes the proof of Theorem 1.

Letting \( H(t,s) = (t-s)^k \) for \( (t,s) \in D \), where \( k > 1 \) is a constant, we have
\[
h(t,s) = (t-s)^{k-1} \left[ k - (t-s) \frac{\rho'(s)}{\rho(s)} \right].
\]
Then, from Theorem 1, we get the following corollary.

**Corollary 1.** In equation (1), let Conditions (a) and (b) hold. If there exist \( \rho \in C^1([t_0, \infty), (0, \infty)) \) and constant \( k > 1 \) such that
\[
\limsup_{t \to \infty} \frac{1}{t^k} \int_{t_0}^{t} \left[ (t-s)^k \rho(s)c(s) - \left( \frac{\rho'}{\rho} \right) \rho(s)r(s)(t-s)^{k-p} \left| k - (t-s) \frac{\rho'(s)}{\rho(s)} \right|^p \right] ds = \infty,
\]
then equation (1) is oscillatory.

**Theorem 2.** Let functions \( H, h, \) and \( \rho \) satisfy Conditions (i)–(iii) in Theorem 1. Moreover, suppose that
\[
0 < \inf_{t \geq t_0} \left\{ \liminf_{t \to \infty} \frac{H(t,s)}{H(t,t_0)} \right\} \leq \infty \tag{7}
\]
and
\[
\limsup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{\rho(s)r(s)|h(t,s)|^p}{H^{(p-1)}(t,s)} ds < \infty. \tag{8}
\]
If there exists a function \( A \in C([t_0, \infty), \mathbb{R}) \) such that
\[
\int_{t_0}^{\infty} \frac{(A_+(s))^{p/(p-1)}}{(\rho(s)r(s))^{1/(p-1)}} ds = \infty, \tag{9}
\]
and for every $T \geq t_0$,

\[
\limsup_{t \to \infty} \frac{1}{H(t, T)} \int_T^t \left[ H(t, s) \rho(s) c(s) - \frac{\gamma \rho(s) r(s) |h(t, s)|^p}{p^p H^{(p-1)}(t, s)} \right] ds \geq A(T),
\]

(10)

where $A_+(t) = \max\{A(t), 0\}$, then equation (1) is oscillatory.

**Proof.** Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1) such that $x(t) \neq 0$ on $[T_0, \infty)$ for some sufficiently large $T_0 \geq t_0$. Define $\omega(t)$ as in (3). As in the proof of Theorem 1, we obtain (5) and (6). It follows that

\[
\limsup_{t \to \infty} \frac{1}{H(t, T)} \int_T^t \left[ H(t, s) \rho(s) c(s) - \frac{\gamma \rho(s) r(s) |h(t, s)|^p}{p^p H^{(p-1)}(t, s)} \right] ds \leq \omega(T),
\]

for all $T \geq T_0$. Thus, by (10), we have

\[
A(T) \leq \omega(T), \quad \text{for all } T \geq T_0
\]

(11)

and

\[
\limsup_{t \to \infty} \frac{1}{H(t, T)} \int_T^t H(t, s) \rho(s) c(s) ds \geq A(T_0).
\]

(12)

Let

\[
F(t) = \frac{1}{H(t, T_0)} \int_T^t |h(t, s)\omega(s)| ds
\]

and

\[
G(t) = \frac{p-1}{H(t, T_0)} \int_T^t H(t, s) \frac{|\omega(s)|^{p/(p-1)}}{\rho(s)r(\rho(s))^{1/(p-1)}} ds,
\]

for $t > T_0$. Then, by (5) and (12), we get that

\[
\liminf_{t \to \infty} [G(t) - F(t)] \leq \omega(T_0) - \limsup_{t \to \infty} \frac{1}{H(t, T_0)} \int_T^t H(t, s) \rho(s) c(s) ds
\]

\[
\leq \omega(T_0) - A(T_0) < \infty.
\]

(13)

Now, we claim that

\[
\int_{T_0}^\infty \frac{|\omega(s)|^{p/(p-1)}}{\rho(s)r(\rho(s))^{1/(p-1)}} ds < \infty.
\]

(14)

Suppose to the contrary that

\[
\int_{T_0}^\infty \frac{|\omega(s)|^{p/(p-1)}}{\rho(s)r(\rho(s))^{1/(p-1)}} ds = \infty.
\]

(15)

