

Semantics of weakening and contraction

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Abstract

The shriek modality $!$ of linear logic performs two tasks: it restores in annotated from both weakening and contraction. We separate these tasks by introducing two modalities: $\overset{w}{!}$ for weakening and $\overset{c}{!}$ for contraction. These give rise to two logics which are “inbetween” linear and intuitionistic logic: in *affine* (or *weakening*) logic one always has weakening and a $\overset{c}{!}$ for contraction and in *relevant* (or *contraction*) logic one always has contraction and a $\overset{w}{!}$ for weakening. The semantics of these logics is obtained from special kinds of monads, introduced by Anders Kock in the early seventies.

As subtle point is how to retrieve the $!$ of linear logic from $\overset{w}{!}$ and $\overset{c}{!}$. Technically this will be achieved in terms of distributive laws—introduced by Jon Beck.^w We find models where one has $! = \overset{w}{!} \overset{c}{!}$ and also models with $! = \overset{c}{!} \overset{w}{!}$. It will be shown that on the category of complete lattices one has comonads $\overset{w}{!}$ and $\overset{c}{!}$ with $\overset{w}{!} \overset{c}{!} = \overset{c}{!} \overset{w}{!} = !$.

1. Introduction

1.1. A logical introduction

In Girard’s linear logic [10] one does not have the structural rules of weakening and contraction, i.e. the following rules cannot be used.

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B} \text{ (weakening)} \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} \text{ (contraction)}$$

But there is a crucial operator $!$ which restores weakening and contraction in annotated form:

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B} \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}$$

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We seek to separate these tasks by introducing operators $\!_w$ and $\!_c$ for which one has

$$\frac{\Gamma \vdash B}{\Gamma, \!_w A \vdash B} \qquad \frac{\Gamma, \!_c A, \!_c A \vdash B}{\Gamma, \!_c A \vdash B}$$

These operations make good sense in combination with certain tensors. One sometimes finds a first informal explanation of (ordinary, linear) tensors \otimes as “cartesian products without diagonals and projections”. In our investigation [16] into the semantics of the λI -calculus, we used *contraction tensors* \otimes_c as “cartesian products without projections”, and, in a sense dually, *weakening tensors* \otimes_w as “cartesian products without diagonals”. In a more positive description:

	cartesian \times	contraction \otimes_c	weakening \otimes_w	linear \otimes
structure	diagonals + projections	diagonals	projections	—
structural rules	weakening + diagonals	contraction	weakening	—

The last row about structural rules is read as: in case the comma used in context concatenation (Γ, Δ) is interpreted as the indicated tensor, then one obtains a logical system with structural rules as displayed. In this table (and throughout the paper) symmetry is taken for granted.

There are of course plenty of examples of cartesian and linear tensors. The smash product is an example of a contraction tensor; it is often used in the description of partial maps (see e.g. [32]). A weakening tensor can be found in the category of metric spaces and non-distance increasing functions, that is, with $f : (X, d) \rightarrow (X', d')$ if for $x, y \in X$ one has $d'(f(x), f(y)) \leq d(x, y)$. This tensor of (X, d) and (X', d') has $X \times X'$ as underlying set and distance between (x, x') and (y, y') given by $d(x, y) + d(x', y')$. The terminal one-point space 1 is then neutral element (see Definition 2.1(ii) below). Weakening tensors also occur in [15].

The above shriek operations can then be understood as restoring in annotated form the logical rules missing in the last row of the above table, i.e.

	Id	$\!_w$	$\!_c$	$\!$
restores	—	weakening	contraction	weakening + contraction

This gives an informal introduction to two “intermediate” logics: in what we call *relevant* or *contraction logic* (see e.g. [9]) one has contraction as a structural rule and an operation $\!_w$ which restores weakening:

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} \qquad \frac{\Gamma \vdash B}{\Gamma, \!_w A \vdash B}$$

In *affine* or *weakening logic* one has weakening as a structural rule and an operation $\!_c$ which restores contraction:

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B} \qquad \frac{\Gamma, !_{\dot{c}}A, !_{\dot{c}}A \vdash B}{\Gamma, !_{\dot{c}}A \vdash B}$$

Such a logic in which one has weakening was first considered in [18]. Semantics for these weakening and contraction logics are described in Section 6.

The next step is to find models in which the ! from linear logic can be retrieved from ! and !_w. This will be achieved in Section 7. There, ! is not a primitive logical operation.

We should point out that the emphasis in this paper lies on semantics. In the end we do formulate a logic, but we don't really do more than write down the rules we find in a specific model. There is no proof-theoretic analysis.

1.2. An operational introduction

Linear logic is a logic of resources. An occurring formula A —or more precisely, a resource of type A —is used exactly once. The ! operation can be read as: ! A means: A , as often as you like. Here we don't distinguish whether this means finitely or infinitely many times; models of both can be found below. Similarly one can read the contraction operation !_w and the weakening operation !_w as

$$!_{\dot{c}}A \text{ means at least once } A \quad \text{and} \quad !_{\dot{w}}A \text{ means at most once } A.$$

Under this reading one can recover ! from !_w and !_c either as ! $A = !_{\dot{w}}!_{\dot{c}}A$ or as ! $A = !_{\dot{c}}!_{\dot{w}}A$. Of these two the latter is less efficient, because it involves doing nothing many times. We'll encounter models for both ways of recovering !.

These considerations suggest a type assignment system in which one has as typical examples

$$K \equiv \lambda xy. x : A \multimap !_{\dot{w}}B \multimap A \qquad (\text{since } y \text{ is used } \leq 1 \text{ times})$$

$$W \equiv \lambda xy. xyy : (A \multimap A \multimap B) \multimap !_{\dot{c}}A \multimap B \quad (\text{since } y \text{ is used } \geq 1 \text{ times}).$$

In such a linear setting with explicit operations for weakening and contraction, an expression $f : !_{\dot{c}}A \multimap B$ will use an argument $a : A$ at least once. Hence in evaluating an application term fa , it seems wise first to evaluate a , and then bind the result as an argument. This is more efficient. On the other hand, an expression $g : !_{\dot{w}}A \multimap B$ uses its argument at most once. This suggests a different strategy: in an application ga it is more efficient first to evaluate g and then bind a as an argument. In first evaluating a one may lose time since a may not be used at all in (the reduct of) g . Thus in a linear setting one may think of

$$!_{\dot{c}}A \multimap B \text{ as strict functions} \quad \text{and of} \quad !_{\dot{w}}A \multimap B \text{ as non-strict functions}$$

where a function is *strict* if fa is defined only if a is defined. Equivalently, if $f\perp = \perp$. Hence for a non-strict function f one may have that fa is defined, but a not. A typical example is a constant function $x \mapsto b$ for $b \neq \perp$.

These observations suggest that the distinction (at least once/at most once) is operationally more significant than the distinction (precisely once/as often as you like), which is found in linear logic with non-decomposed !. These matters only serve as a motivation and are not further pursued.

1.3. A denotational introduction

The operations $!$ and $\overset{!}{!}$ first emerged in the semantics of “substructure” versions of the untyped λ -calculus (as reported in [16]). Soon it became clear that $!$ and $\overset{!}{!}$ have a wider significance and that they arise in many more situations. It turned out that the category theory needed to capture these operations has all been developed around 1970, notably by Anders Kock and Jon Beck cf. [19–23,6]. Exaggerating a bit, one can say that all the notions are there, but not their logical significance. This will be provided in the present paper, see especially 5.2(iii)—and the similar result 4.3(ii)—and 7.5.

The models investigated here are all categories of algebras of certain monads. A lot of the effort goes into the description of the technical aspects of such monads and their categories of algebras, see Sections 3, 4 and 5. On categories of algebras one always has a comonad, induced by the adjunction with the underlying category. It is this comonad that is particularly interesting from a logical perspective: it often appears as a weakening and/or contraction operation.

There are alternative decompositions of $!$, but they are on a syntactic level and lack a clear denotational semantics. In [13] one finds an infinite family $\{!_n \mid n \in \mathbb{N}\}$ of shrieks with operational meaning of $!_n A$ being n uses of A . Thus our $\overset{!}{!}$ corresponds to $\{!_n \mid n \leq 1\}$ and $\overset{!}{!}$ to $\{!_n \mid n \geq 1\}$. And in [7] one finds a less discriminating distinction of uses which is closer to ours; it is motivated by operational considerations like in Subsection 1.2.

The first four sections contain preliminary expositions about diagonals and projections in monoidal categories, about monads, especially the “affine” and “relevant” monads of Kock, and about tensors in categories of algebras. These enable us to describe models of affine and relevant logic. Putting these together requires the notion of distributive law, which can be found in Section 7. The subsequent section concentrates on the category **CL** of complete lattices. It supports a logic with features from linear, affine, relevant and intuitionistic logic.

Finally we put our (categorically motivated) notation in contrast with Girard’s.

here		Girard	
name	notation	name	notation
(cartesian) products	$(\times, 1)$	direct product, with	$(\&, \top)$
coproducts	$(+, 0)$	direct sum, plus	$(\oplus, 0)$
tensor products	(\otimes, I)	times	$(\otimes, 1)$
tensor sums	(\oplus, J)	par	(\wp, \perp)

2. Diagonals and projections

Let’s recall the basic notions. A *monoidal category* is a 6-tuple $(\mathbf{C}, I, \otimes, \alpha, \lambda, \rho)$ where \mathbf{C} is a category containing a neutral element I for a bifunctor $\otimes: \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ equipped with natural isomorphisms α, λ, ρ having components

$$\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z,$$

$$\lambda_X : I \otimes X \xrightarrow{\sim} X, \quad \rho_X : X \otimes I \xrightarrow{\sim} X.$$

These are required to satisfy the so-called Kelly–Mac Lane equations (given by the pentagon law $(\alpha \otimes id) \circ \alpha \circ (id \otimes \alpha) = \alpha \circ \alpha$, the triangle law $(\rho \otimes id) \circ \alpha = id \otimes \lambda$ and by $\lambda_I = \rho_I$). Such a monoidal category is *symmetric*—and is then called an *SMC*—if there is an additional “symmetry” natural isomorphism γ with $\gamma_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ satisfying some additional equations (to be complete, $\gamma \circ \gamma = id$, $\rho = \lambda \circ \gamma$ and $\alpha \circ \gamma \circ \alpha = (\gamma \otimes id) \circ \alpha \circ (id \otimes \gamma)$).

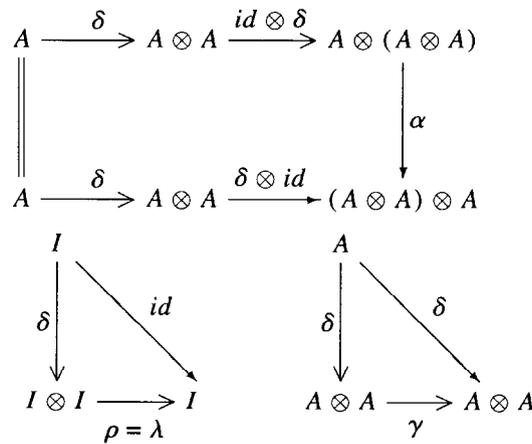
A *monoidal functor* F from $(\mathbf{C}, I, \otimes, \alpha, \lambda, \rho)$ to $(\mathbf{C}', I', \otimes', \alpha', \lambda', \rho')$ is a functor $F : \mathbf{C} \rightarrow \mathbf{C}'$ equipped with a map $\zeta : I' \rightarrow FI$ and a natural transformation $\xi : F(-) \otimes' F(+)$ \rightarrow $F(- \otimes +)$ which match the structure involved (i.e. $F\alpha \circ \xi \circ (id \otimes \xi) = \xi \circ (\xi \otimes id) \circ \alpha'$, $F\lambda \circ \xi \circ (\zeta \otimes id) = \lambda'$ and $F\rho \circ \xi \circ (\zeta \otimes id) = \rho'$). It is a *symmetric monoidal functor* if additionally $F\gamma \circ \xi = \xi \circ \gamma'$. We'll say that F *preserves* the (symmetric) monoidal structure (or: F is a morphism of SMC's) if these ζ and ξ are isomorphisms.

A *monoidal transformation* between monoidal functors $F, F' : \mathbf{C} \rightarrow \mathbf{C}'$ is a natural transformation $\sigma : F \rightarrow F'$ satisfying $\sigma \circ \xi = \xi' \circ (\sigma \otimes' \sigma)$ and $\sigma_I \circ \zeta = \zeta'$. In this way one obtains 2-categories of monoidal and of symmetric monoidal categories.

In this paper we work exclusively with *symmetric* monoidal categories. The next definition comes from [16].

Definition 2.1. Consider an SMC as described above.

(i) The monoidal structure has *diagonals* if there is a natural transformation $\delta : Id \rightarrow (-) \otimes (-)$ making the following diagrams commute.



(ii) *Projections* for the monoidal structure are given whenever the neutral element I is terminal. Then one defines

$$\pi_{X,Y} = \rho_X \circ (id \otimes !_Y) : X \otimes Y \rightarrow X \otimes I \rightarrow X,$$

$$\pi'_{X,Y} = \lambda_Y \circ (!_X \otimes id) : X \otimes Y \rightarrow I \otimes Y \rightarrow Y.$$

Such an SMC with terminal object as neutral element may be called an *SMC with projections* or an *affine SMC*.

In an SMC with projections as above, one easily establishes that $\pi \circ \pi \circ \alpha = \pi$, $\pi' \circ \pi \circ \alpha = \pi \circ \pi'$, $\pi' \circ \alpha = \pi' \circ \pi'$, $\pi \circ \gamma = \pi'$ and $\pi' \circ \gamma = \pi$. Cartesian categories (i.e. with finite products) correspond to SMC's with both diagonals and projections such that $\pi \circ \delta = \pi$, $\pi' \circ \delta = \pi'$ and $\delta \circ (\pi \otimes \pi') = id$.

The “contraction tensors” \otimes mentioned in the Introduction 1.1 are tensors in an SMC with diagonals and the “weakening tensors” \otimes_w are the ones in an SMC with projections. Strictly speaking, the presence of diagonals or projections is not a property of the tensors but of the SMC structure as a whole. In case we want to stress that we don't assume that a specific SMC has diagonals or projections, we'll call it *linear*.

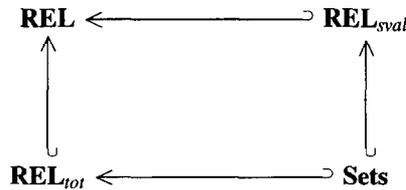
For completeness we add the following.

Definition 2.2. A morphism $F : C \rightarrow C'$ of SMC's as described above *preserves diagonals* if it additionally satisfies $F\delta = \xi \circ \delta'$.

Notice that a morphism of SMC's always preserves projections.

Example 2.3. In the examples (i)–(iii) below, we don't focus on the precise form of the monoidal structure, but rather on the setting in which they appear. The occurrence of the structure is a consequence of some abstract results which will appear in the sequel.

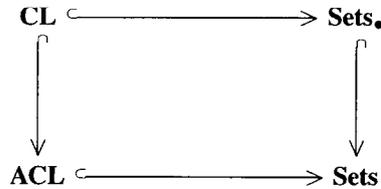
(i) Having seen the above Definition 2.1, André Joyal suggested the following elementary example. Let **REL** denote the usual category of sets and relations. A relation $R \subseteq X \times Y$ is called *single valued* if $\forall x, y, y'. [xRy \ \& \ xRy' \Rightarrow y = y']$ and *total* if $\forall x. \exists y. xRy$. Let **REL_{sva}** and **REL_{tot}** denote the subcategories of **REL** with corresponding maps. Of course, the category of sets **Sets** can be considered as the subcategory of **REL** with relations which are both single valued and total. There is the diagram of inclusions,



in which the inclusions except the one on the left have left adjoints. One has that the cartesian product \times of sets together with a singleton set 1 form an SMC structure which (1) is linear in **REL**; (2) has diagonals in **REL_{sva}**; (3) has projections in **REL_{tot}**, and (4) is of course cartesian in **Sets**. Notice that **REL_{sva}** is the Kleisli category of the lift monad \perp on **Sets**, **REL_{tot}** of the non-empty powerset monad \mathcal{P}^+ and **REL** of the “composite” powerset monad \mathcal{P} .

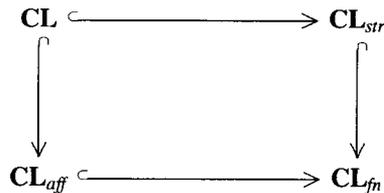
(ii) A poset is called a *complete lattice* if every subset has a supremum. A function between complete lattices is *linear* if it preserves all suprema. This yields a category **CL**. Similarly, a poset is called an *affine complete lattice* if every *non-empty* subset has a supremum; and a function between such lattices is *affine* if it preserves all these non-

empty suprema. Thus we have a category **ACL**. Notice that a complete lattice is an affine complete lattice with a bottom element. Finally, let **Sets_•** denote the category of pointed sets (objects are sets with a distinguished base point; morphisms preserve these points). Alternatively, one may think of **Sets_•** as the category of sets and partial functions or as the category of flat domains with strict (i.e. bottom-preserving) functions. Again there is a diagram of inclusions



This time all inclusions have left adjoints. Moreover, (1) **CL** is a linear SMC; (2) **Sets_•** is an SMC with diagonals; (3) **ACL** is an SMC with projections, and trivially (4) **Sets** is a cartesian SMC. The diagram arises by taking the categories of algebras of the monads mentioned in (i).

(iii) Let **CL_{aff}** (resp. **CL_{str}**, **CL_{fn}**) denote the category of complete lattices with affine functions—which preserve suprema of non-empty sets only—(resp. with strict functions—which preserve bottoms only, resp. with ordinary functions—which preserve no order structure at all). The diagram of inclusions is



in which all functions have left adjoints. Further, (1) **CL** is an SMC, as already mentioned above; (2) **CL_{str}** is an SMC with diagonals; (3) **CL_{aff}** is an SMC with projections; (4) **CL_{fn}** is a cartesian SMC. This diagram is obtained by taking the Kleisli categories of the comonads induced on **CL** in the diagram in (ii).

Definition 2.4. A symmetric monoidal closed category (SMCC) is an SMC where each functor $- \otimes X$ has a right adjoint; it will be denoted by $X \multimap (-)$.

A cartesian closed category (CCC) is an SMCC in which the monoidal structure is cartesian. The operations involved will then be denoted by $(1, \times$ and $\Rightarrow)$.

3. A recap on monads

This section contains a brief exposition of the basic results on monads which we need later. Recall that a monad on a category **C** consists of an endofunctor $T: \mathbf{C} \rightarrow \mathbf{C}$ together with two natural transformations: the “unit” $\eta: Id \rightarrow T$ and the “multiplication” $\mu: T^2 \rightarrow T$ of the monad. These are required to satisfy the equations $\mu \circ \eta_T = id = \mu \circ T\eta$ and $\mu \circ \mu T = \mu \circ T\mu$. A morphism between monads $S, T: \mathbf{C} \rightarrow \mathbf{C}$ is a natural

transformation $\sigma: S \rightarrow T$ such that $\sigma \circ \eta^S = \eta^T$ and $\sigma \circ \mu^S = \mu^T \circ \sigma^2$. One calls S a *submonad* of T if σ is monic. Dually, a *comonad* on \mathbf{C} is a monad on the opposite category \mathbf{C}^{op} .

Since the notions of monad and comonad can be described in an arbitrary 2-category, we can speak of a *monoidal monad/comonad*, whereby we mean a monad/comonad in the 2-category of monoidal categories. This terminology will only be used in passing, see 3.7 and 5.7.

Example 3.1. (i) On the category **Sets** we are particularly interested in the following three monads. Although we don't do so, they can be studied in an arbitrary topos. First the *lift monad* \perp is given by

$$\begin{aligned} \perp X &= \{a \subseteq X \mid a \text{ contains at most one element}\} \\ &= \{\{x\} \mid x \in X\} \cup \{\emptyset\} \\ &= \{a \subseteq X \mid \forall x, y \in a. x = y\}, \end{aligned}$$

the latter being the topos-theoretic definition. Elements of $\perp X$ are sometimes called subsingletons. On functions, one defines $\perp(f)(\{x\}) = \{f(x)\}$ and $\perp(f)(\emptyset) = \emptyset$. The unit $X \rightarrow \perp X$ is given by $x \mapsto \{x\}$ and multiplication $\perp \perp X \rightarrow \perp X$ by $A \mapsto \bigcup A$.

Related is the *non-empty powerset monad* \mathcal{P}^+ described by

$$\begin{aligned} \mathcal{P}^+(X) &= \{a \subseteq X \mid a \text{ contains at least one element}\} \\ &= \{a \subseteq X \mid a \neq \emptyset\}. \end{aligned}$$

The function part of \mathcal{P}^+ is given by images: $\mathcal{P}^+(f)(a) = \{f(x) \mid x \in a\}$. Unit and multiplication are by singletons and unions as before.

The ordinary *powerset monad* \mathcal{P} can be seen as the composite $\perp \mathcal{P}^+$ —in a sense to be made precise in Section 7. Notice that \perp and \mathcal{P}^+ are submonads of \mathcal{P} . There are obvious finite versions \mathcal{P}_f^+ and \mathcal{P}_f of these powerset monads.

(ii) The *free monoid monad* on **Sets**, denoted here by *list*, assigns to a set X the collection $\text{list}(X) = \bigcup_{n \geq 0} X^n$ of all finite sequences of elements of X (including the empty one in X^0). Unit and multiplication are respectively $x \mapsto \langle x \rangle$ and $\langle \langle \dots \rangle, \dots, \langle \dots \rangle \rangle \mapsto \langle \dots \rangle$, where in the latter case, the inner braces are removed. There is also the *free semi-group monad* list^+ with $\text{list}^+(X) = \bigcup_{n \geq 1} X^n$. One can view *list* as the composite $\perp \text{list}^+$.

For the next few examples, it is convenient to introduce some notation. Let K be a set containing a distinguished element 'zero'; in each of the examples below, it will be clear what zero is. We define the "*K-span*" functor $K\langle - \rangle: \mathbf{Sets} \rightarrow \mathbf{Sets}$ as follows

$$K\langle X \rangle = \{\varphi: X \rightarrow K \mid \varphi \text{ is almost everywhere zero.}\}$$

An element $\varphi \in K\langle X \rangle$ can be described in a unique way as a formal sum $k_1 x_1 + \dots + k_n x_n$ where each factor k_i is $\varphi(x_i)$. Thus one makes $K\langle - \rangle$ into a functor by putting $K\langle f \rangle(k_1 x_1 + \dots + k_n x_n) = k_1 f(x_1) + \dots + k_n f(x_n)$.

(iii) The *free commutative monoid monad* on **Sets** is given by $\mathbb{N}\langle - \rangle$. Thus $\mathbb{N}\langle X \rangle$ is the set of finite multisets of elements of X . The unit $X \rightarrow \mathbb{N}\langle X \rangle$ is $x \mapsto 1x$ and multiplication $\mathbb{N}\langle \mathbb{N}\langle X \rangle \rangle \rightarrow \mathbb{N}\langle X \rangle$ is described by $k_1 \varphi_1 + \dots + k_n \varphi_n \mapsto \lambda x \in X. k_1 \varphi_1(x) + \dots + k_n \varphi_n(x)$.

(iv) The *free abelian group* monad on **Sets** is $\mathbb{Z}\langle - \rangle$. Unit and multiplication are as before.

(v) Let K be a field. The *free vector space* (or *linear span*) monad is $K\langle - \rangle$. The *affine vector space* monad is $K_a\langle X \rangle = \{\varphi \in K\langle X \rangle \mid \sum_{x \in X} \varphi(x) = 1\}$.

The above are the main examples which will serve as illustrations of the theory. Occasionally we shall refer to the following additional ones. A monoid (A, e, m) in an SMC yields a *left actions* monad given by $X \mapsto X \otimes A$. Similarly, a comonoid (A, u, d) on an SMCC induces a *copy monad* $X \mapsto (A \multimap X)$. And an object A in a category with binary coproducts $+$ gives rise to the *A-lift* monad $X \mapsto (X + A)$. The name ‘lift’ is appropriate since for $A = 1$ we get the lift monad $X \mapsto \perp X = X + 1$ from (i). By duality $X \mapsto X \times A$ is a comonad on a category with cartesian products.

Associated with a monad $T : \mathbf{C} \rightarrow \mathbf{C}$ are two categories: the category of algebras (or Eilenberg–Moore category) \mathbf{C}^T and the Kleisli category \mathbf{C}_T . These come equipped with the familiar adjunctions $\mathbf{C}^T \rightleftarrows \mathbf{C} \rightleftarrows \mathbf{C}_T$. For details, see [26] or [5].

Let’s look at some examples. Algebras for the lift monad \perp on **Sets** are pointed sets: suppose $\varphi : \perp X \rightarrow X$ is a structure map, then $\varphi(\emptyset)$ is the base point in X . It is preserved by algebra maps. Hence $\mathbf{Sets}^\perp \cong \mathbf{Sets}_\bullet$. Incidentally, also the Kleisli category of \perp is \mathbf{Sets}_\bullet ; it can also be described as $\mathbf{REL}_{\text{sym}}$, see Example 2.3(i).

Algebras for the powerset monad \mathcal{P} on **Sets** are complete lattices: if $\varphi : \mathcal{P}X \rightarrow X$ is a structure map, then there is a partial order structure on X by $x \leq y \Leftrightarrow \varphi(\{x, y\}) = y$. The supremum of $a \subseteq X$ is $\varphi(a)$. One easily verifies that algebra maps preserve suprema and thus that $\mathbf{Sets}^\mathcal{P} \cong \mathbf{CL}$. Similarly $\mathbf{Sets}^{\mathcal{P}^+} \cong \mathbf{ACL}$, see 2.3(ii). Algebras for the \mathcal{P}_f^+ monad are posets with binary joins; for the \mathcal{P}_f monad one obtains posets with binary joins plus a bottom element (i.e. with all finite joins).

The Kleisli categories of these two monads occurred in Example 2.3(i): $\mathbf{Sets}_\mathcal{P} \cong \mathbf{REL}$ and $\mathbf{Sets}_{\mathcal{P}^+} \cong \mathbf{REL}_{\text{tot}}$.