By (7), there is a positive constant $\eta$ satisfying

\[
\inf_{s \geq t_0} \left\{ \liminf_{t \to \infty} \frac{H(t, s)}{H(t, t_0)} \right\} > \eta > 0.
\]

(16)

On the other hand, by (15), for any positive number $\mu$, there exists a $T_1 > T_0$ such that

\[
\int_{T_0}^t \frac{|\omega(s)|^{p/(p-1)}}{\rho(s)r(\rho(s))^{1/(p-1)}} ds \geq \frac{\mu}{(p-1)\eta}, \quad \text{for all } t \geq T_1,
\]
so for all \( t \geq T_1 \),

\[
G(t) = \frac{p-1}{H(t, T_1)} \int_{T_0}^{t} H(t, s) \left[ \int_{T_0}^{s} \frac{|\omega(u)|^{p/(p-1)}}{(\rho(u)r(u))^{1/(p-1)}} \, du \right] \left[ \int_{T_0}^{s} \frac{|\omega(u)|^{p/(p-1)}}{(\rho(u)r(u))^{1/(p-1)}} \, d\nu(u) \right] \, ds
\]

\[
\leq \frac{(p-1)}{H(t, T_1)} \int_{T_1}^{t} \left[ \frac{\partial H}{\partial s}(t, s) \right] \left[ \int_{T_0}^{s} \frac{|\omega(u)|^{p/(p-1)}}{(\rho(u)r(u))^{1/(p-1)}} \, du \right] \, ds
\]

\[
\geq \frac{(p-1)\mu}{(p-1)\eta H(t, T_0)} \int_{T_1}^{t} \left[ \frac{\partial H}{\partial s}(t, s) \right] \, ds = \frac{\mu H(t, T_1)}{\eta H(t, T_0)}.
\]

From (16), we have

\[
\lim_{t \to \infty} \frac{H(t, T_1)}{H(t, T_0)} > \eta > 0,
\]

so there exists \( T_2 \geq T_1 \) such that \( H(t, T_1)/H(t, T_0) \geq \eta, \) for all \( t \geq T_2 \). Therefore, by (17), \( G(t) \geq \mu, \) for all \( t \geq T_2 \), and since \( \mu \) is arbitrary constant, we conclude that

\[
\lim_{t \to \infty} G(t) = \infty.
\]

Next, consider a sequence \( \{t_n\}_{n=1}^{\infty} \) in \((T_0, \infty)\) with \( \lim_{n \to \infty} t_n = \infty \) and such that

\[
\lim_{n \to \infty} [G(t_n) - F(t_n)] = \liminf_{t \to \infty} [G(t) - F(t)].
\]

In view of (13), there exists a constant \( M \) such that

\[
G(t_n) - F(t_n) \leq M, \quad \text{for all sufficiently large } n.
\]

It follows from (18) that

\[
\lim_{n \to \infty} G(t_n) = \infty.
\]

This and (19) give

\[
\lim_{n \to \infty} F(t_n) = \infty.
\]

Then, by (19) and (20),

\[
\frac{F(t_n)}{G(t_n)} - 1 \geq -\frac{M}{G(t_n)} > -\frac{1}{\frac{1}{2}}, \quad \text{for all } n \text{ large enough.}
\]

Thus,

\[
\frac{F(t_n)}{G(t_n)} > \frac{1}{2}, \quad \text{for all } n \text{ large enough.}
\]

This and (21) imply that

\[
\lim_{n \to \infty} \frac{F'(t_n)}{G(t_n)} = \infty.
\]

On the other hand, by the Hölder inequality, we have

\[
F(t_n) = \frac{1}{H(t_n, T_0)} \int_{T_0}^{t_n} |h(t_n, s)\omega(s)| \, ds
\]

\[
\leq \left\{ \frac{p-1}{H(t_n, T_0)} \int_{T_0}^{t_n} H(t_n, s) \left[ \frac{|\omega(s)|^{p/(p-1)}}{(\rho(s)r(s))^{1/(p-1)}} \right]^{(p-1)/p} \, ds \right\}^{1/p}
\]

\[
\times \left\{ \frac{1}{(p-1)^{(p-1)/p} H(t_n, T_0)} \int_{T_0}^{t_n} \rho(s)r(s) |h(t_n, s)|^p \left( H(t_n, s) \right)^{(p-1)} \, ds \right\}^{1/p}
\]