Algebras of the list monad are monoids and of the list⁺ monad are semi-groups. Similarly, the names used in 3.1(v)–(vii) suggest which algebraic structures are algebras for these monads.

Whereas limits in a category of algebras are created by the forgetful functor, colimits are much more evasive. The kind of colimits we often need in the sequel are *coequalizers of reflexive pairs* (see e.g. [25]; we mean coequalizers of algebra maps $f, g : \varphi \rightrightarrows \psi$ which have a common right inverse algebra map $s : \psi \rightarrow \varphi$ with $f \circ s = id = g \circ s$). We don’t worry too much about this requirement because in the examples we are interested in, the categories of algebras are always cocomplete. For example, when the underlying category is **Sets** this is always the case (see e.g. [5, Section 9.3, Proposition 4]). Whenever we need such coequalizers we shall indicate this by writing an extra requirement “... such that the category of algebras has CRP’s”.

The next lemma shows why these CRP’s are pivotal. It is due to Linton [25, Corollary 2].

Lemma 3.2. *If a category of algebras has CRP’s, then it is as cocomplete as its underlying category.*

Proof. We'll have a look at finite coproducts. If 0 is initial in a category \mathbf{C} with a monad T on it, then the free algebra on 0 is easily seen to be initial in \mathbf{C}^T . If \mathbf{C} has coproducts and $\varphi: TX \rightarrow X$, $\psi: TY \rightarrow Y$ are structure maps, one forms their coproduct $\varphi +^T \psi$ in \mathbf{C}^T as the coequalizer

$$\left(\begin{array}{c} T^2(TX + TY) \\ \downarrow \mu \\ T(TX + TY) \end{array} \right) \xrightarrow[T(\varphi + \psi)]{\mu \circ T[T(in), T(in')]} \left(\begin{array}{c} T^2(X + Y) \\ \downarrow \mu \\ T(X + Y) \end{array} \right) \longrightarrow \left(\begin{array}{c} TW \\ \downarrow \\ W \end{array} \right)$$

The common right inverse of the parallel pair is

$$T(\eta_X + \eta_Y) : T(X + Y) \rightarrow T(TX + TY). \quad \square$$

The above construction yields the one-point set 1 as initial object in **Sets**• (it is at the same time terminal) and the *coalesced sum* as coproduct (given by $X + Y = (X - \bullet) \cup (Y - \bullet) \cup \{\bullet\}$). In the category **CL** of complete lattices one has that (finite) products and coproducts coincide.

Here is a further (standard) application of CRP's.

Lemma 3.3. *Let $\sigma: S \dashrightarrow T$ be a morphism between two monads on a category \mathbf{C} . Suppose \mathbf{C}^T had CRP's; then the functor $(-\circ\sigma) : \mathbf{C}^T \rightarrow \mathbf{C}^S$ has a left adjoint $[\sigma]$.*

Proof. For a structure map $\varphi: SX \rightarrow X$, let $[\sigma](\varphi)$ be the T -algebra obtained as coequalizer in

$$\left(\begin{array}{c} T^2 SX \\ \downarrow \mu \\ TSX \end{array} \right) \xrightarrow[T_\varphi]{\mu^T \circ T\sigma} \left(\begin{array}{c} T^2 X \\ \downarrow \mu \\ TX \end{array} \right) \longrightarrow \left(\begin{array}{c} TX' \\ \downarrow [\sigma](\varphi) \\ X' \end{array} \right) \quad \square$$

Next we consider a special kind of monads, which will be crucial in the sequel.

Definition 3.4. Let (\mathbf{C}, I, \otimes) be an SMC and T a monad on \mathbf{C} .

(i) This monad T is called *strong* if there is a "strength" natural transformation with components

$$st_{X,Y} : X \otimes TY \rightarrow T(X \otimes Y)$$

satisfying the following equations (see also [28])

$$\begin{aligned} T\lambda \circ st &= \lambda, & T\alpha \circ st \circ id \otimes st &= st \circ \alpha, \\ st \circ id \otimes \eta &= \eta, & \mu \circ T(st) \circ st &= st \circ id \otimes \mu. \end{aligned}$$

(ii) Suppose T is a strong monad; put

$$st'_{X,Y} = T(\gamma) \circ st_{X,Y} \circ \gamma : TX \otimes Y \rightarrow T(X \otimes Y).$$

Then there are two "double strength" maps $TX \otimes TY \rightrightarrows T(X \otimes Y)$; namely

$$\begin{aligned} dst_{X,Y} &= \mu_{X \otimes Y} \circ T(st'_{X,Y}) \circ st_{TX,Y}, \\ dst'_{X,Y} &= \mu_{X \otimes Y} \circ T(st_{X,Y}) \circ st'_{X,TY}. \end{aligned}$$

The monad T is called *commutative* if $dst = dst'$. This definition is due to Kock, see [19, 3.1].

(iii) Let S and T both be strong monads. A *morphism of strong monads* is a morphism of monads $\sigma : S \rightarrow T$ which satisfies $\sigma \circ st^S = st^T \circ \sigma$. In case σ is monic, we'll say that S is a *strong submonad* of T .

Most of the monads mentioned in Examples 3.1 are strong but among these, not all are commutative. The monads \perp , \mathcal{P}^+ and \mathcal{P} (and their finite versions) are commutative and the double strength $dst = dst'$ is described in these cases by $(a, b) \mapsto a \times b$.

As one may expect, the monoid monad is not commutative, but the commutative monoid and abelian group monad are. Also the linear and affine span monads are commutative.

As to the examples mentioned briefly after 3.1, the left actions monad has the associativity isomorphism $X \otimes (Y \otimes A) \rightarrow (X \otimes Y) \otimes A$ as strength. By chasing some diagrams one obtains that the monad is commutative if and only if the underlying monoid A is commutative. The copy monad has a strength map $\Lambda((id \otimes ev) \circ \alpha^{-1}) : X \otimes (A \multimap Y) \rightarrow (A \multimap (X \otimes Y))$. In case the underlying comonoid is commutative, one has a commutative monad. Finally, we'll have a closer look at the “ A -lift” monad $X \mapsto X + A$ on the category of sets. Strength is given by $X \times (Y + A) \cong (X \times Y) + (X \times A) \rightarrow (X \times Y) + A$. We claim that this monad is commutative if and only if the set A is either initial (empty) or terminal (a singleton). The (if)-part of the statement is obvious: if A is empty, the monad is trivial and if A is a singleton, the monad is lift. As to the reverse implication, one uses that for any commutative monad T on a cartesian category with an initial object 0 , there is at most one “constant” $1 \rightarrow T0$.

The next lemma establishes some technical properties of the various strength maps. Proofs are by straightforward calculations.

Lemma 3.5. (i) *The natural transformation st' satisfies the “dual” properties of those for st in Definition 3.4(i):*

$$\begin{aligned} T\rho \circ st' &= \rho, & T\alpha \circ st' &= st' \circ st' \otimes id \circ \alpha, \\ st' \circ \eta \otimes id &= \eta, & \mu \circ T(st') \circ st' &= st' \circ \mu \otimes id. \end{aligned}$$

(ii) *Further, one can retrieve st and st' from dst and dst' :*

$$\begin{aligned} dst \circ \eta \otimes id &= st = dst' \circ \eta \otimes id, \\ dst \circ id \otimes \eta &= st' = dst' \circ id \otimes \eta, \\ dst \circ \eta \otimes \eta &= \eta = dst' \circ \eta \otimes \eta. \end{aligned}$$

(iii) *The double strength maps are related via symmetry:*

$$T(\gamma) \circ dst = dst' \circ \gamma \quad \text{and} \quad T(\gamma) \circ dst' = dst \circ \gamma.$$

(iv) *Both dst and dst' , together with η_I make T into a monoidal functor. By (iii) T is a symmetric monoidal functor if and only if T is a commutative monad. Further, $\eta : Id \rightarrow T$ is a monoidal transformation. And a morphism of monads $\sigma : S \rightarrow T$ is a morphism of strong monads if and only if it is a monoidal transformation. \square*

The main result of [19] and [23] is the characterization of commutative monads as mentioned below in Corollary 3.7. We first extract a useful lemma.

Lemma 3.6. *Let T be a strong monad; it is commutative if and only if the following diagram commutes.*

$$\begin{array}{ccc}
 T^2X \otimes T^2Y & \xrightarrow{dst} & T(TX \otimes TY) \xrightarrow{T(dst)} T^2(X \otimes Y) \\
 \mu \otimes \mu \downarrow & & \downarrow \mu \\
 TX \otimes TY & \xrightarrow{dst} & T(X \otimes Y).
 \end{array}$$

Corollary 3.7 (Kock). *A monad on an SMC is commutative if and only if it is a monoidal monad.*

Proof. Suppose T is a commutative monad; Lemmas 3.5(iv) and 3.6 yield that T (with η_I and dst) is a monoidal functor and that the unit and multiplication are monoidal transformations. Conversely, suppose that (T, ζ, ξ) is a monoidal functor forming a monoidal monad. Then $st = \xi \circ (\eta \otimes id)$ makes T a commutative monad (with $dst = \xi$). \square

For symmetric monoidal *closed* categories, strength is usually expressed in terms of internal hom's. This can be obtained as follows, see also [21, Definition 2.1].

Lemma 3.8. *Let T be strong monad on a SMCC. There is then a “representation” r of T ; it is a natural transformation with components $r_{X,Y} = \Lambda(T(ev) \circ st) : (X \multimap Y) \rightarrow (TX \multimap TY)$. Similarly one has $r'_{X,Y} = \Lambda(T(ev) \circ st') : T(X \multimap Y) \rightarrow (X \multimap TY)$.*

These in turn give rise to two maps $T(X \multimap Y) \rightrightarrows (TX \multimap TY)$; namely $dr = (id \multimap \mu) \circ r \circ r'$ and $dr' = (id \multimap \mu) \circ r' \circ Tr$. One has that T is commutative if and only if $dr = dr'$.

Proof. Because the “double strengths” and “double representations” are related: $dr_{X,Y} = \Lambda(T(ev) \circ dst_{X \multimap Y, X})$ and $dst_{X,Y} = ev \circ (dr_{X, X \otimes Y} \circ T\Lambda(id_{X \otimes Y})) \otimes id$ and similarly for the primed versions. \square

With this lemma one can show that if an object X carries an algebra structure (of a strong monad), then so does each exponent object $Y \multimap X$.

4. Affine and relevant monads

The notions in the following definition are due to Anders Kock [22].

Definition 4.1. Let T be a strong monad on a category with finite products.

(i) The monad T is called *affine* if the following diagram commutes.

$$\begin{array}{ccc}
 TX \times TY & \xrightarrow{dst} & T(X \times Y) \\
 & \searrow id & \downarrow \langle T\pi, T\pi' \rangle \\
 & & TX \times TY
 \end{array}$$

(ii) Similarly, T is called *relevant* if, in reverse order,

$$\begin{array}{ccc}
 T(X \times Y) & \xrightarrow{\langle T\pi, T\pi' \rangle} & TX \times TY \\
 & \searrow id & \downarrow dst \\
 & & T(X \times Y)
 \end{array}$$

(iii) And T is *cartesian* if it is both affine and relevant, that is when T preserves cartesian products.

The name ‘affine’ used in (i) above is introduced in [22], but the notion in (ii) is not given a name there. We have chosen ‘relevant’ because it gives a link with relevant logic, see Section 6. Kock uses the name ‘cartesian closed’ where we simply use ‘cartesian’ in (iii). Comparable structures are investigated in [17] under the name ‘hyperaffine’. Notice that the monad T in the above definition is not required to be commutative. Hence using dst instead of dst' seems arbitrary. After the next result we see that this is not the case.

The non-empty powerset monads \mathcal{P}^+ and \mathcal{P}_f^+ are affine. Also the affine span monad $K_a\langle - \rangle$ is affine. The general A -lift monads (which include the standard lift) are relevant. Obviously, the identity monad is cartesian; the example used in [22] is the monad of directed downsets (or ideals) on the category of posets.

The next lemma also occurs in [22].

Lemma 4.2. *Let T be a strong monad on a cartesian category \mathbf{C} ; then*

- (i) T is affine \Leftrightarrow the unit $\eta_1 : 1 \rightarrow T1$ is an isomorphism;
- (ii) T is relevant $\Leftrightarrow dst \circ \delta = T\delta \Leftrightarrow dst \circ \langle Tf, Tg \rangle = T\langle f, g \rangle$.

Proof. (i) (\Leftarrow) We only do the first projection; the second is handled similarly.

$$\begin{aligned}
 T\pi \circ dst_{X,Y} &= T\rho \circ T(id \times !_Y) \circ dst_{X,Y} && \text{see 2.1(ii)} \\
 &= T\rho \circ dst_{X,1} \circ id \times T(!_Y) \\
 &= T\rho \circ dst_{X,1} \circ id \times \eta_1 \circ id \times !_T \circ id \times T(!_Y) && \text{by assumption} \\
 &= T\rho \circ st'_{X,1} \circ id \times !_T && \text{by 3.5(ii)} \\
 &= \rho \circ id \times !_T && \text{by 3.5(i)} \\
 &= \pi.
 \end{aligned}$$

(\Rightarrow) We need to show $\eta_1 \circ !_{T1} = id : T1 \rightarrow T1$. One has $\pi = \pi' : 1 \times 1 \rightarrow 1$. Using the assumption, we obtain $\pi = \pi' : T1 \times T1 \rightarrow T1$. Hence

$$\eta_1 \circ !_{T1} = \pi \circ \langle \eta_1 \circ !_{T1}, id_{T1} \rangle = \pi' \circ \langle \eta_1 \circ !_{T1}, id_{T1} \rangle = id_{T1}.$$

(ii) Suppose T is relevant; then $dst \circ \delta = dst \circ \langle T\pi, T\pi' \rangle \circ T\delta = T\delta$. Suppose next that $dst \circ \delta = T\delta$; then $dst \circ \langle Tf, Tg \rangle = dst \circ \langle Tf \times Tg \rangle \circ \delta = T(f \times g) \circ dst \circ \delta = T(f \times g) \circ T\delta = T\langle f, g \rangle$. This last result in turn yields that T is relevant by taking $f = \pi$, $g = \pi'$. \square

As in the proof of (i) above one obtains that $\langle T\pi, T\pi' \rangle \circ dst' = id$ iff η_1 is iso. Further, if T is relevant then using (ii) one gets $dst' \circ \delta = T\gamma \circ dst \circ \delta = T\gamma \circ T\delta = T\delta$ and vice-versa. Hence describing ‘affine’ and ‘relevant’ in terms of dst' instead of dst in Definition 4.1 leads to the same notions.

These preliminaries lead to our first application. It explains the situation in Example 2.3(i). Part (i) of the next result is folklore.

Theorem 4.3. *Let T be a commutative monad on an SMC \mathbf{C} .*

(i) *The Kleisli category \mathbf{C}_T then also has an SMC structure (\otimes_T, I_T) and the free functor $\mathbf{C} \rightarrow \mathbf{C}_T$ preserves this structure (on-the-nose).*

(ii) *Suppose now the monoidal structure on \mathbf{C} is cartesian. If T is a relevant / affine / cartesian monad, then the induced monoidal structure on \mathbf{C}_T has diagonals / has projections / is cartesian. Moreover, the free functor preserves this structure.*

Proof. (i) Define a tensor \otimes_T on \mathbf{C}_T by $X \otimes_T Y = X \otimes Y$ and $f \otimes_T g = dst \circ (f \otimes g)$. Identities are preserved by \otimes_T , since by Lemma 3.5(ii): $\eta \otimes_T \eta = dst \circ \eta \otimes \eta = \eta$ and composition is preserved by Lemma 3.6:

$$\begin{aligned} (f \otimes_T g) \bullet (h \otimes_T k) &= \mu \circ T(dst) \circ T(f \otimes g) \circ dst \circ h \otimes k \\ &= \mu \circ T(dst) \circ dst \circ Tf \otimes Tg \circ h \otimes k \\ &= dst \circ \mu \otimes \mu \circ Tf \otimes Tg \circ h \otimes k \\ &= dst \circ (f \bullet h) \otimes (g \bullet k) \\ &= (f \bullet h) \otimes_T (g \bullet k), \end{aligned}$$

where \bullet denotes the composition in \mathbf{C}_T . The neutral element I in \mathbf{C} works also in \mathbf{C}_T .

(ii) The affine case is handled by Lemma 4.2(i): it yields that $T1 \cong 1$ is terminal in \mathbf{C}_T

In the relevant case, take $\delta_T = \eta \circ \delta$. Then indeed,

$$\begin{aligned} (f \otimes_T f) \bullet \delta_T &= \mu \circ T(dst) \circ T(f \times f) \circ \eta \circ \delta \\ &= \mu \circ \eta \circ dst \circ f \times f \circ \delta \\ &= \mu \circ \eta \circ dst \circ \delta \circ f \\ &= \mu \circ T\eta \circ T\delta \circ f \\ &= \delta_T \bullet f. \end{aligned} \quad \square$$

Lemma 4.4. *Let S, T be commutative monads such that S is a strong submonad of T , say via $\sigma : S \rightarrow T$. Then T is affine/relevant implies S is affine/relevant.*

Proof. Suppose T is affine. Then

$$\begin{aligned} \sigma \times \sigma \circ \langle S\pi, S\pi' \rangle \circ dst^S &= \langle T\pi, T\pi' \rangle \circ \sigma \circ dst^S \\ &= \langle T\pi, T\pi' \rangle \circ dst^T \circ \sigma \times \sigma \quad \text{because } \sigma \text{ is monoidal} \\ &= \sigma \times \sigma. \end{aligned}$$

Using that $\sigma \times \sigma$ is monic, we get that S is affine. The other implication is handled similarly. \square

Next it will be shown how to extract affine, relevant and cartesian parts from a given monad.

Definition 4.5. Let T be a commutative monad on a category with finite limits.

(i) The *affine part* T_a of T is given by the pullback diagram

$$\begin{array}{ccc} T_a(X) & \xrightarrow{\quad} & T(X) \\ \downarrow & & \downarrow T(!_X) \\ 1 & \xrightarrow{\eta_1} & T(1) \end{array}$$

(ii) The *relevant part* T_r of T is given by the equalizer

$$T_r(X) \xrightarrow{\quad} T(X) \begin{array}{c} \xrightarrow{dst \circ \delta} \\ \xrightarrow{T\delta} \end{array} T(X \times X)$$

(iii) The *cartesian part* T_c of T is obtained by intersection as in the pullback diagram

$$\begin{array}{ccc} T_c(X) & \xrightarrow{\quad} & T_a(X) \\ \downarrow & & \downarrow \\ T_r(X) & \xrightarrow{\quad} & T(X) \end{array}$$

Proposition 4.6. (i) *The affine/relevant/cartesian parts as defined above extend to commutative monads, which are strong submonads of T . Moreover the cartesian part is a strong submonad, both of the affine and of the relevant part. Further,*

$$\begin{aligned} T_a \text{ is an affine monad and } T \text{ is affine iff } T_a &\xrightarrow{\sim} T \\ T_r \text{ is a relevant monad and } T \text{ is relevant iff } T_r &\xrightarrow{\sim} T \\ T_c \text{ is a cartesian monad and } T \text{ is cartesian iff } T_c &\xrightarrow{\sim} T. \end{aligned}$$

(ii) *The affine/relevant/cartesian part of T is the greatest affine/relevant/cartesian strong submonad of T .*

(iii) *One has $(T_a)_r \cong T_c \cong (T_r)_a$.*

Proof. (i) The extensions are obtained in a straightforward way using the universality of the definitions. One obtains that T_a is affine from the fact that $T_a(1) \cong 1$ by construction. In a similar way one has that T_r is relevant. Finally T_c is cartesian by the previous lemma using $T_c \multimap T_a$ and $T_c \multimap T_r$.

(ii) and (iii) are left to the reader. \square

Remark 4.7. (i) One easily verifies that the affine parts of the monads \mathcal{P} , \mathcal{P}_f , $K(-)$ are \mathcal{P}^+ , \mathcal{P}_f^+ and $K_a(-)$, see Example 3.1. The relevant parts of these three monads is \perp .

(ii) The affine part of a monad T appears in [24] in slightly different form, namely as equalizer of $\eta_1 \circ !_{TX}, T(!_X) : TX \rightrightarrows T1$, but that leads to the same notion as in (i) above. The formulation we use is closer to the one found in [27, Section 1.3, Exercise 5]. As far as we know, the relevant and cartesian parts of a monad are first identified in general in (ii) and (iii) above.

(iii) Commutativity of the monad T is not needed in order to obtain the affine part, but it is needed for the relevant part to be a monad.

5. Tensors of algebras

The aim in this section is to obtain a result similar to Theorem 4.3 for categories of algebras. The constructions involved come from papers by Day and Kock ([8] and [21]), but see also [14]. In the presence of certain coequalizers one has that “algebras of commutative monads have tensors”. We recall these basic results in Lemmas 5.1–5.3. The final three results of this section are probably a bit less familiar.

The following is exemplaric for what follows. For complete lattices X, Y, Z , a function $f: X \times Y \rightarrow Z$ is called *bilinear* if it is linear in each of its variables separately, i.e. $f(\bigvee a, y) = \bigvee \{f(x, y) \mid x \in a\}$ and $f(x, \bigvee b) = \bigvee \{f(x, y) \mid y \in b\}$. In the category **CL** of complete lattices there is an object $X \otimes Y$ such that linear functions $X \otimes Y \rightarrow Z$ correspond to bilinear functions $X \times Y \rightarrow Z$. This phenomenon will be investigated at a more abstract level.

Let T be a strong monad on an SMC. Suppose we have algebras $\varphi: TX \rightarrow X$, $\psi: TY \rightarrow Y$ and $\chi: TZ \rightarrow Z$. One says that a map $f: X \otimes Y \rightarrow Z$ is a *bimorphism* $[\varphi, \psi] \rightarrow \chi$ one has

$$\chi \circ Tf \circ dst = f \circ id \otimes \psi, \quad \chi \circ Tf \circ dst = f \circ \varphi \otimes id.$$

In the first one, the X -input is kept fixed and in the second one the Y -input. A bit less conspicuously, this can be said in one equation: $\chi \circ Tf \circ dst = f \circ \varphi \otimes \psi$. Or equivalently, with dst' instead of dst . Notice that Lemma 3.6 says that T is a commutative monad if and only if dst is a bimorphism $[\mu, \mu] \rightarrow \mu$. It is also worth noticing that for a bimorphism $f: [\varphi, \psi] \rightarrow \chi$ one has

- if $g: \chi \rightarrow \chi'$ is an algebra map, then $g \circ f$ is a bimorphism $[\varphi, \psi] \rightarrow \chi'$;
- if $h: \varphi' \rightarrow \varphi$ and $k: \psi \rightarrow \psi'$ are algebra maps, then $f \circ h \otimes k$ is a bimorphism $[\varphi', \psi'] \rightarrow \chi$.

The next three lemmas contain the main results we need; the proofs will be given in some detail. The construction used in the proof of the first result comes from [8, Proposition 4.4].

Lemma 5.1. *Let T be a strong monad on an SMC \mathbf{C} such that the category of algebras \mathbf{C}^T has CRP's (i.e. coequalizers of reflexive pairs). Then there are universal bimorphisms.*

This means that for each pair of algebras φ, ψ there is an algebra $\varphi \otimes^T \psi$ and a bimorphism $u: [\varphi, \psi] \rightarrow (\varphi \otimes^T \psi)$ such that any bimorphism $f: [\varphi, \psi] \rightarrow \chi$ factorizes as $f = \bar{f} \circ u$ for a unique $\bar{f}: (\varphi \otimes^T \psi) \rightarrow \chi$.

Proof. Suppose $\varphi: TX \rightarrow X$ and $\psi: TY \rightarrow Y$ are structure maps. Form the algebra $\varphi \otimes^T \psi: TW \rightarrow W$ as coequalizer,

$$\left(\begin{array}{c} T^2(TX \otimes TY) \\ \downarrow \mu \\ T(TX \otimes TY) \end{array} \right) \xrightarrow[T(\varphi \otimes \psi)]{\mu \circ T(dst)} \left(\begin{array}{c} T^2(X \otimes Y) \\ \downarrow \mu \\ T(X \otimes Y) \end{array} \right) \xrightarrow{e} \left(\begin{array}{c} TW \\ \downarrow \\ W \end{array} \right)$$

and put $u = e \circ \eta: X \otimes Y \rightarrow W$. One easily verifies that it is a bimorphism $[\varphi, \psi] \rightarrow \varphi \otimes^T \psi$. If $f: [\varphi, \psi] \rightarrow \chi$ is another bimorphism, then $\chi \circ Tf: \mu_{X \otimes Y} \rightarrow \chi$ coequalizes the above parallel pair. This yields the unique $\bar{f}: (\varphi \otimes^T \psi) \rightarrow \chi$ with $\bar{f} \circ u = f$. \square

Lemma 5.2. *Let T be a commutative monad on an SMC \mathbf{C} such that \mathbf{C}^T has CRP's. Then*

(i) *The free algebra μ_1 on the neutral element I of the tensor in \mathbf{C} is neutral for \otimes^T .*

(ii) $\mu_X \otimes^T \mu_Y \cong \mu_{X \otimes Y}$.

As a result of (i) and (ii), \mathbf{C}^T is an SMC and the free functor $\mathbf{C} \rightarrow \mathbf{C}^T$ preserves the SMC-structure.

(iii) *In case T is an affine / relevant / cartesian monad, then the monoidal structure induced on \mathbf{C}^T has projections / has diagonals / is cartesian.*

Proof. For (i) one needs a bijective correspondence between bimorphisms $f: [\mu_1, \psi] \rightarrow \chi$ and algebra maps $g: \psi \rightarrow \chi$. Similarly for (ii) one uses a correspondence between bimorphisms $f: [\mu_X, \mu_Y] \rightarrow \chi$ and morphisms $g: X \otimes Y \rightarrow Z$ —where Z is the carrier of χ . This is more or less standard, and so we concentrate on (iii). The affine case is easy: by Lemma 4.2(i), $T1$ is a terminal object in \mathbf{C} , so the neutral element μ_1 for the tensor in \mathbf{C}^T is terminal.