\[
\leq \frac{G^{(p-1)/p}(t_n)}{(p-1)^{(p-1)/p}} \left\{ \frac{1}{H(t_n, T_0)} \int_{T_0}^{t_n} \rho(s)r(s) |h(t_n, s)|^p \left( H(t_n, s) \right)^{(p-1)} \, ds \right\}^{1/p}.
\]
and therefore,
\[ \frac{F^p(t_n)}{G^{(p-1)}(t_n)} \leq \frac{1}{(p-1)(p-1)} \int_{t_n}^{t_{n+1}} \frac{\rho(s)r(s)|h(t_n, s)|^p}{(H(t_n, s))^{(p-1)}} ds \]
\[ \leq \frac{1}{(p-1)(p-1)} \eta H(t_n, t_0) \int_{t_0}^{t_n} \frac{\rho(s)r(s)|h(t_n, s)|^p}{(H(t_n, s))^{(p-1)}} ds, \]
for all large \( n \). It follows from (22) that
\[ \lim_{n \to \infty} \frac{1}{H(t_n, t_0)} \int_{t_0}^{t_n} \frac{\rho(s)r(s)|h(t_n, s)|^p}{(H(t_n, s))^{(p-1)}} ds = \infty, \] (23)
that is,
\[ \limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \frac{\rho(s)r(s)|h(t, s)|^p}{H^{(p-1)}(t, s)} ds = \infty, \]
which contradicts (8). Hence, (14) holds. Then, it follows from (11) that
\[ \int_{\infty}^{\infty} \frac{(A_+(s))^{p/(p-1)}}{(\rho(s)r(s))^{1/(p-1)}} ds \leq \int_{T_0}^{\infty} \frac{|\omega(s)|^{p/(p-1)}}{(\rho(s)r(s))^{1/(p-1)}} ds < \infty, \]
which contradicts (9).

This completes the proof of Theorem 2.

**Theorem 3.** Let functions \( H, h, \) and \( \rho \) satisfy Conditions (i)-(iii) in Theorem 1. Moreover, suppose that (7) holds and
\[ \liminf_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} H(t, s)\rho(s)c(s) ds < \infty. \] (24)
If there exists a function \( A \in C([t_0, \infty), \mathbb{R}) \) such that (9) holds and for every \( T \geq t_0, \)
\[ \liminf_{t \to \infty} \frac{1}{H(t, T)} \int_{T}^{t} \left[ H(t, s)\rho(s)c(s) - \frac{\gamma\rho(s)r(s)|h(t, s)|^p}{p\rho H^{(p-1)}(t, s)} \right] ds \geq A(T), \] (25)
where \( A_+(s) = \max\{A(s), 0\} \), then equation (1) is oscillatory.

**Proof.** Without loss of generality, we may assume that there exists a solution \( x(t) \) of equation (1) such that \( x(t) \neq 0 \) on \([T_0, \infty)\) for some sufficiently large \( T_0 \geq t_0 \). Define \( \omega(t) \) as in (3). As in the proofs of Theorems 1 and 2, we can obtain (5), (6), and (11). From (24), it follows that
\[ \limsup_{t \to \infty} [G(t) - F(t)] \leq v(T_0) - \liminf_{t \to \infty} \frac{1}{H(t, T_0)} \int_{T_0}^{t} H(t, s)\rho(s)c(s) ds < \infty, \] (26)
where \( F(t) \) and \( G(t) \) are defined as in the proof of Theorem 2. By (25), we have
\[ A(t_0) \leq \liminf_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} H(t, s)\rho(s)c(s) ds \]
\[ - \liminf_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \frac{\gamma\rho(s)r(s)|h(t, s)|^p}{p\rho H^{(p-1)}(t, s)} ds. \]
This and (24) imply that
\[ \liminf_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \frac{\rho(s)r(s)}{H^{(p-1)}(t, s)}|h(t, s)|^p ds < \infty. \]
Considering a sequence \( \{ t_n \} \) in \((T_0, \infty)\) with \( \lim_{n \to \infty} t_n = \infty \) and such that
\[
\lim_{n \to \infty} \frac{1}{H(t_n, t_0)} \int_{t_0}^{t_n} \frac{\rho(s)r(s)}{H(t_n, s)^{(p-1)}} |h(t_n, s)|^p \, ds = \lim_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \frac{\rho(s)r(s)}{H(t)^{(p-1)}} |h(t, s)|^p \, ds < \infty. \tag{27}
\]

Now, suppose that (15) holds. With the same argument as in Theorem 2, we conclude that (18) is satisfied. By (26), there exists a constant \( M \) such that (19) is fulfilled. Then, following the procedure of the proof of Theorem 2, we see that (23) holds, which contradicts (27). This contradiction proves that (15) fails. The remainder of the proof is similar to that of Theorem 2, so we omit the details.