In the relevant case, put $\delta_\varphi^T = u_\varphi \circ \delta = e \circ \eta \circ \delta$, where u_φ is the universal bimorphism $[\varphi, \varphi] \rightarrow \varphi \otimes^T \varphi$ determined in the proof of the previous lemma. Then δ_φ^T is an algebra map $\varphi \rightarrow \varphi \otimes^T \varphi$:

$$\begin{aligned}
\delta_\varphi^T \circ \varphi &= e \circ \eta \circ \delta \circ \varphi \\
&= e \circ T(\varphi \times \varphi) \circ \eta \circ \delta \\
&= e \circ \mu \circ T(dst) \circ \eta \circ \delta \quad e \text{ is coequalizer} \\
&= e \circ \mu \circ \eta \circ dst \circ \delta \\
&= e \circ \mu \circ T\eta \circ T\delta \quad T \text{ is relevant, see Lemma 4.2(ii)} \\
&= (\varphi \otimes^T \varphi) \circ Te \circ T\eta \circ T\delta \\
&= (\varphi \otimes^T \varphi) \circ T(\delta_\varphi^T).
\end{aligned}$$

Finally, δ_φ^T is a natural transformation: for $f: \varphi \rightarrow \psi$ one has

$$\begin{aligned}
(f \otimes^T f) \circ \delta_\varphi^T &= (f \otimes^T f) \circ u_\varphi \circ \delta \\
&= u_\psi \circ (f \otimes f) \circ \delta \\
&= u_\psi \circ \delta \circ f \\
&= \delta_\psi^T \circ f. \quad \square
\end{aligned}$$

Having seen a monoidal structure in a category of algebras, we proceed with internal hom's. The construction below comes from [21], but here it is linked directly to the above tensor. In [21] one can find the description of this internal hom “on its own” in terms of closed categories.

Lemma 5.3. *Let T be a commutative monad on an SMCC \mathbf{C} which has equalizers and is such that the category of algebras \mathbf{C}^T has CRP's. Then there is an internal hom functor $\multimap^T: (\mathbf{C}^T)^{op} \times \mathbf{C}^T \rightarrow \mathbf{C}^T$ such that $\psi \multimap^T (-)$ is right adjoint to $(-) \otimes^T \psi$. This makes \mathbf{C}^T into an SMCC as well.*

Proof. Assume $\psi: TY \rightarrow Y$ and $\chi: TZ \rightarrow Z$ are structure maps. Form the equalizer

$$W \xrightarrow{e} (Y \multimap Z) \begin{array}{c} \xrightarrow{(id \multimap \chi) \circ r} \\ \xrightarrow{(\psi \multimap id)} \end{array} (TY \multimap Z)$$

and put $h = (id \multimap \chi) \circ r' \circ T(e): TW \rightarrow (Y \multimap Z)$, where r and r' are as introduced in Lemma 3.8. This h equalizes the above pair and thus gives rise to the map $\psi \multimap^T \chi: TW \rightarrow W$; it is not hard to verify it is an algebra. Further, there is a bijective correspondence between bimorphisms $f: [\varphi, \psi] \rightarrow \chi$ and algebra maps $g: \varphi \rightarrow (\psi \multimap^T \chi)$: given such an $f: X \otimes Y \rightarrow Z$ one has $\Lambda(f): X \rightarrow (Y \multimap Z)$ equalizing the above two maps; thus one obtains $f^\vee: X \rightarrow W$. In the other direction one takes for a given g the map $g^\wedge = ev \circ ((e \circ g) \otimes id)$. \square

After Lemma 5.5 below one finds how \multimap and \multimap^T are related.

Example 5.4. (i) The above lemmas justify the claims made in Example 2.3(ii) about the monoidal structures in the categories **CL**, **ACL** and **Sets_{*}**. Let's describe the tensors as given in the proof of Lemma 5.1 explicitly in these cases.

For pointed sets X, Y one obtains the well-known *smash product*

$$\begin{aligned} X \otimes_c Y &\cong \{c \in \perp(X \times Y) \mid \forall x \in X. \forall y \in Y. \{(x, y)\} \in c \Leftrightarrow [x \neq \bullet \ \& \ y \neq \bullet]\} \\ &\cong ((X - \bullet) \times (Y - \bullet)) \cup \{\bullet\} \end{aligned}$$

with associated universal bi-strict function $X \times Y \rightarrow X \otimes_c Y$ given by

$$(x, y) \mapsto \text{if } x \neq \bullet \ \& \ y \neq \bullet \ \text{then } (x, y) \ \text{else } \bullet.$$

Diagonals $X \rightarrow (X \otimes_c X)$ are defined by $\bullet \mapsto \bullet$ and $\bullet \neq x \mapsto (x, x)$.

For affine complete lattices X, Y one obtains

$$X \otimes_w Y \cong \{c \in \mathcal{P}^+(X \times Y) \mid \forall a \in \mathcal{P}^+X. \forall b \in \mathcal{P}^+Y. a \times b \subseteq c \Leftrightarrow (\bigvee a, \bigvee b) \in c\}.$$

The neutral element for \otimes_w is terminal since $\mathcal{P}^+1 \cong 1$. Hence there are projections $X \leftarrow X \otimes_w Y \rightarrow Y$.

The construction for complete lattices is very similar:

$$X \otimes Y \cong \{c \in \mathcal{P}(X \times Y) \mid \forall a \in \mathcal{P}X. \forall b \in \mathcal{P}Y. a \times b \subseteq c \Leftrightarrow (\bigvee a, \bigvee b) \in c\}.$$

Notice that every $c \in X \otimes Y$ contains all elements of the form (\perp, y) and (x, \perp) . These latter two tensors \otimes and \otimes come equipped with a universal bi-affine map $X \times Y \rightarrow X \otimes_w Y$ and a universal bi-linear map $X \times Y \rightarrow X \otimes Y$.

(ii) Since the abelian group monad on **Sets** is commutative, one obtains the well-known fact that the category **Ab** of abelian groups is a SMCC and that the free functor **Sets** \rightarrow **Ab** preserves the monoidal structure. Similarly the category Vect_K of vector spaces over a field K is an SMCC.

(iii) Here are two more categories with the structure of an affine SMCC: affine vector spaces over a field K and posets with binary joins. These arise as categories of algebras of the affine monads $K_a(-)$ and \mathcal{P}_f^+ .

(iv) Let R be a commutative ring, i.e. a commutative monoid in **Ab**. It induces a commutative “left actions monad” on **Ab**. The resulting category of algebras is the category $R\text{-Mod}$ of left R -modules (see [26, VI, 2]). Because $R\text{-Mod}$ has CRP’s—it is in fact cocomplete—the above constructions can be performed and yield the usual symmetric monoidal structure of $R\text{-Mod}$.

The proofs of the next few technical results are left to the reader. They can be obtained by using the detailed form of the SMCC structure in a category of algebras.

Lemma 5.5. *Let S and T be commutative monads on a SMCC \mathbf{C} such that the corresponding categories of algebras have CRP’s. Assume $\sigma : S \rightarrow T$ is a morphism of strong monads and let $[\sigma] : \mathbf{C}^S \rightarrow \mathbf{C}^T$ be the left adjoint to the functor $(- \circ \sigma)$ determined in Lemma 3.3.*

(i) *For an S -algebra ψ and a T -algebra χ one has $\psi \dashv\circ^S (\chi \circ \sigma) \cong ([\sigma](\psi) \dashv\circ^T \chi) \circ \sigma$ in \mathbf{C}^S .*

(ii) *The functor $[\sigma] : \mathbf{C}^S \rightarrow \mathbf{C}^T$ preserves the SMC-structure. \square*

Applying the above result to the unit morphism of monads $\eta: Id \rightarrow T$ yields a relation between the induced internal hom \dashv^T in a category of algebras to the internal hom \dashv in the underlying category—since the induced functor $[\eta]$ is the free functor $\mathbf{C} \rightarrow \mathbf{C}^T$ and $-\circ\eta$ is the forgetful functor $\mathbf{C}^T \rightarrow \mathbf{C}$.

Proposition 5.6. *Let T be a commutative monad on a SMC \mathbf{C} . The full and faithful functor $\mathbf{C}_T \rightarrow \mathbf{C}^T$ from the Kleisli category to the category of algebras preserves the SMC-structure. Moreover if T is a relevant / affine / cartesian monad, then $\mathbf{C}_T \rightarrow \mathbf{C}^T$ preserves the additional structure. \square*

Proposition 5.7. *Let T be a commutative monad on a SMC \mathbf{C} such that \mathbf{C}^T has CRP's. The endofunctor L on \mathbf{C}^T induced by the adjunction $(\mathbf{C}^T \rightleftarrows \mathbf{C})$ forms a monoidal comonad.*

Proof. The required map $(L\varphi \otimes^T L\psi) \rightarrow L(\varphi \otimes^T \psi)$ is obtained from the bimorphism $T(u) \circ dst: [L\varphi, L\psi] \rightarrow L(\varphi \otimes^T \psi)$ —where u is the universal map from Lemma 5.1. Further, one needs $T(\eta_I): \mu_I \rightarrow L(\mu_I)$. \square

6. A bit of logic

Let's start by putting some results from the previous two sections together. Assume

- (a) \mathbf{C} is a cartesian closed category with equalizers and finite coproducts;
- (b) T is a commutative monad on \mathbf{C} such that its category of algebras has CRP's.

Then one has the following.

(1) The category of algebras \mathbf{C}^T is an SMCC and the free functor $\mathbf{C} \rightarrow \mathbf{C}^T$ preserves the symmetric monoidal structure.

(2) \mathbf{C}^T has finite products and coproducts. The forgetful functor $\mathbf{C}^T \rightarrow \mathbf{C}$ preserves the products.

Put L for the comonad induced on \mathbf{C}^T . The picture we get is as follows.

$$\begin{array}{ccc} \mathbf{C}^T & \xrightarrow{\quad} & (\mathbf{C}^T)_L \\ L \circlearrowleft & \xleftarrow{\quad} & \xrightarrow{\text{full \& faithf.}} \mathbf{C} \\ & & \circlearrowright T \end{array}$$

(3) L is a functor $(\mathbf{C}^T, 1^T, \times^T) \rightarrow (\mathbf{C}^T, I^T, \otimes^T)$ preserving the SMC-structure.

(4) The Kleisli category $(\mathbf{C}^T)_L$ induced by L is cartesian closed. Finite products are as in \mathbf{C}^T and the exponent $\psi \Rightarrow \chi$ is $L\psi \dashv^T \chi$ since

$$\begin{aligned} (\mathbf{C}^T)_L(\varphi \times^T \psi, \chi) &= \mathbf{C}^T(L(\varphi \times^T \psi), \chi) \\ &\cong \mathbf{C}^T(L\varphi \otimes^T L\psi, \chi) \\ &\cong \mathbf{C}^T(L\varphi, L\psi \dashv^T \chi) \\ &= (\mathbf{C}^T)_L(\varphi, \psi \Rightarrow \chi). \end{aligned}$$

(5) Algebras of the form $L\varphi$ in \mathbf{C}^T carry by (3) a natural comonoid structure, namely

$$L\varphi \xrightarrow{L^!} L(1^T) \cong I^T, \quad L\varphi \xrightarrow{L\delta} L(\varphi \times^T \varphi) \cong L\varphi \otimes^T L\varphi.$$

(6) In case T is relevant (resp. affine), the monoidal structure on \mathbf{C}^T has diagonals (resp. projections).

6.1. Intuitionistic linear logic. As a basis, we consider the fragment with the multiplicative connectives (I, \otimes, \multimap) and with the additive finite products $(1, \times)$ and co-products $(0, +)$. Rules for these (including the cut-rule) may be found in any standard text on linear logic. Just for convenience, we call this fragment our Basic Intuitionistic Logic (BIL); the commonly used reason for calling such a system ‘intuitionistic’ is that on the right hand side of a turnstile one has precisely one formula.

Let \square be a unary operation on formulae. One says the \square satisfies the (S4)-rules (or the comonad rules) if both

$$\frac{\Gamma, A \vdash B}{\Gamma, \square A \vdash B} \qquad \frac{\square \Gamma \vdash A}{\square \Gamma \vdash \square A}$$

where in the latter case $\square \Gamma$ denotes the sequence obtained from Γ by applying \square componentwise. We call \square a *weakening operation* if it satisfies the (S4)-rules and additionally

$$\frac{\Gamma \vdash B}{\Gamma, \square A \vdash B}$$

Similarly, \square will be called a *contraction operation* if one has the (S4)-rules, plus

$$\frac{\Gamma, \square A, \square A \vdash B}{\Gamma, \square A \vdash B}$$

Notice that in (BIL) one can always define a weakening operation \downarrow_w by $\downarrow_w A = A \times I$.

What we call Basic Intuitionistic Linear Logic (BILL, for convenience) here, is (BIL) plus a $!$ which is both a weakening and a contraction operation. If \mathbf{C} is a category with a monad T satisfying (a) + (b) above, then \mathbf{C}^T is by (1)–(5) a model of (BILL), see e.g. [33]. As examples one can think of the (finite) powerset monad, the abelian group monad or the linear span monad on **Sets**. The fact that examples of (BILL) can be obtained in such a simple way was also noted by Gordon Plotkin.