This completes the proof of Theorem 3.

Similarly, we have the following.

**Theorem 4.** Let functions \( H, h, \) and \( \rho \) satisfy Conditions (i)-(iii) in Theorem 1. Moreover, suppose that (7) holds and
\[
\liminf_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \frac{\rho(s)r(s)}{H(t)^{(p-1)}} |h(t, s)|^p \, ds < \infty.
\]

If there exists a function \( A \in C([t_0, \infty), \mathbb{R}) \) such that (9) and (25) hold, then equation (1) is oscillatory.

**Remark.** In [1], the authors used the following inequality to get (7) in [1]:
\[
\frac{\partial H}{\partial s}(t, s)\omega(s) + \frac{\rho'(s)}{\rho(s)} H(t, s)\omega(s) = -h(t, s)\left[H(t, s)\right]^{1/q}\omega(s) + \frac{\rho'(s)}{\rho(s)} H(t, s)\omega(s) \leq h(t, s)\left[H(t, s)\right]^{1/q}\omega(s) + \frac{\rho'(s)}{\rho(s)} H(t, s)\omega(s).
\]

Comparing this with inequality (5), our method presented in this paper seems more reasonable. The important point to note here is that when \( \rho'(t) \leq 0 \), Theorems 1, 2, 3, and 4 reduce to Theorems 2, 4, 5, and 6 of [1], respectively.

When \( \Phi(x) \equiv 1 \) for all \( x \in \mathbb{R} \), Theorems 1–3 reduce to Theorems 1–3 of [2]. Our results also extend and improve Theorems 1–6 of [12], the oscillation criteria of [8–10] and a number of other existing results.

From Theorems 1–4, we can present different explicit sufficient conditions for the oscillation of equation (1) by appropriate choices of \( H(t, s) \) and \( \rho(s) \). For instance, if we choose \( H(t, s) = (t-s)^k \), \( H(t, s) = [R(t)-R(s)]^k \), or \( H(t, s) = [\log(Q(t)/Q(s))]^k \), \( H(t, s) = [\int_{s}^{t} \frac{ds}{R(s)}]^k \), for \( t \geq s \geq t_0 \), \( \rho(s) \) may choose 1, s, etc., where \( k > 1 \) is a constant, \( R(t) = \int_{t_0}^{t} \frac{ds}{R(s)} \), \( Q(t) = \int_{t_0}^{\infty} \frac{ds}{R(s)} < \infty \), for \( t \geq t_0, \theta \in C([t_0, \infty), (0, \infty)) \) satisfying \( \int_{t_0}^{\infty} \frac{ds}{R(s)} = \infty \), then from Theorems 1–4, we can derive some explicit oscillation criteria.

Let us consider the following examples to better understand our criteria. We note that the conclusions do not appear to follow from the known oscillation criteria in the literature [1–18].