6.2. Relevant logic. A different system is obtained by considering (BIL) with the contraction rule and a weakening operation \downarrow_w . We find it useful to decorate the tensor in this case with a subscript: \otimes_c . Here are some formulae which are derivable from the empty context.

$$\begin{array}{ll} A \multimap (A \otimes_c A) & (A \otimes_c \downarrow_w B) \multimap A \\ A \multimap \downarrow_w B \multimap A & (A \multimap B \multimap C) \multimap (A \multimap B) \multimap A \multimap C \\ \downarrow_w (A \times B) \multimap (\downarrow_w A \otimes_c \downarrow_w B) & (\downarrow_w A \otimes_c \downarrow_w B) \multimap \downarrow_w (A \times B) \\ I \multimap \downarrow_w 1 & \downarrow_w 1 \multimap I \end{array}$$

Models of this relevant logic can be obtained from a relevant monad T on a category \mathbf{C} such that (a) + (b) above are satisfied. The canonical example is the lift monad on **Sets**.

(ii) Such a distributive law $\lambda: ST \rightarrow TS$ gives rise to the *composite monad* TS with

$$\begin{aligned}\eta^{TS} &= (\eta^T)S \circ \eta^S = T(\eta^S) \circ \eta^T, \\ \mu^{TS} &= (\mu^T)S \circ T^2(\mu^S) \circ T\lambda S = T(\mu^S) \circ (\mu^T)S^2 \circ T\lambda S.\end{aligned}$$

(iii) Suppose S, T are strong monads. We say $\lambda: ST \rightarrow TS$ is a *distributive law of strong monads* if in addition to the above four equations, one also has

$$\lambda \circ S(st^T) \circ st^S = T(st^S) \circ st^T \circ id \otimes \lambda.$$

And this λ will be called a *distributive law of commutative monads* if it is a distributive law of strong monads for which one has

$$\lambda \circ S(st^T) \circ st^S = T(st^S) \circ st^T.$$

Example 7.2. (i) Our basic example is the distributive law $\lambda_X: \mathcal{P}^+ \perp X \rightarrow \perp \mathcal{P}^+ X$ given by

$$\{\emptyset\} \mapsto \emptyset \quad \text{and} \quad \{\emptyset\} \neq A \mapsto \{\{x \mid \{x\} \in A\}\}.$$

It yields as composite monad $\perp \mathcal{P}^+ \cong \mathcal{P}$, the ordinary powerset monad. Notice that \mathcal{P} is thus the composite of its own affine and relevant parts.

Similarly there is in the finite case a distributive law $\mathcal{P}_f^+ \perp \rightarrow \perp \mathcal{P}_f^+$.

(ii) There are also distributive laws in the reverse direction: one has $\kappa_X: \perp \mathcal{P}^+ X \rightarrow \mathcal{P}^+ \perp X$ by $\emptyset \mapsto \{\emptyset\}$ and $\{a\} \mapsto \{\{x\} \mid x \in a\}$. Notice that $\lambda \circ \kappa = id$, but $\kappa \circ \lambda \neq id$. The latter follows from a simple cardinality argument. Similarly one can describe a map $\perp \mathcal{P}_f^+ \rightarrow \mathcal{P}_f^+ \perp$. All these are distributive laws of commutative monads.

(iii) There is a similar situation where $list \cong \perp list^+$ comes from a distributive law $list^+ \perp \rightarrow \perp list^+$.

(iv) The linear span $K(-)$ is also an example of a monad which can be written as composite of its relevant and affine part, but in an order which is different from powerset. There is a distributive law $\kappa_X: \perp K_a(X) \rightarrow K_a(\perp X)$ given by

$$\emptyset \mapsto 1\emptyset \quad \text{and} \quad \{k_1x_1 + \dots + k_nx_n\} \mapsto k_1\{x_1\} + \dots + k_n\{x_n\}.$$

The resulting composite monad $K_a(\perp -)$ is $K(-)$ since there is an isomorphism $K(X) \xrightarrow{\sim} K_a(\perp X)$ described by

$$k_1x_1 + \dots + k_nx_n \mapsto k_1\{x_1\} + \dots + k_n\{x_n\} + (1 - \sum_i k_i)\emptyset.$$

(v) One of the basic examples used by Beck in [6] is the distributivity of the monoid monad $list(-)$ over the abelian group monad $\mathbb{Z}(-)$. The distributive law involves the distributivity of multiplication over addition. The resulting composite monad $\mathbb{Z}(list(-))$ is the free ring monad.

The above are our main examples, but there are many more that could be mentioned. For example, the monoid monad list distributes over the copy monad $A \Rightarrow (-)$ (see [29]) and the quantale (with unit) monad results from distribution of list over powerset \mathcal{P} . In similar vein, in [30] the Plotkin powerdomain is described using a distributive law. Also composition of closure operations on a poset can be understood in terms of distributive laws.

Lemma 7.3. *Let S and T be strong monads and $\lambda: ST \rightarrow TS$ a distributive law of strong monads.*

- (i) *The composite monad TS is strong with strength $st^{TS} = T(st^S) \circ st^T$.*
- (ii) *The natural transformations $T(\eta^S): T \rightarrow TS$ and $(\eta^T)S: S \rightarrow TS$ are morphisms of strong monads.*
- (iii) *Suppose now λ is a distributive law of commutative monads. Then S and T are commutative implies that TS is commutative.*

Proof. (i) + (ii) Easy.

(iii) One has $dst^{TS} = T(dst^S) \circ dst^T$ and $dst'^{TS} = T(dst'^S) \circ dst'^T$. \square

The next thing we need are *liftings of monads*, as described in [6]. Assume $\lambda: ST \rightarrow TS$ is a distributive law of S over T on a category \mathbf{C} . The monad T can then be lifted to a monad \tilde{T} on the category \mathbf{C}^S of S -algebras. This is done as follows; define \tilde{T} by

$$(SX \xrightarrow{\varphi} X) \mapsto (STX \xrightarrow{\lambda_X} TSX \xrightarrow{T\varphi} TX) \quad \text{and} \quad f \mapsto Tf.$$

Unit and multiplication for \tilde{T} at component $SX \xrightarrow{\varphi} X$ are η_X^T and μ_X^T .

Thus one can form the category of algebras $(\mathbf{C}^S)^{\tilde{T}}$. It turns out that the comparison functor Ψ is an isomorphism in

$$\begin{array}{ccc}
 (\mathbf{C}^S)^{\tilde{T}} & \xrightarrow{\Psi} & \mathbf{C}^{TS} \\
 \uparrow \dashv & \sim & \uparrow \dashv \\
 \mathbf{C}^S & \xrightarrow{\quad \top \quad} & \mathbf{C} \\
 & \longleftarrow &
 \end{array}$$

Thus one obtains a picture

$$\begin{array}{ccc}
 \mathbf{C}^{TS} & \xrightarrow{- \circ T(\eta^S)} & \mathbf{C}^T \\
 \uparrow \dashv & \xleftarrow{\quad \top \quad} & \uparrow \dashv \\
 \mathbf{C}^S & \xrightarrow{- \circ \eta_S^T} & \mathbf{C} \\
 & \xleftarrow{\quad \top \quad} &
 \end{array}$$

in which the dashed arrow denotes a left adjoint to $(- \circ T(\eta^S))$, which exists by Lemma 3.3 in case \mathbf{C}^{TS} has CRP's. Let's assume such coequalizers exists; then one obtains two comonads on \mathbf{C}^{TS} , namely one induced by the adjunction $(\mathbf{C}^{TS} \rightleftarrows \mathbf{C}^S)$ — which will be denoted by \check{S} —and one by $(\mathbf{C}^{TS} \rightleftarrows \mathbf{C}^T)$ —denoted by \check{T} .

The next result establishes some basic properties in this situation. The proof follows from unravelling the constructions involved; it is somewhat technical and left to the interested reader.

Proposition 7.4. (i) *In the above situation, there is a distributive law of comonads $\check{T}\check{S} \rightarrow \check{S}\check{T}$.*

(ii) *The resulting composite comonad $\check{T}\check{S}$ is the one induced by the adjunction $(\mathbf{C}^{TS} \rightleftarrows \mathbf{C})$.*

(iii) *If the distributive law of monads $\lambda : ST \rightarrow TS$ is an isomorphism, then the induced distributive law of comonads $\check{T}\check{S} \rightarrow \check{S}\check{T}$ is an isomorphism as well. \square*

Proposition 7.5. *Consider again the above situation.*

(i) *If S is an affine and T a relevant monad, then \check{S} is a weakening and \check{T} a contraction operation.*

(ii) *The other way round, if S is relevant and T affine, then \check{S} is a contraction and \check{T} a weakening operation.*

Proof. (i) Using the projections in \mathbf{C}^S and diagonals in \mathbf{C}^T one obtains “counits” $\check{S}(\varphi) \rightarrow I^{TS}$ and “comultiplications” $\check{T}(\varphi) \rightarrow \check{T}(\varphi) \otimes^{TS} \check{T}(\varphi)$. The same argument applies for (ii). \square

In the following examples we first describe a model where one has $! = \underset{w}{!!}$ and then one with $! = \underset{wc}{!!}$.

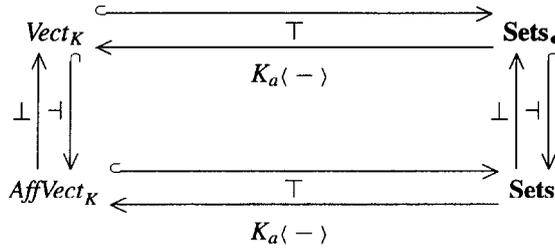
Example 7.6. (i) By now a lot more is understood about the situation in Example 2.3 (ii). Using the distributive law of monads $\lambda : \mathcal{P}^+ \perp \rightarrow \perp \mathcal{P}^+$, we find that the forgetful functor $\mathbf{CL} \rightarrow \mathbf{ACL}$ has a left adjoint given by lifting; it yields a weakening comonad $!$ on \mathbf{CL} , by (i) in the previous proposition. The forgetful functor $\mathbf{CL} \rightarrow \mathbf{Sets}_\bullet$ also has a left adjoint. The proof of Lemma 3.3 yields the following construction: take a pointed set X , form the free complete lattice $\mathcal{P}X$ on the set X and identify in all its members the base point \perp of X with \emptyset . This yields $!X = \mathcal{P}(X - \perp)$ as a contraction comonad on \mathbf{CL} . And indeed, as Proposition 7.4(ii) tells us, we have that $!X = \mathcal{P}X = \underset{c}{!!} \underset{w}{X}$ is the weakening and contraction comonad on \mathbf{CL} induced by the adjunction $(\mathbf{CL} \rightleftarrows \mathbf{Sets})$.

Notice that by Lemma 5.5(ii) one has

$$\underset{w}{!}(X \otimes_w Y) \cong \underset{w}{!}X \otimes \underset{w}{!}Y \quad \text{and} \quad \underset{c}{!}(X \otimes_c Y) \cong \underset{c}{!}X \otimes \underset{c}{!}Y$$

where \otimes_w, \otimes_c and \otimes_c are as introduced in Example 5.4(i).

(ii) Another interesting example arises from the distributive law $\kappa_X : \perp K_a(X) \rightarrow K_a(\perp X)$ in Example 7.2(v). It give rise to the following diagram.



The free functor $K_a(-) : \mathbf{Sets}_\bullet \rightarrow \mathbf{Vect}_K$ gives—by (ii) in the previous proposition—rise to a contraction comonad $!_c$ on \mathbf{Vect}_K . The functor $\mathbf{AffVect}_K \rightarrow \mathbf{Vect}_K$ turns an affine vector space into an ordinary one by adding a point. The proof of Lemma 3.3 gives therefore the recipe: take an affine vector space X , consider the free vector space $K(X)$ on the set X and identify therein all affine (i.e. non-zero) points. This yields $\perp X$ with a suitable vector space structure. One obtains a weakening operation $!_w$ on \mathbf{Vect}_K . Then $! = K(-) = !_{wc}$.

It is not true that the additional distributive law of monads $\kappa : \perp \mathcal{P}^+ \rightarrow \mathcal{P}^+ \perp$ from Example 7.2(ii) induces a distributive law of comonads $!_{wc} \rightarrow !_{cw}$ in the above example (i): suppose there is a map $\nu_X : !_{wc}(X) \rightarrow !_{cw}(X)$ commuting with the counits of $!_w$ and $!_c$. Then one has that $\nu_X : \perp \mathcal{P}(X - \perp) \rightarrow \mathcal{P}X$ acts as follows

$$\emptyset \mapsto \emptyset, \quad \{\emptyset\} \mapsto \{\perp\}, \quad \{a\} \mapsto a$$

where in the latter case $a \neq \emptyset$. But such a map ν_X is not linear: for $x \neq \perp$ in X , take $A = \{\{\emptyset\}, \{x\}\}$ in $\perp \mathcal{P}(X - \perp)$. Then $(\bigcup \circ \mathcal{P}(\nu_X))(A) = \bigcup \{\{\perp\}, \{x\}\} = \{\perp, x\}$ but $(\nu_X \circ \bigvee)(A) = \nu_X(\{\emptyset \cup \{x\}\}) = \{x\}$.