**Example 1.** Consider the differential equation of the form
\[
[t^\alpha \left(1 + e^{-|x(t)|}\right)] |x'(t)|^{p-2} x'(t) + [\beta t^\theta \sin t] |x(t)|^{p-2} x(t) = 0, \tag{28}
\]
for \( t \geq t_0 \geq 1 \), where \( p > 1 \), \( \beta > 1 \), and \( \alpha \) are constants such that \( \alpha < p - 2 \). Here we take \( \rho(t) = t^2 \) and \( H(t, s) = (t - s)^2 \) for \( t \geq s \geq t_0 \). Since
\[
\int_{t_0}^{t} \rho(s)c(s)\,ds = \int_{t_0}^{t} s^2 [\beta s^{\beta-3}(2 - \cos s) + s^{\beta-2} \sin s] \,ds
\]
\[
= \int_{t_0}^{t} [\beta s^{\beta-1}(2 - \cos s) + s^\beta \sin s] \,ds
\]
\[
= t^\beta (2 - \cos t) - t_0^\beta (2 - \cos t_0)
\]
\[
= t^\beta (2 - \cos t) - k_0 \geq t^\beta - k_0,
\]
it follows that
\[
\limsup_{t \to \infty} \frac{1}{t^2} \int_{t_0}^{t} \left\{ (t - s)^2 \rho(s)c(s) - \left( \frac{\gamma}{p^p} \right) \rho(s)r(s)(t - s)^{2-p} \left| 2 - (t - s) \frac{\rho'(s)}{\rho(s)} \right| \right\} \,ds
\]
\[
= \limsup_{t \to \infty} \frac{1}{t^2} \int_{t_0}^{t} \left( 2(t - s) \left( \int_{t_0}^{s} \rho(u)c(u)\,du \right) - \left( \frac{2}{pp} \right) s^{\alpha+2}(t - s)^{2-p} \left| 4 - 2\frac{t^p}{s} \right| \right) \,ds
\]
\[
\geq \limsup_{t \to \infty} \frac{1}{t^2} \int_{t_0}^{t} \left( 2(t - s) (s^\beta - k_0) - \left( \frac{2p+1}{pp} \right) t^p(t - s)^{2-p} \right) \,ds
\]
\[
\geq \limsup_{t \to \infty} \left\{ \frac{2t^\beta}{(\beta + 1)(\beta + 2)} + \frac{k_1}{t^2} + \frac{k_2}{t^2} + k_0 - \frac{k_2t}{3-p} \left( 1 - t_0 \right)^{3-p} \right\} = \infty,
\]
where
\[
k_1 = \frac{2t_0^{\beta+2}}{\beta + 2} - k_0 t_0^\beta, \quad k_2 = 2k_0 t_0 - \frac{2t_0^{\beta+1}}{\beta + 1}, \quad k_3 = \frac{2p+1}{pp}.
\]
All conditions of Corollary 1 are satisfied, and hence, equation (28) is oscillatory.

**Example 2.** Consider the differential equation of the form
\[
\left[ t^{-1/2} \left( 1 + e^{-|x(t)|} \right) x'(t) \right]' + t^{-5/2} x(t) = 0, \quad t \geq t_0 > 1.
\]
(29)

Here we take \( \rho(s) = s \) and \( H(t, s) = (t - s)^2 \). Note that
\[
\liminf_{t \to \infty} \frac{1}{(t - t_0)^2} \int_{t_0}^{t} \frac{1}{4} s^{1/2} \left( 3 - \frac{t}{s} \right)^2 \,ds < \infty
\]
and
\[
\liminf_{t \to \infty} \frac{1}{(t - T)^2} \int_{t}^{T} \left\{ (t - s)^2 s^{-3/2} - \frac{1}{2} s^{1/2} \left( 3 - \frac{t}{s} \right)^2 \right\} \,ds \geq T^{-1/2}, \quad T \geq t_0 > 1.
\]
Setting \( A(T) = T^{-1/2} \), it is clear that
\[
\limsup_{t \to \infty} \int_{t_0}^{t} \frac{(A_+(s))^{p/(p-1)}}{\rho(s)r(s)^{1/(p-1)}} \,ds = \limsup_{t \to \infty} \int_{t_0}^{t} \frac{(A_+(s))^2}{s^{1/2}} \,ds = \limsup_{t \to \infty} \int_{t_0}^{t} s^{-1/2} \,ds = \infty.
\]
All conditions of Theorem 4 are satisfied, and hence, equation (29) is oscillatory.
3. ASYMPTOTICS FOR FORCED EQUATIONS

In this section, we discuss the asymptotic behavior of solutions of the forced equation

$$r(t)\Phi(x(t))\varphi(x'(t))' + c(t)\varphi(x(t)) = e(t), \quad t \geq t_0,$$

where $r(t), \Phi, c, \varphi \in C(\mathbb{R}, \mathbb{R})$, and $e(t)$ is defined as in equation (1), and $e(t) \in C([t_0, \infty), \mathbb{R})$.

**Theorem 5.** Let functions $H, h, \rho$ satisfy Conditions (i)-(ii1) in Theorem 1. If (2) holds and

$$\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} H(t, s)\rho(s)|e(s)| \, ds < \infty,$$

then every solution $x(t)$ of equation (30) satisfies $\liminf_{t \to \infty} |x(t)| = 0$.

**Proof.** Let $x(t)$ be a solution of equation (30) and suppose by contradiction that

$$\limsup_{t \to \infty} |x(t)| > 0,$$

so $x(t)$ is nonoscillatory. Without loss of generality, we may assume that $x(t) \neq 0$ on $[T_0, \infty)$ for some sufficiently large $T_0 \geq t_0$. Define $\omega(t)$ as in (3), then differentiating (3) and making use

for all $t \geq T_0$, where $\beta = \inf_{t \geq T_0} |x(t)| > 0$. Hence, as in the proof of Theorem 1, we can obtain

$$\int_{T_0}^{t} \left[ H(t, s)\rho(s)c(s) - \frac{\gamma(t)\rho(s)|h(t, s)|^p}{p^p H^{(p-1)}(t, s)} \right] \, ds \leq \frac{1}{\beta^{(p-1)}} \int_{T_0}^{t} H(t, s)\rho(s)|e(s)| \, ds.$$

It follows that

$$\int_{T_0}^{t} \left[ H(t, s)\rho(s)c(s) - \frac{\gamma \rho(s)r(s)|h(t, s)|^p}{p^p H^{(p-1)}(t, s)} \right] \, ds$$

$$= \int_{T_0}^{t} \left[ H(t, s)\rho(s)c(s) - \frac{\gamma \rho(s)r(s)|h(t, s)|^p}{p^p H^{(p-1)}(t, s)} \right] \, ds$$

$$+ \int_{T_0}^{t} \left[ H(t, s)\rho(s)c(s) - \frac{\gamma \rho(s)r(s)|h(t, s)|^p}{p^p H^{(p-1)}(t, s)} \right] \, ds$$

$$\leq H(t, t_0) \int_{t_0}^{t_0} \rho(s) |c(s)| \, ds$$

$$+ H(t, t_0) |\omega(T_0)| + \frac{1}{\beta^{(p-1)}} \int_{t_0}^{t} H(t, s)\rho(s)|e(s)| \, ds.$$
This completes the proof of Theorem 5.

EXAMPLE 3. Consider the nonlinear differential equation of the form

$$2 \frac{x''(t)}{1 + x'(t)^2} x'(t) + t^{-2} x(t)^2 x(t) = 5 t^{-6} \frac{2t^2 + 1}{t^2 + 1} - \frac{2t^{-4}}{(t^2 + 1)^2} + t^{-5}, \tag{32}$$

where $t \geq t_0 > 1$.

Taking $\rho(t) = t^2$ and $H(t, s) = (t - s)^4$, then we obtain

$$\limsup_{t \to \infty} \frac{1}{t^4} \int_{t_0}^{t} \left( (t - s)^4 \rho(s) c(s) - \left( \frac{\gamma}{p'} \right) \rho(s) r(s)(t - s)^{k - p} \left| k - (t - s)^{p'}(s) \right| \right) ds$$

$$= \limsup_{t \to \infty} \frac{1}{t^4} \int_{t_0}^{t} \left( (t - s)^4 - \frac{2}{4^4} s^{3} \left( 6 - 2 \frac{t}{s} \right)^4 \right) ds$$

$$\geq \limsup_{t \to \infty} \frac{1}{t^4} \int_{t_0}^{t} \left( (t - s)^4 - \frac{1}{8} s^{-1} (3s - t)^4 \right) ds = \infty.$$

On the other hand,

$$\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} H(t, s) \rho(s) |e(s)| ds$$

$$= \limsup_{t \to \infty} \frac{1}{t^4} \int_{t_0}^{t} (t - s)^4 s^2 \left( 5s^{-6} \frac{2s^2 + 1}{s^2 + 1} - \frac{2s^{-4}}{(s^2 + 1)^2} + s^{-3} \right) ds$$

$$\leq \limsup_{t \to \infty} \frac{1}{t^4} \int_{t_0}^{t} (t - s)^4 \left( 5s^{-6} \frac{2s^2 + 1}{s^2 + 1} + \frac{2s^{-2}}{(s^2 + 1)^2} + s^{-3} \right) ds < \infty.$$

Thus, by Theorem 5, we conclude that every solution $x(t)$ of equation (32) satisfies $\liminf_{t \to \infty} |x'(t)| = 0$. Observe that $x(t) = t^{-1}$ is a such a solution.

REFERENCES