There are more examples like the above two: the category of event spaces introduced in [31] can be understood as the category of algebras of the composite monad on \mathbf{PoSets} which is obtained from the monad which adds joins of non-empty sets, followed by the monad which adds a top element.

8. The example of complete lattices over posets

In the examples that we have seen so far, we have separate weakening and contraction comonads $!_w$ and $!_c$ with only one distributive law between them—either $!_{wc} \rightarrow !_{cw}$ or $!_{cw} \rightarrow !_{wc}$. In this section we shall see that the category \mathbf{CL} of complete lattices carries weakening and contraction comonads $!_w$ and $!_c$ with $!_{wc} \cong !_{cw}$. These are not induced by the (monadic) adjunction $(\mathbf{CL} \rightleftarrows \mathbf{Sets})$, that we have studied so far, but by $(\mathbf{CL} \rightleftarrows \mathbf{PoSets})$.

On the (cartesian closed) category \mathbf{PoSets} of posets and monotone functions we consider the following three monads. The lift monad \perp which adds a bottom element \perp to a poset. The original elements $x \in X$ will in $\perp X$ be written as $[x]$. Thus the unit is $x \mapsto [x]$. The multiplication $\perp \perp X \rightarrow \perp X$ can then be described by $[[x]] \mapsto [x]$ and $[\perp], \perp \mapsto \perp$. Further, we consider the monad \mathcal{D}^+ of non-empty downsets and

the monad \mathcal{D} of arbitrary downsets. The category of algebras of the lift monad is the category \mathbf{PoSets}^\perp of posets with bottom element and monotone “strict” (i.e. bottom preserving) functions. The category of algebras of \mathcal{D}^+ is the category \mathbf{ACL} of affine complete lattices and the category of algebras of \mathcal{D} is \mathbf{CL} . It is easy to verify that these three monads are commutative; hence the resulting categories of algebras have SMCC structures which are preserved by the free functors from \mathbf{PoSets} .

There are distributive laws $\lambda : \mathcal{D}^+ \perp \rightarrow \perp \mathcal{D}^+$ and $\kappa : \perp \mathcal{D}^+ \rightarrow \mathcal{D}^+ \perp$ given by

$$\lambda(a) = \begin{cases} \perp, & \text{if } a = \{\perp\}, \\ \{[x \mid [x] \in a]\}, & \text{else,} \end{cases}$$

$$\kappa(b) = \begin{cases} \{\perp\}, & \text{if } b = \perp, \\ \{[x] \mid x \in c\} \cup \{\perp\}, & \text{if } b = [c]. \end{cases}$$

We show that λ and κ are each other’s inverses. The first cases are obvious and so we concentrate on the second ones. If $b = [c]$, then

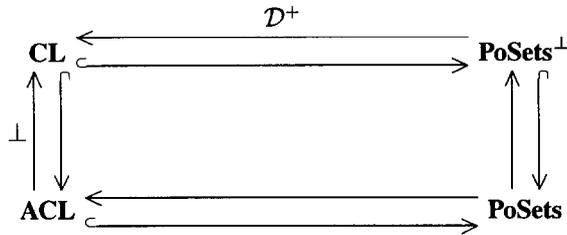
$$(\lambda \circ \kappa)(b) = \lambda(\{[x] \mid x \in c\} \cup \{\perp\}) = \{[x \mid x \in c]\} = b$$

and if $a \neq \perp$, then

$$(\kappa \circ \lambda)(a) = \kappa(\{[x \mid [x] \in a]\}) = \{[x] \mid [x] \in a\} \cup \{\perp\} = a$$

the latter because a is a non-empty downset, so it must contain \perp .

We conclude that the monads \mathcal{D}^+ and \perp commute and that $\mathcal{D} \cong \perp \mathcal{D}^+ \cong \mathcal{D}^+ \perp$. The resulting decomposition diagram is as follows.



One obtains from \mathcal{D}^+ a contraction comonad $!$ on \mathbf{CL} and from \perp a weakening comonad $\overset{w}{!}$ on \mathbf{CL} . Then $!! \overset{cw}{\cong} ! \overset{wc}{\cong} !!$ where $!$ is the comonad induced by the adjunction $(\mathbf{CL} \rightleftarrows \mathbf{PoSets})$, see Proposition 7.4(iii).

Next we study the associated Kleisli categories. The SMCC structure on \mathbf{CL} will be written as (I, \otimes, \multimap) .

Proposition 8.1. *The Kleisli category of the weakening comonad $\overset{w}{!}$ on \mathbf{CL} is the category \mathbf{CL}_{aff} of complete lattices and affine functions (see Example 2.3(iii)). It is a model of affine logic because*

- (i) the tensor \otimes of \mathbf{ACL} (cf. Example 5.4(i)) extends to \mathbf{CL}_{aff} ; the associated internal hom \multimap is given by $Y \multimap Z = !Y \multimap Z$. Thus \mathbf{CL}_{aff} is an SMCC with projections.
- (ii) the comonad $!$ on \mathbf{CL} restricts to a contraction comonad on \mathbf{CL}_{aff} .

Proof. First note that for $X, Y \in \mathbf{CL}$, linear functions $!X \rightarrow Y$ correspond to affine ones $X \rightarrow Y$.

(i) The main point is that the tensor \otimes from Example 5.4(i) applied to $X, Y \in \mathbf{CL}$ yields again a complete lattice $X \otimes Y$: it is by construction an affine complete lattice, but it has a bottom element, namely $\{(\perp_X, \perp_Y)\}$. Hence it is a complete lattice. Then

$$\begin{aligned} \mathbf{CL}_{\text{aff}}(X \otimes Y, Z) &= \mathbf{CL}(\underset{w}{!}(X \otimes Y), Z) \\ &\cong \mathbf{CL}(\underset{w}{!}(X) \otimes \underset{w}{!}(Y), Z) \\ &\cong \mathbf{CL}(\underset{w}{!}(X), \underset{w}{!}(Y) \multimap Z) \\ &= \mathbf{CL}_{\text{aff}}(X, Y \multimap Z). \end{aligned}$$

(ii) Obviously $!$ is also a comonad on \mathbf{CL}_{aff} . A comultiplication $!X \rightarrow !X \otimes !X$ in \mathbf{CL}_{aff} is obtained as the following composite.

$$\begin{aligned} \underset{w}{!} \underset{c}{!} \underset{w}{!} X &\xrightarrow{\sim} \underset{c}{!} \underset{w}{!} \underset{w}{!} X \\ &\longrightarrow \underset{c}{!} \underset{w}{!} \underset{w}{!} X \otimes \underset{c}{!} \underset{w}{!} \underset{w}{!} X \quad \text{using that } ! \text{ is a contraction operation on } \mathbf{CL} \\ &\xrightarrow{\sim} \underset{w}{!} \underset{c}{!} \underset{w}{!} X \otimes \underset{w}{!} \underset{c}{!} \underset{w}{!} X \\ &\xrightarrow{\sim} \underset{w}{!} (\underset{c}{!} \underset{c}{!} X \otimes \underset{w}{!} \underset{c}{!} X) \\ &\longrightarrow \underset{c}{!} \underset{w}{!} X \otimes \underset{w}{!} \underset{c}{!} X \quad \text{using the counit of } !. \quad \square \end{aligned}$$

There is a similar result for $!$.

Proposition 8.2. *The Kleisli category of the contraction comonad $!$ on \mathbf{CL} is the category \mathbf{CL}_{str} of complete lattices and strict functions. It is a model of relevant logic because*

- (i) *the smash product \otimes of \mathbf{PoSets}_{\perp} extends to \mathbf{CL}_{str} ; the associated internal hom \multimap is given by $Y \multimap Z = !Y \multimap Z$. Thus \mathbf{CL}_{str} is an SMCC with diagonals.*
- (ii) *the comonad $!$ on \mathbf{CL} restricts to a weakening comonad on \mathbf{CL}_{str} . \square*

Thus we know what structure the Kleisli categories of $!$ and $!$ have. The analysis at the beginning of Section 6 yields that the Kleisli category of $! = \underset{w}{!} = \underset{c}{!}$ is cartesian closed—because $!$ is induced by the adjunction $(\mathbf{CL} \rightleftarrows \mathbf{Sets})$.

Next we show how dual versions of the above structure on complete lattices can be obtained. If we write X^{\perp} for the poset obtained from X by reverting the order (i.e. $x \leq y$ in X^{\perp} iff $y \leq x$ in X), then X^{\perp} is a complete lattice in case X is. Moreover $X^{\perp\perp} = X$. Alternatively, X^{\perp} can be described as the internal hom $X \multimap I$ in \mathbf{CL} —where $I = \{\perp, \top\}$ is neutral for \otimes . One has that \mathbf{CL} is a $*$ -autonomous category, see [3,4]. A useful property in such categories is the bijective correspondence between morphisms $X \rightarrow Y$ and $Y^{\perp} \rightarrow X^{\perp}$. Thus one obtains De Morgan isomorphisms $X + Y \cong (X^{\perp} \times Y^{\perp})^{\perp}$ and $0 \cong 1^{\perp}$.

The following two results are very much like in [1, Chapter 8, Theorem 4.6].

Lemma 8.3. *Consider the following endofunctors on \mathbf{CL}*

$$? = (-)^{\perp} \circ ! \circ (-)^{\perp}, \quad ?_w = (-)^{\perp} \circ \underset{w}{!} \circ (-)^{\perp}, \quad ?_c = (-)^{\perp} \circ \underset{c}{!} \circ (-)^{\perp}.$$

Then $\overset{?}{\oplus}$, $\overset{?}{\otimes}$ and $\overset{?}{\circ}$ are monads on \mathbf{CL} .

Proof. If (L, ε, δ) is a comonad on a $*$ -autonomous category \mathbf{C} , then $T = (-)^\perp \circ L \circ (-)^\perp$ is a monad on \mathbf{C} with unit $\eta_X = (\varepsilon_{X^\perp})^\perp$ and multiplication $\mu_X = (\delta_{X^\perp})^\perp$. \square

Lemma 8.4. For $X, Y \in \mathbf{CL}$ put

$$\begin{aligned} X \oplus Y &= (X^\perp \otimes Y^\perp)^\perp, & X \overset{?}{\oplus} Y &= (X^\perp \overset{?}{\otimes} Y^\perp)^\perp, \\ J &= I^\perp, & X \overset{?}{\otimes} Y &= (X^\perp \overset{?}{\circ} Y^\perp)^\perp. \end{aligned}$$

- (i) $(\overset{?}{\oplus}, J)$ yields a symmetric monoidal structure on \mathbf{CL} .
- (ii) $(\overset{?}{\oplus}, 0)$ yields a SMC-structure on the Kleisli category of $\overset{?}{\circ}$ and similarly $(\overset{?}{\otimes}, J)$ on the Kleisli category of $\overset{?}{\circ}$.
- (iii) One has the following isomorphisms.

$$\begin{array}{ccc} \overset{?}{\circ}(X + Y) \cong \overset{?}{\circ}X \overset{?}{\oplus} \overset{?}{\circ}Y & \overset{?}{\circ}(X \overset{?}{\oplus} Y) \cong \overset{?}{\circ}X \overset{?}{\oplus} \overset{?}{\circ}Y & \overset{?}{\circ}(X \overset{?}{\otimes} Y) \cong \overset{?}{\circ}X \overset{?}{\otimes} \overset{?}{\circ}Y \\ \overset{?}{\circ}0 \cong J & \overset{?}{\circ}0 \cong 0 & \overset{?}{\circ}J \cong J \end{array}$$

Proof. (i) Obvious.

(ii) As in the proof of the previous lemma, using Propositions 8.1 and 8.2: suppose (\otimes, I) forms an SMC structure on the Kleisli category of a comonad L and put $X \overset{?}{\oplus} Y = (X^\perp \otimes Y^\perp)^\perp$ and $J = I^\perp$. Then $\overset{?}{\oplus}$ extends to a functor on the Kleisli category of $T = (-)^\perp \circ L \circ (-)^\perp$,

$$\frac{\frac{\frac{X \longrightarrow U \text{ in } \mathbf{C}_T}{L(U^\perp) \longrightarrow X^\perp \text{ in } \mathbf{C}}{U^\perp \longrightarrow X^\perp \text{ in } \mathbf{C}_L}}{U^\perp \otimes V^\perp \longrightarrow X^\perp \otimes Y^\perp \text{ in } \mathbf{C}_L}}{X \overset{?}{\oplus} Y \longrightarrow U \overset{?}{\oplus} V \text{ in } \mathbf{C}_T} \quad \frac{\frac{\frac{Y \longrightarrow V \text{ in } \mathbf{C}_T}{L(V^\perp) \longrightarrow Y^\perp \text{ in } \mathbf{C}}{V^\perp \longrightarrow Y^\perp \text{ in } \mathbf{C}_L}}{U^\perp \otimes V^\perp \longrightarrow X^\perp \otimes Y^\perp \text{ in } \mathbf{C}_L}}{X \overset{?}{\oplus} Y \longrightarrow U \overset{?}{\oplus} V \text{ in } \mathbf{C}_T}$$

Further $X \overset{?}{\oplus} J = (X^\perp \otimes I)^\perp \cong X^{\perp\perp} = X$.

(iii) $\overset{?}{\circ}(X + Y) = [!(X + Y)^\perp]^\perp \cong [!(X^\perp \times Y^\perp)]^\perp \cong [!(X^\perp) \otimes !(Y^\perp)]^\perp = \overset{?}{\circ}X \overset{?}{\oplus} \overset{?}{\circ}Y$. The rest follows from a similar argument. \square

Lemma 8.5. (i) The tensor sum $\overset{?}{\oplus}$ has coprojections and $\overset{?}{\otimes}$ has codiagonals.

(ii) The monad $\overset{?}{\circ}$ supports “co-weakening” and $\overset{?}{\circ}$ supports “co-contraction”.

Proof. (i) Because the initial object 0 is neutral for $\overset{?}{\oplus}$ one obtains coprojections like in Definition 2.1(ii),

$$X \xrightarrow{\sim} X \overset{?}{\oplus} 0 \longrightarrow X \overset{?}{\oplus} Y, \quad Y \xrightarrow{\sim} 0 \overset{?}{\oplus} Y \longrightarrow X \overset{?}{\oplus} Y$$

Since there are diagonals $X^\perp \rightarrow (X^\perp \overset{?}{\otimes} X^\perp)$ one gets codiagonals $(X \overset{?}{\oplus} X) \rightarrow X$ using $(-)^\perp$.

(ii) Using (i) above and (iii) from the previous lemma, one obtains a unit for $\overset{?}{\circ}$ and a multiplication for $\overset{?}{\circ}$ with respect to the the monoidal structure $(\overset{?}{\oplus}, J)$ on \mathbf{CL} . \square

$$J \cong \underset{w}{?}(0) \longrightarrow \underset{w}{?}(X), \quad \underset{c}{?}X \oplus \underset{c}{?}X \cong \underset{c}{?}(X \oplus X) \longrightarrow \underset{c}{?}(X). \quad \square$$

Let’s recapitulate the main categorical aspects of **CL** that we are using. **CL** comes equipped with a weakening comonad $\underset{w}{!}$ and a contraction comonad $\underset{c}{!}$ such that $\underset{w}{!} \cong \underset{c}{!}$. Moreover, the Kleisli category of $\underset{w}{!}$ is an SMCC with projections, the Kleisli category of $\underset{c}{!}$ is an SMCC with diagonals and the Kleisli category of $\underset{w}{!}$ is cartesian closed. Finally, the fact that **CL** is $*$ -autonomous enables yields the De Morgan duals of the shrieks and tensors by involution.

This structure on **CL** suggests a logical system which combines linear, affine, relevant and intuitionistic logic. We concentrate on a version with solely one formula on the right hand side of a turnstile. The extension to a “classical version” (with also sequents on the right) is rather straightforward using the involution $(-)^{\perp}$, but notationally cumbersome.

The judgements we use have the basic form

$$\rangle_1 \Gamma \rangle_a \Delta \rangle_r \Theta \rangle_i \Xi \vdash B$$

where \rangle_x works as a separator; it should not suggest an order. We call Γ the linear context, Δ the affine context, Θ the relevant context and Ξ the intuitionistic context. Permutation is allowed in all of these. The idea is that formulae in Γ are used exactly once, in Δ at most once, in Θ at least once and in Ξ arbitrary many times. Thus we allow weakening in the affine and in the intuitionistic context and contraction in the relevant and in the intuitionistic context.

Semantically we think of such an entailment as a map $\Gamma \otimes \underset{w}{!}\Delta \otimes \underset{c}{!}\Theta \otimes \Xi \longrightarrow B$, where a modality applied to a context means componentwise application. The modality $!$ can be read as a defined operation, namely as $! = \underset{w}{!} = \underset{c}{!}$. In principle one can do without $!$ and without the intuitionistic context, but that seems less natural.

Whenever one of the delimiters \rangle_x is missing, we mean that the corresponding context is empty, e.g. in $\rangle_1 \Gamma \rangle_r \Theta \vdash B$, the affine and intuitionistic contexts are empty. There will be a number of rules about moving formulae from one context to another. Girard [11,12] calls them *permeability rules*. Because of these, other rules can be formulated with most of the contexts empty; this is not a restriction. But we don’t always give the minimal version.

Axiom.

$$\rangle_1 A \vdash A$$

Cut.

$$\frac{\rangle_1 \Gamma \vdash A \quad \rangle_1 \Gamma', A \vdash B}{\rangle_1 \Gamma, \Gamma' \vdash B}$$

Cut rules for the other compartments are then derivable.

Weakening and contraction.

$$\frac{\rangle_1 \Gamma \rangle_a \Delta \rangle_r \Theta \rangle_i \Xi \vdash B}{\rangle_1 \Gamma \rangle_a \Delta, A \rangle_r \Theta \rangle_i \Xi \vdash B} \quad \frac{\rangle_1 \Gamma \rangle_a \Delta \rangle_r \Theta, A, A \rangle_i \Xi \vdash B}{\rangle_1 \Gamma \rangle_a \Delta \rangle_r \Theta, A \rangle_i \Xi \vdash B}$$

Weakening and contraction in the intuitionistic context are derived rules.

Free moves from left to right.

$$\frac{\lambda_1 \Gamma, A \lambda_a \Delta \lambda_r \Theta \lambda_i \Xi \vdash B}{\lambda_1 \Gamma \lambda_a \Delta, A \lambda_r \Theta \lambda_i \Xi \vdash B} \qquad \frac{\lambda_1 \Gamma, A \lambda_a \Delta \lambda_r \Theta \lambda_i \Xi \vdash B}{\lambda_1 \Gamma \lambda_a \Delta \lambda_r \Theta, A \lambda_i \Xi \vdash B}$$

$$\frac{\lambda_1 \Gamma \lambda_a \Delta, A \lambda_r \Theta \lambda_i \Xi \vdash B}{\lambda_1 \Gamma \lambda_a \Delta \lambda_r \Theta \lambda_i \Xi, A \vdash B} \qquad \frac{\lambda_1 \Gamma \lambda_a \Delta \lambda_r \Theta, A \lambda_i \Xi \vdash B}{\lambda_1 \Gamma \lambda_a \Delta \lambda_r \Theta \lambda_i \Xi, A \vdash B}$$

Annotated moves.

$$\frac{\lambda_1 \Gamma, \underset{w}{!}A \lambda_a \Delta \lambda_r \Theta \lambda_i \Xi \vdash B}{\lambda_1 \Gamma \lambda_a \Delta, A \lambda_r \Theta \lambda_i \Xi \vdash B} \qquad \frac{\lambda_1 \Gamma, \underset{c}{!}A \lambda_a \Delta \lambda_r \Theta \lambda_i \Xi \vdash B}{\lambda_1 \Gamma \lambda_a \Delta \lambda_r \Theta, A \lambda_i \Xi \vdash B}$$

$$\frac{\lambda_1 \Gamma \lambda_a \Delta \lambda_r \Theta, \underset{w}{!}A \lambda_i \Xi \vdash B}{\lambda_1 \Gamma \lambda_a \Delta \lambda_r \Theta \lambda_i \Xi, A \vdash B} \qquad \frac{\lambda_1 \Gamma \lambda_a \Delta, \underset{c}{!}A \lambda_r \Theta \lambda_i \Xi \vdash B}{\lambda_1 \Gamma \lambda_a \Delta \lambda_r \Theta \lambda_i \Xi, A \vdash B}$$

The double bar means that the rules may be used in both directions.

Exponential introduction on the right.

$$\frac{\lambda_a \Delta \vdash B}{\lambda_a \Delta \vdash \underset{w}{!}B} \qquad \frac{\lambda_r \Theta \vdash B}{\lambda_r \Theta \vdash \underset{c}{!}B}$$

A similar rule for $! = \underset{wc}{!!} = \underset{cw}{!!}$ in the intuitionistic context is then derivable.

Tensor introduction on the right.

$$\frac{\lambda_1 \Gamma \vdash B \qquad \lambda_1 \Gamma' \vdash B'}{\lambda_1 \Gamma, \Gamma' \vdash B \otimes B'}$$

$$\frac{\lambda_a \Delta \vdash B \qquad \lambda_a \Delta' \vdash B'}{\lambda_a \Delta, \Delta' \vdash B \underset{w}{\otimes} B'} \qquad \frac{\lambda_r \Theta \vdash B \qquad \lambda_r \Theta' \vdash B'}{\lambda_r \Theta, \Theta' \vdash B \underset{c}{\otimes} B'}$$

Tensor introduction on the left.

$$\frac{\lambda_1 \Gamma, A, A' \lambda_a \Delta \lambda_r \Theta \lambda_i \Xi \vdash B}{\lambda_1 \Gamma, A \otimes A' \lambda_a \Delta \lambda_r \Theta \lambda_i \Xi \vdash B}$$

$$\frac{\lambda_1 \Gamma \lambda_a \Delta, A, A' \lambda_r \Theta \lambda_i \Xi \vdash B}{\lambda_1 \Gamma \lambda_a \Delta, A \underset{w}{\otimes} A' \lambda_r \Theta \lambda_i \Xi \vdash B} \qquad \frac{\lambda_1 \Gamma \lambda_a \Delta \lambda_r \Theta, A, A' \lambda_i \Xi \vdash B}{\lambda_1 \Gamma \lambda_a \Delta \lambda_r \Theta, A \underset{c}{\otimes} A' \lambda_i \Xi \vdash B}$$

The form of the last two rules comes from the fact that $\underset{w}{\otimes}$ and $\underset{c}{\otimes}$ are tensors in the Kleisli categories of $\underset{w}{!}$ and $\underset{c}{!}$. Since the cartesian product \times and the coproduct $+$ live in the linear world, we have the following three rules.

Product introduction on the right and left

$$\frac{\lambda_1 \Gamma \vdash A \qquad \lambda_1 \Gamma \vdash B}{\lambda_1 \Gamma \vdash A \times B} \qquad \frac{\lambda_1 \Gamma, A \vdash C}{\lambda_1 \Gamma, A \times B \vdash C} \qquad \frac{\lambda_1 \Gamma, B \vdash C}{\lambda_1 \Gamma, A \times B \vdash C}$$

Coproduct introduction on the right and left

$$\frac{\lambda_1 \Gamma \vdash A}{\lambda_1 \Gamma \vdash A + B} \quad \frac{\lambda_1 \Gamma \vdash B}{\lambda_1 \Gamma \vdash A + B} \quad \frac{\lambda_1 \Gamma, A \vdash C \quad \lambda_1 \Gamma, B \vdash C}{\lambda_1 \Gamma, A + B \vdash C}$$

Also the constants I , 1 and 0 live in the linear world.

$$\frac{\lambda_1 \Gamma \vdash B}{\lambda_1 \Gamma, I \vdash B} \quad \vdash I \quad \lambda_1 \Gamma \vdash 1$$

Linear implication introduction on the right and on the left.

$$\frac{\lambda_1 \Gamma, A \vdash B}{\lambda_1 \Gamma \vdash A \multimap B} \quad \frac{\lambda_1 \Gamma \vdash A \quad \lambda_1 \Gamma', B \vdash C}{\lambda_1 \Gamma, \Gamma', A \multimap B \vdash C}$$

Let's first notice that one has annotated weakening and contraction in the linear context:

$$\frac{\lambda_1 \Gamma \vdash B}{\lambda_1 \Gamma \rangle_a A \vdash B} \quad \frac{\lambda_1 \Gamma, !A, !A \vdash B}{\lambda_1 \Gamma \rangle_r A, A \vdash B} \\ \frac{\lambda_1 \Gamma \rangle_a A \vdash B}{\lambda_1 \Gamma, !A \vdash B} \quad \frac{\lambda_1 \Gamma \rangle_r A \vdash B}{\lambda_1 \Gamma, !A \vdash B}$$

Also, both for $!$ and for $!$ one has the (S4) rules from Section 6 in the linear context. Similarly $!$ is a contraction operation in the affine context and $!$ a weakening operation in the relevant context. Further, notice that the following rule is derivable.

$$\frac{\lambda_1 \Gamma \rangle_a \Delta, !A \rangle_r \Theta \rangle_i \Xi \vdash B}{\lambda_1 \Gamma \rangle_a \Delta \rangle_r \Theta, !A \rangle_i \Xi \vdash B} \quad (\text{distributive laws})$$

by using the annotated moves with respect to the intuitionistic context. These distributive laws have to be added explicitly in case one prefers a formulation without the intuitionistic context.

The following abbreviations for affine, relevant and intuitionistic implication are convenient.

$$A \multimap_w B = (!A) \multimap B, \quad A \multimap_c B = (!A) \multimap B, \quad A \Rightarrow B = (!A) \multimap B.$$

The expected left and right introduction rules are derivable for these: one has

- (i) linear logic in the linear context;
- (ii) affine logic in the affine context (with connectives \otimes , 1 , \multimap_w , \times and $!$), see Proposition 8.1;
- (iii) relevant logic in the relevant context (with \otimes , I , \multimap_c , \times , 1 and $!$), see Proposition 8.2;
- (iv) intuitionistic logic in the intuitionistic context (with \times , 1 and \Rightarrow)

Thus we understand the above system as a combination of linear, affine, relevant and classical logic. We emphasize that the above collection of rules is obtained by looking at what holds in the category of complete lattices. At this stage, it is presented without proof-theoretic analysis.

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